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PROGRESS REPORT.

The committee notes with great pleasure that Mr. S. Ramanujan, a member of the society, who went to Cambridge as a research scholar from Madras, has recently obtained the rare distinction of being elected a Fellow of the Royal Society. This is a great honour of which the whole of India may be proud, and the fact that he was elected to the Fellowship on the first occasion when his name was proposed, and at the comparatively early age of thirty, is eloquent testimony to the excellent work that he has done in the field of Mathematics. It is the earnest hope of the Committee that Mr. Ramanujan may be blessed with long life and health to enable him to engage in further research work, winning fresh laurels for himself and bringing credit and lustre to his motherland, while at the same time serving humanity at large by valuable contributions to the advancement of science.

D. D. Kapadia,

Honorary Secretary.

Complex Roots of Equations.

By M. T. Naraniengar.

Introduction.

It is proposed to discuss in this paper *graphic methods* of visualising the complex roots of an equation. To this end Argand diagrams are freely used, and complex roots obtained as the *real* intersections of two plane curves. The method of the Theory of Functions of Complex Variables is briefly indicated towards the close of the paper.

The paper is conveniently divided into three Sections: (1) The First Section treats of equations up to the fifth degree by means of the elementary methods of curve tracing; (2) In the second Section, Approximations to Complex Roots of a general algebraical equation, are developed by means of Cauchy's expansion and Taylor's theorem, and *incidentally* a "Method of dealing with the Intersections of Plane Curves" is referred to and its connection with the properties of Polar Curves explained; (3) The third and last Section is devoted to a brief discussion of Transcendental equations.

The Post Script deals with simultaneous equations involving two Complex Variables.

The computation of imaginary roots of Numerical Algebraic Equations has not received much attention on the part of mathematicians. There is a reference to a modification of Horner's method so as to apply to imaginary roots on page 12 of Mathematical Monographs, No. 10.—*The Solution of Equations* by M. Merriman*. Mc Clintock in 1894 published a method of development in series of the roots of an equation by means of his Calculus of Enlargement†. His method can be used for approximate computations when the series is convergent. Lambert in 1903 gave out a similar method of expansion by Maclaurin's formula, applicable to Algebraic, as well as, Transcendental Equations‡. This method consists in introducing a second variable into all the terms but two of an equation and putting it equal to unity after the expansion of the first variable is obtained by Maclaurin's theorem.

Dr. F. S. Macaulay in his *Algebraic Theory of Modular Systems*, 1916, No. 19 of Cambridge Tracts, gives the following further references: Transactions of the American Mathematical Society, No. 3, 1902; and No. 5, 1904.

* Sheffler, Die Auflösung der algebraischen and transzendenten Gleichungen Braunschweig, 1859; and Jelink, Die Auflosung der hoheren numerisehen Gleichungen, Leipzig, 1865.

† Bulletin of American Mathematical Soc., 1894, Vol. I, p. 37; also, American Journal of Mathematics, Vol XVII, pp. 89—110.

‡ American Philosophical Society Proceedings, Vol. 42.

Section (1).

1. Consider the quadratic equation

$$z^2+2pz+q=0. \qquad (1)$$

We may write its roots in the form $(x+iy)$ without loss of generality. Substituting in the equation and separating real and imaginary parts, we have

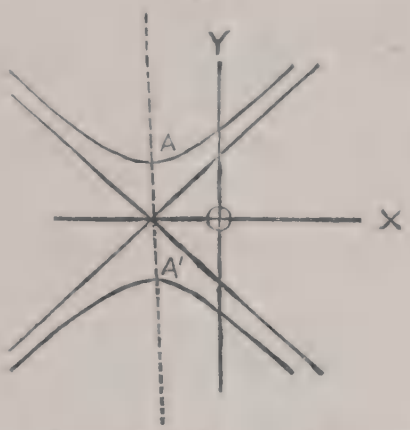
$$(x^2-y^2+2px+q)+2iy(x+p)=0,$$

which is equivalent to the two equations

$$\left.\begin{aligned} x^2-y^2+2px+q&=0 \\ y(x+p)&=0. \end{aligned}\right\} \qquad (2)$$

Drawing the curves (2), we find (x,y) as their intersection. In other words, the Argand representations of the complex roots $(x+iy)$ are the *real* intersections of the Cartesian curves (2). It is quite easy to see that these intersect outside the axis of x only when the roots of

Fig. 1



the quadratic are complex. In this case the graphic points A, A′ will represent the complex roots

$$\{-p\pm i\sqrt{q-p^2}\} \qquad \text{[see Fig. 1.]}$$

2. Next, suppose the equation is

$$f(z)\equiv z^3+3pz^2+3qz+r=0. \qquad \ldots \qquad \ldots \qquad (3)$$

Proceeding as before, we have

$$f(x\pm iy)=f(x)\pm iy\,f'(x)-\frac{y^2}{2}f''(x)\mp\frac{iy^3}{6}f'''(x)=0,$$

which breaks up into the two separate equations

$$\left.\begin{aligned} f(x)-\tfrac{1}{2}\,y^2\,f''(x)&=0 \\ y[f'(x)-\tfrac{1}{6}\,y^2\,f'''(x)]&=0 \end{aligned}\right\}$$

The factor $y=0$, in the second equation leads to $f(x)=0$, determining the real roots of the proposed equation. The complex roots (when they exist) are represented graphically as the real intersections of

$$\left.\begin{aligned} f(x)-\tfrac{1}{2}\,y^2\,f''(x)&=0 \\ f'(x)-\tfrac{1}{6}\,y^2\,f'''(x)&=0. \end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad (4)$$

Now the cubic and conic represented by (4) can be readily traced and their intersections obtained; or (4) may be reduced to a cubic equation in x by eliminating y^2 and the real roots of this resulting cubic obtained. The corresponding real values of y will then give the graphic points, required.

[*cf.* MacRobert's *Theory of Functions*, pp. 16—19.]

3. The biquadratic equation $f(z)\equiv 0$, treated similarly leads to the intersections of the curves

$$\left.\begin{aligned} f(x)-\tfrac{1}{2}y^2f''(x)+y^4=0,\\ f'(x)-\tfrac{1}{6}y^2f'''(x)=0,\end{aligned}\right\} \quad \ldots \quad \ldots \quad (5)$$

which are of the 4th and 3rd degrees respectively.

The elimination of y^2 between these gives an equation in x which may also be used for locating the real values of x, but the process is tedious.

4. Instead of the Cartesian $(x+iy)$ for the complex root, we may take the standard form $r(\cos\theta\pm i\sin\theta)$. The auxiliary curves corresponding to the complex roots will then reduce to

$$r^n\cos n\theta+p_1r^{n-1}\cos(n-1)\theta+p_2r^{n-2}\cos(n-2)\theta+\ldots+p_n=0,$$
$$r^n\sin n\theta+p_1r^{n-1}\sin(n-1)\theta+p_2r^{n-2}\sin(n-2)\theta+\ldots+p_n=0.$$

5. In connection with the method of this section the following hints on *curve-tracing* may be of use:

Suppose $F(x,y)=0$ is any plane curve. Then in relation to the curve the set of points in the plane of the curve may be grouped into three classes:

(i) The group of points for which $F(x,y)$ takes a positive value; say, the *positive group*.

(ii) The group of points for which $F(x,y)$ takes a negative value; say, the *negative group*.

(iii) The group of points for which $F(x,y)$ takes a zero value; say the *zero group*.

The last group defines the curve, and the other two groups are separated by it. Thus the whole plane is divided into *compartments* of *positive* and *negative* points, whose boundary is the curve itself.

Now, let P be any positive point and Q any negative point. Then by the principle of continuity we infer that there must be some point between P and Q lying on the curve, which can be approximately

found. The method, though laborious, is useful in locating a curve consisting of several branches. Further, its essential importance as a method consists in its negative character; it does not require any knowledge of points on the curve, as in the usual method of curve tracing. By this method we can locate a curve of any degree in relation to two points taken at random.

Ex. 1.—As an illustration, let us consider the equation $z^5-4z-2=0$ [Cajori: *Theory of Equations*, p. 29].

The auxiliary curves being denoted by P and Q, we have

$P\equiv r^5\cos 5\theta-4r\cos\theta-2\equiv x^5-10x^3y^2+5xy^4-4x-2=0,$

$Q\equiv r^5\sin 5\theta-4r\sin\theta\equiv y(5x^4-10x^2y^2+y^4-4)=0.$

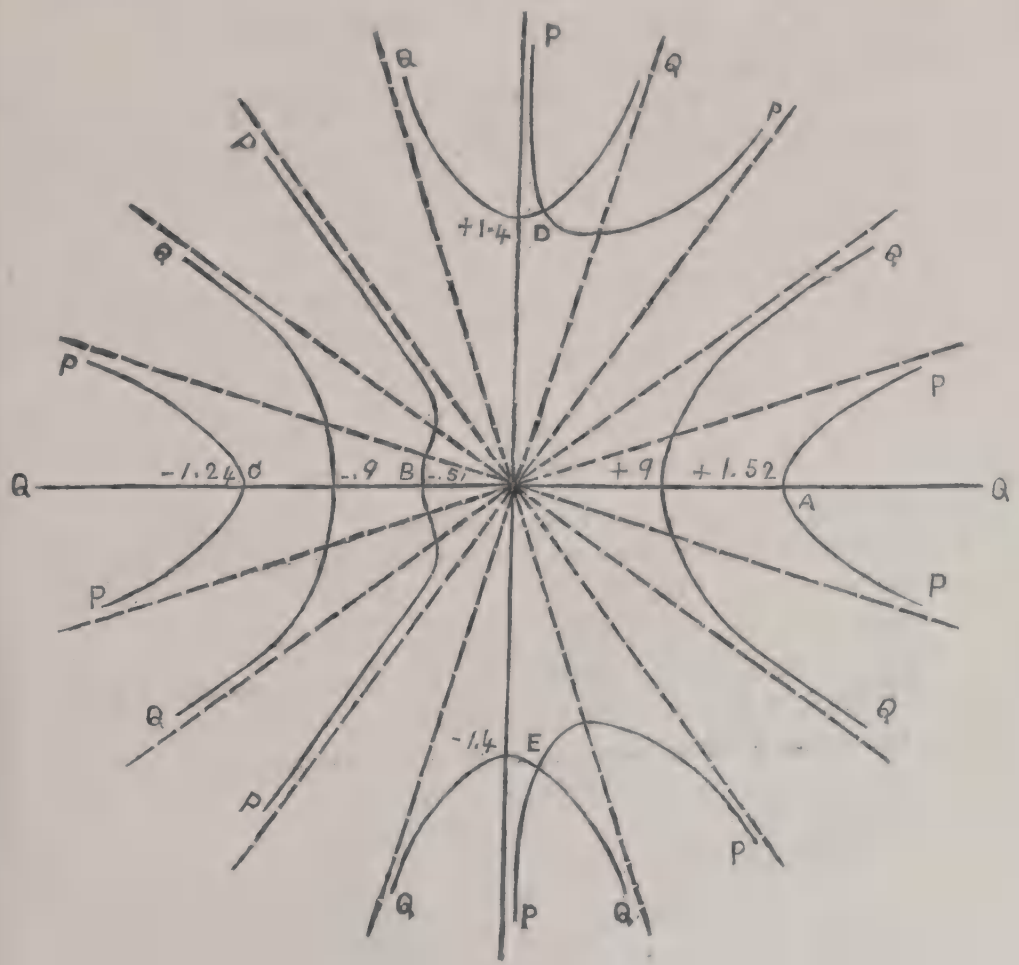
(I) *The P—curve*:—Considering P as a quadratic in y^2, we find that the discriminant

$D=100x^6-20x(x^5-4x-2)=40(2x^6+2x^2+x)$ and is positive for all values of x not lying between 0 and $-\cdot48$, since $D=0$ for $x=-\cdot48$ nearly.

Hence y has real values except when x lies between these limits.

Again all the values of y will be real only, if $\phi\equiv(x^5-4x-2)/x$ is positive. The latter changes sign when $x=1\cdot52,-\cdot51,-1\cdot24$ nearly.

Fig. 2.



Thus the following cases arise

(i) $\phi = +$, when $x > 1{\cdot}52$;
(ii) $\phi = -$, when $1{\cdot}52 > x > 0$;
(iii) $\phi = +$, when $0 > x > -{\cdot}51$;
(iv) $\phi = -$, when $-{\cdot}51 > x > -1{\cdot}24$;
(v) $\phi = +$, when $-1{\cdot}24 > x$.

In case (i) y has *four real* values,
(ii) y has *two real* and *two imaginary* values,
(iii) y has *four imaginary* values, if $0 > x > -{\cdot}48$, and *four real* values, if $-{\cdot}48 > x > -{\cdot}51$,
(iv) y has *two real* and *two imaginary* values,
(v) y has *four real* values.

(2) *The Q—Curve* :—This consists of the axis of x and two hyperbolic branches as is easily seen.

[*N.B.*—The asymptotes to the P and Q curves are $\cos 5\theta = 0$, $\sin 5\theta = 0$ respectively].

A tracing of the curves is given in Fig. 2.

Their *real intersections* furnish the roots of the quintic, and they are

A, B, C, corresponding to the real roots;
D, E, corresponding to the imaginary roots.

Their approximations are

$$+1{\cdot}52, \quad -{\cdot}51, \quad -1{\cdot}24; \quad +{\cdot}12 \pm 1{\cdot}44\sqrt{-1}$$

The approximation to D is found by changing the origin to $(0, \sqrt{2})$, so that Q comes

$$y' = \frac{5\sqrt{2}}{4} x'^2, \text{ nearly};$$

and P becomes (on substitution for y)

$$50x'^5 + gx' - 1 \doteq 0,$$

whence
$$x' = \tfrac{1}{8} = {\cdot}12$$
$$y' = \frac{5\sqrt{2}}{4} \cdot \frac{1}{64} = {\cdot}03$$

or
$$x = x' = .12!\ ; \quad y = \sqrt{2} + y' = 1{\cdot}44.$$

It may be remarked here, that the P and Q curves are orthogonal always; and that the P—curve cuts the axis of x in $f(x) = 0$, and the Q—curve in $f'(x) = 0$.

By Rolle's theorem, therefore, it follows that if P crosses the axis p times, Q crosses it in *at least* $(p-1)$ times. If all the roots of $f(z)$ are real, P and Q will never cut in real points.

Ex. 2. As a second example, we may take the cubic

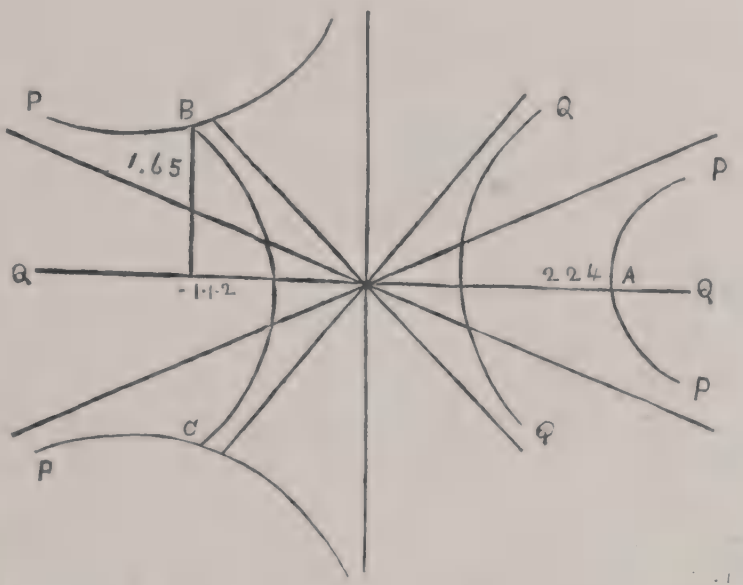
$$z^3 - z - 9 = 0.$$

Here $P \equiv x^3 - 3xy^2 - x - 9 = 0$,

$Q \equiv y(3x^2 - y^2 - 1) = 0$.

The graphs are as under.

Fig. 3.



The roots are $+2{\cdot}24$; $-1{\cdot}12 \pm 1{\cdot}65\sqrt{-1}$.

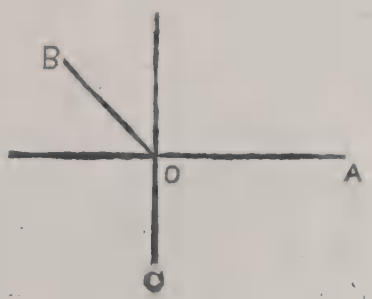
6. The method of '*vectors*' may be employed in a few typical cases, as the following examples will show :—

Ex. (1) : To solve $2z^2 + 2z + 1 = 0$ (i)

Let OB represent the vector z, then z^2 is represented by OC, where $\hat{AOB} = \hat{BOC}$, and $OA.OC = OB^2$; so that the equation is written

$$\overline{OA} + 2\overline{OB} + 2\overline{OC} = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Fig. 4.



Now, since OB bisects the angle AOC, equation (ii) requires that 2OC should be equal to OA.

Putting $OB=r$, we have

$$2r^2=1, \text{ or } r^2=\tfrac{1}{2}.$$

$$\therefore \quad r=\frac{1}{\sqrt{2}}.$$

Also $\quad 2r=-2\cos AOB$, from (ii.)

$$\therefore \quad \cos AOB=-\frac{1}{\sqrt{2}}.$$

$$\therefore \quad AOB=135^\circ.$$

Hence, the value of z corresponding to OB is

$$r(\cos 135^\circ+i\sin 135^\circ)=-\frac{1}{2}(1-i).$$

Ex. (2) $z^3-9z-12=0$.

Here z is easily written in the form $[\sqrt[3]{3}+\sqrt[3]{9}]$ and the several values of z are therefore

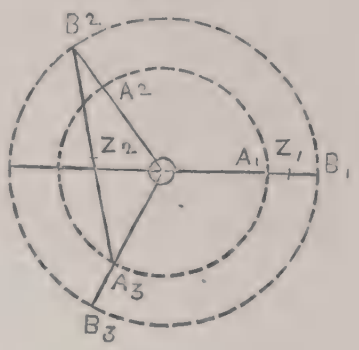
$$a+b,\ a\alpha+b\alpha^2,\ a\alpha^2+b\alpha,$$

where $a=\sqrt[3]{3}$, $b=\sqrt[3]{9}$, α = cube root of unity.

Hence, we obtain the following construction for the roots of the cubic.

Describe concentric circles of radii a, b; inscribe equilateral triangles $A_1A_2A_3$, $B_1B_2B_3$ in them as in the figure.

Fig. 5.



Then the roots are represented by

$$2OZ_1,\ 2OZ_2,\ 2OZ_3$$

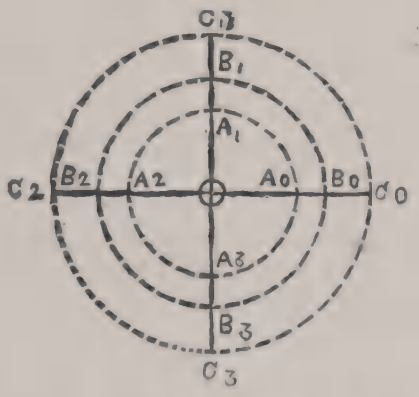
where Z_1 Z_2 Z_3 are the mid-points of A_1B_1, A_2B_3, A_3B_2

Ex. 3. $z^4-18z^2-48z-39=0$.

In this case $z=a\alpha+b\alpha^2+c\alpha^3$, where $a=3^{\frac{1}{4}}$, $b=3^{\frac{1}{2}}$, $c=3^{\frac{3}{4}}$, and $\alpha^4=1$. construction for the several values of z corresponding to the four values of α is as follows:

Draw concentric circles of radii a, b, c; inscribe squares in them as shewn in the figure. Then the four roots are represented by $3\,OZ_1$, $3\,OZ_2$, $3\,OZ_3$, $3\,OZ_4$, where $Z_1\,Z_2\,Z_3\,Z_4$ are the mean centres of $A_1B_2C_3$, $A_2B_0C_2$, $A_3B_2C_1$, $A_0B_0C_0$

Fig. 6



7. The *vector* method can be successfully employed in the case of a *circulant* equation of any degree whatever.

For, we know that the roots of such an equation of the n^{th} degree can be written

$$a\alpha + b\alpha^2 + \ldots \qquad l\alpha^{n-1}$$

where $\alpha^n = 1$. [Burnside and Panton : *Theory of Equations*, Vol. II, p. 62.]

To represent the roots graphically draw concentric circles of radii $a,b,c,\ldots l$ and place in them regular n—sided polygons

$$A_1A_2\ldots A_n,\ B_1B_2\ldots B_n,\ldots\ldots L_1L_2\ldots L_n.$$

Then the roots of the circulant equation are given by

$$nOZ_1,\ nOZ_2,\ldots\ldots nOZ_n$$

$Z_1, Z_2,\ldots\ldots Z_n$ being the mean centres of $A_1B_2\ldots L_n, A_2B_1C_6\ldots, A_3B_6C_9,\ldots$, the suffixes having a period n.

8. The *general* cubic and biquadratic equations can also be solved by the above method.

The circulant equation of the *third* degree is

$$\begin{vmatrix} -x & a & b \\ b & -x & a \\ a & b & -x \end{vmatrix} = 0,$$

which is the same as

$$x^3 - 3abx - a^3 - b^3 = 0\,;$$

comparing with $x^3 + qx + r = 0$, we have

$$q = -3ab,\ r = -(a^3 + b^3).$$

Thus a^3, b^3 are the roots of

$$t^2+rt-q^3/27=0;$$

and the graphical representation of x is as in § 6, Ex. 2.

Again, the circulant equation of the *fourth* degree is

$$\begin{vmatrix} -x & a & b & c \\ c & -x & a & b \\ b & c & -x & a \\ a & b & c & -x \end{vmatrix} = 0,$$

which reduces to

$$x^4-2x^2(b^2+2ac)-4bx(a^2+c^2)+b^2(b^2+4ac)-(a^2-c^2)^2=0.$$

Comparing with $x^4+qx^2+rx+s=0$, we have

$$q=-2(b^2+2ac),\ r=-4b(a^2+c^2),$$
$$s=b^2(b^2-4ac)-(a^2-c^2)^2.$$

Eliminating a and c, the cubic equation for b^2 is

$$t^3+\tfrac{1}{2}qt^2+(\tfrac{1}{16}q^2-\tfrac{1}{4}s)t-\tfrac{1}{64}r^2=0,$$

which is identical with the '*auxiliary cubic*' of the biquadratic.

The difference between Euler's method and this method, however, consists in the expression for the roots of the quartic in terms of those of the cubic. According to Euler's method the roots of the quartic are

$$\sqrt{t_1}+\sqrt{t_2}+\sqrt{t_3}$$
$$\sqrt{t_1}-\sqrt{t_2}-\sqrt{t_3}$$
$$-\sqrt{t_1}+\sqrt{t_2}-\sqrt{t_3}$$
$$-\sqrt{t_1}-\sqrt{t_2}+\sqrt{t_3};$$

whereas our method gives them in the form

$$-b\pm(a-c)i,\ b\pm(a+c),$$

b, a, c standing for

$$-\sqrt{t_1},\ \tfrac{1}{2}\sqrt{t_3}(1-i)-\tfrac{1}{2}\sqrt{t_2}(1+i),\ \tfrac{1}{2}\sqrt{t_3}(1+i)-\tfrac{1}{2}\sqrt{t_2}(1-i)$$

respectively.

Section (II).

9 We shall consider in this Section *approximations* to complex roots of an algebraic equation by means of Cauchy's theorem, *viz.*:

$$f(z)=f(a)+(z-a)f'(a)+\frac{(z-)}{2!}f''(a)+\ldots$$

This series leads to the following approximations, provided there is a root in the neighbourhood of a point a.

The *first* approximation is $(a+h_1)$, such that

$$f(a)+h_1f'(a)=0.$$
$$\therefore\ h_1=-f(a)/f'(a). \qquad \text{(i)}$$

The *second* approximation is $(a+h_1+h_2)$, such that

$$f(a+h_1)+h_2 f'(a+h_1)=0,$$

i.e. $$f(a)+h_1 f'(a)+\frac{h_1^2}{2!}f''(a)+h_2 f'(a)=0.$$

$$\therefore \qquad h_2=-\tfrac{1}{2}\,[f(a)]^2.\ f''(a)/[f'(a)]^3. \qquad \text{(ii)}$$

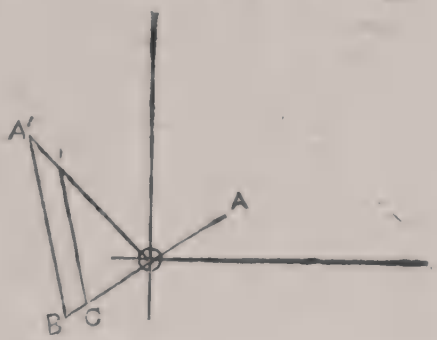
And so on.

These approximations are *graphically* equivalent to the following construction:

Corresponding to a in the z—plane, let

A be the point $f(a)$ in the w—plane,

A'..... $f'(a)$ in the same,



then $$h_1=-f(a)/f'(a)$$

$$=\overline{OB}/\overline{OA'}.\ \text{[Make } OA'.\ OI=OB^2=OA^2.$$

In other words, h_1 is the vector equal to OC in the figure whose vectorial angle is equal to A'OB.

The *first* approximation to the root is thus

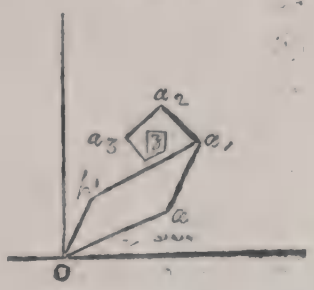
$$a+h_1=a_1 \text{ say.}$$

Similarly, the *second* approximation is

$$a+h_1+h_2=a_2, \text{ say.}$$

And so on, for further approximations.

Fig. 8.



The actual complex root z of the proposed equation is therefore graphically determined as the point of convergence of the polygon $o\ a\ a_1\ a_2 \ldots$

9·1. The assumption that

$$a_1 = a + h_1$$

is a better approximation than a depends upon the following results :

(i) $$\frac{f'(z)}{f(z)} \equiv \sum \left(\frac{1}{z-\alpha}\right)$$

where α is a complex root or real root of $f(z)$. Now, when a is an approximate complex root, the corresponding partial fraction on the right side will have its modulus very great when $z = a$.

Hence $\left|\frac{f'(a)}{f(a)}\right|$ is large when a is an approximate complex root.

that is, $|h_1| = -\left|\frac{f(a)}{f'(a)}\right|$ is small.

(ii) $$f(a_1) = f(a) + h_1 f'(a) + \frac{h_1^2}{2!} f''(a) +$$

$$= \frac{h_1^2}{2} f''(a), \text{ since } h_1 = -f(a)/f'(a).$$

$$\therefore \quad \frac{f(a_1)}{f(a)} = \frac{h_1^2}{2} \frac{f''(a)}{f(a)}.$$

And in general $\left|\frac{f''(a)}{f(a)}\right|$ is small for a similar reason as in (i).

Hence $\left|\frac{f(a_1)}{f(a)}\right|$ is small.

Thus $f(a_1)$ is a better approximation than $f(a)$ under the circumstances.

9·2. The degree of approximation in taking $a_1 = a + h_1$ instead of a may be investigated as in the usual methods for real roots.

10. The approximations found in the last article may also be *directly* obtained by separating the real and imaginary parts at each stage of the approximations.

Thus, putting

$$f(a) = \alpha + i\beta,$$
$$f(a) = \alpha' + i\beta', \text{ \&c.,}$$

we have

$$w = u + iv = f(x+iy)$$
$$= \alpha + i\beta + (x+iy-\alpha-i\beta)(\alpha' + i\beta') + \ldots$$

Equating real and imaginary parts :

$$u = \alpha + [(x-\alpha)\alpha' - (y-\beta)\beta'] + \ldots$$
$$= \alpha + \alpha_1 + \alpha_2 + \ldots, \text{ say ;}$$
$$v = \beta + [\beta'(x-\alpha) + \alpha'(y-\beta)] + \ldots$$
$$= \beta + \beta_1 + \beta_2 + \ldots, \text{ say.}$$

The *first* approximation is thus the intersection of the straight lines

$$\alpha+\alpha_1=0,\ \beta+\beta_1=0,$$

which are obviously rectangular.

It may be observed that *all* the successive approximations are *orthogonal*, being *conjugate* functions.

11. The approximations here investigated are capable of interpretation by means of 'the theory of polars' of a binary form.

For, corresponding to any point a, the $(n-1)th$ polars with respect to u and v are easily seen to be identical with the *first* approximations in § 10; and therefore, the first approximation is geometrically interpreted as the point of intersection of these polar lines, which are at right angles. Similarly, for further approximations.

12. A general method of dealing with the intersections of two plane algebraic curves suggests itself from the preceding.

Take *any* point in the neighbourhood of a point of intersection and construct the *polar lines* of the point with respect to each curve. Then the point of intersection of these polar lines will be a *first approximation*; similarly we can construct further approximations.

Section (III).

13. In this last Section we shall apply the method of approximations to investigate the complex roots of a *Transcendental Equation*.

Consider the equation

$$z=\cos z,$$

which readily breaks up into the two equations

$$P\equiv x-\cos x\cosh y=0,$$
$$Q\equiv y+\sin x\sinh y=0.$$

The P and Q curves are traced by writing them in the forms

$$\cosh y=x\sec x,$$
$$\sin x=-y\operatorname{cosech} y;$$

and using the *lambda* table (*vide*: A. Lodge's *Diffl. Calc.*, p. 82), along with the ordinary mathematical tables.

A tracing of portions of P and Q, prepared with reference to the subjoined tabular values, follows:

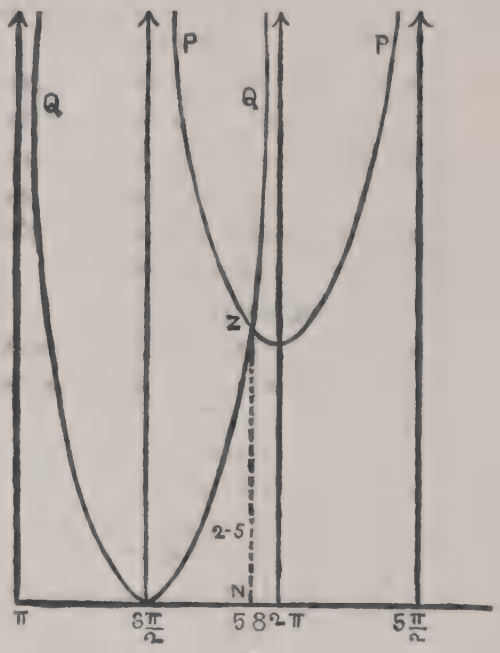
Table (1): $\cosh y=\sec\theta=x.\sec x$.

x	$3\pi/2$	6·11	2π	$5\pi/2$	For values of x between $3\pi/2$ & $5\pi/2$.
$\cos\theta$	0	·16	·156	0	
θ	90°	80°40′	81°	90°	
y	∞	2·5 (min.)	2·54	∞	

Table (2) : $\sin x = -y \operatorname{cosech} y$; $\sinh y = \tan \theta$

y	0	1	2	3	∞	
θ	0°	49°36′	74°40′	84°17′	90°	For values of x between $\frac{3\pi}{2}$ and 2π.
$\sin x$	−1	−·851	−·558	−·3003	0	
x	$3\pi/2 = 270°$	301°40′ 238°20′	326°3′ 213°57′	342°31′ 197°29′	360° 180°	

Fig. 9.



From the diagram it is obvious that the P and Q curves cross in the neighbourhood of the point

$$[x = 5{\cdot}8\,;\ y = 2{\cdot}5].$$

To determine the *first* approximation, we take the equations

$$P' + h\frac{dP'}{d\alpha} + k\frac{dP'}{d\beta} = 0,$$

$$Q' + h\frac{dQ'}{d\alpha} + k\frac{dQ'}{d\beta} = 0,$$

where P′, Q′ stand for the values of P and Q, when $x = 5{\cdot}8 = \alpha$, say ; $y = 2{\cdot}5 = \beta$, say.

Writing $$\sin \alpha - \sin 5{\cdot}8 = \sin 332^\circ\ 19' = -\ {\cdot}4645$$
$$\cos \alpha = \cos 332^\circ\ 19' = {\cdot}8855$$
$$\operatorname{sech} \beta = \operatorname{sech} 2{\cdot}5 = \cos 80^\circ\ 40' = {\cdot}1622$$
$$\tanh \beta = \sin 80^\circ\ 40' = {\cdot}9868$$
from the tables and solving the two equations for h and k, we get
$$h = {\cdot}074,\ k = {\cdot}039.$$

Hence, a first approximation to the complex root in the neighbourhood of (α, β) is
$$x = 5{\cdot}874;\ y = 2{\cdot}539;$$
or
$$z = 5{\cdot}874 + 2{\cdot}539\, i.$$

14. We might have proceeded more *directly* thus—

$f(z) \equiv z - \cos z = (a - \cos a) + (z - a) f'(a)$, approximately,

where $a = \alpha + i\beta$; so that
$$z - a = -(a - \cos a)/f'(a)$$
$$= -(a - \cos a)/(1 + \sin a)$$
$$\therefore\ z = a - (a - \cos a)/(1 + \sin a)$$
$$= (a \sin a + \cos a)/(1 + \sin a)$$
$$= [(\alpha + i\beta) \sin (\alpha + i\beta) + \cos (\alpha + i\beta)]/[1 + \sin(\alpha + i\beta)]$$
$$= \alpha_1 + i\beta_1, \text{ say.}$$

Calculating α_1, β_1, we get for the first approximation
$$5{\cdot}874 + 2{\cdot}537i$$

Other complex roots may be obtained similarly by drawing the remaining branches of the P and Q curves and noting their intersections.

Postscript.

15. Two simultaneous equations involving *two variables* are usually represented by plane graphs and their real solutions obtained as the intersections of the graphs. When there are no *real* intersections, however, the corresponding *complex* solutions of the equations may be visualized as follows :—

(i) *First Method.*—Consider z, z' as the unknown complex quantities satisfying the equations
$$f(z, z') = 0,\ \phi(z, z') = 0.$$

Write $z = x + iy$, $z' = x' + iy'$ in the above and separate the real and imaginary parts, so that, we have *four* equations in the *four* unknowns (x, y, x', y').

Now, these four unknowns may be regarded as determining a *real point* in space of four dimensions, and the resulting equations in them as relating to four hyper-surfaces. The common points of these hyper-

surfaces would consequently represent the complex solution of the proposed equations.

The difficulty of conceiving four dimensional space detracts from the usefulness of this method of representation.

(ii) *Second Method.*—In this method z, z' may be associated with a line in space in as much as *four co-ordinates* are necessary to determine a line completely. Accordingly the two proposed simultaneous equations on being split up into real and imaginary parts specify four *complexes* in space, and the complex solutions of the equations are geometrically represented by the lines common to the four complexes. The intuitive difficulties of the previous method exist in this case also, in so far as line-geometry and hyper-geometry are intimately connected.

(iii) *Third Method.*—This is only a relic of the line-representation, being a representation of the two variables z and z' by *a point pair* in the same or in different planes. A few simple properties of the two plane representation are discussed in Forsyth's *Theory of Functions of Two Complex Variables*, § § 15—20.

(iv) *Fourth Method.*—Lastly, we may eliminate one of the variables z', between the proposed equations and treat the resulting equation by the methods applicable to functions of a single variable.

16. In simple cases, however, the *vector method* will be found quite suitable, as the following examples will prove :—

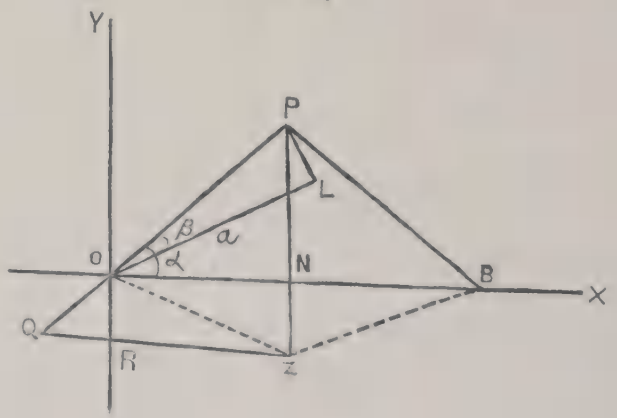
Example 1.—Solve $z^2+z'^2=a^2$, $z+z'=b$; where $b^2>2a^2$.

Let $z=a\cos\theta$, $z'=a\sin\theta$, where θ is a complex quantity. Then we may write

$$z=a(\cos\alpha\sec\beta-i\sin\alpha\tan\beta),$$
$$z'=a(\sin\alpha\sec\beta+i\cos\alpha\tan\beta),$$

α, β being real quantities.

Fig. 10.



Hence the following construction for the vectors z, z'.—

In the figure $\overline{OL}=a$, $OB=b$, $OQ=PL$, &c.; and we have

$$ON=OL \sec \cos \alpha = a \sec \beta \cos \alpha,$$
$$NZ=OQ \sin \alpha = PL \sin \alpha = a \tan \beta \sin \alpha.$$

$$\therefore \quad \overline{OZ}=\overline{ON}+\overline{NZ}=a(\cos \alpha \sec \beta - i (\tan \beta \sin \alpha)$$
$$=z.$$

Again $NP=a \sec \beta \sin \alpha$.
$QR=a \tan \beta \cos \alpha$;

$$\therefore \quad iz'=\overline{NP}+\overline{RQ}.$$

But $z'=b-z=\overline{ZB}=\overline{ZN}+\overline{NB}$

$\therefore$ $NB=NP$, and $ZN=RQ$.

In other words, $\alpha=\pi/4$, $ON=\frac{1}{2}b$.

The vector solutions are thus OZ, ZB, corresponding to the complex quantities

$$[\tfrac{1}{2} b \pm i \sqrt{(\tfrac{1}{2} b^2-a^2)}]$$

Example 2. Solve $z^2/a^2+z'^2/b^2=1$, $z+z'=c$; where $c^2 > a^2+b^2$.

Here, we can write

$$z=a (\cos \alpha \sec \beta - i \sin \alpha \tan \beta)$$
$$z'=b (\sin \alpha \sec \beta + i \cos \alpha \tan \beta).$$

Proceeding, as before,

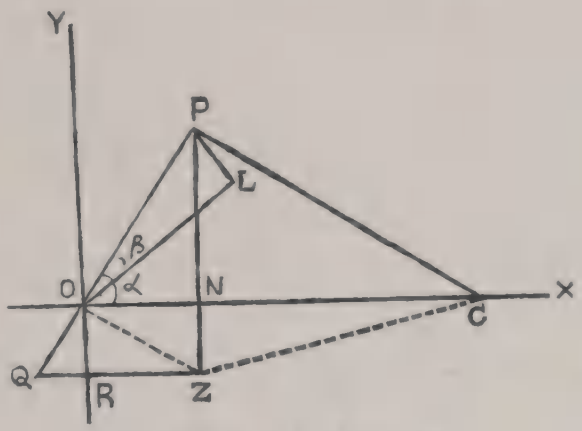


Fig. 11.

$$z=\overline{OZ},\ z'=ZC=\overline{ZN}+\overline{NC}.$$

Also $$i z'=\frac{b}{a}\left(\overline{NP}+\overline{RQ}\right).$$

Hence $$NC=\frac{b}{a} NP,\ ZN=\frac{b}{a} RQ.$$

$$\therefore OPC=90°,\ \tan \alpha=\frac{b}{a},\ a=c \cos \alpha \cos \beta ;\ \&c.$$

The vector solutions are thus OZ, ZC, corresponding to the complex quantities

$$\frac{a^2 c}{(a^2+b^2)} \pm i \frac{ab \sqrt{(c^2-a^2-b^2)}}{(a^2+b^2)} \quad \ldots \ldots$$

The Function of Mathematics in Scientific Research.

By Professor G. A. Miller.

(*Concluded from p. 225 of Vol. IX.*)

This common ground of investigators may serve to explain the fact that many of the most influential research organizations, like the National Academy of Sciences in our own country, embrace all the sciences. In recent decades there has been a tendency to organize research separately in the various subjects in the form of national societies named after these subjects. In fact, there are those who think that the latter have assumed such a preponderant sphere of influence as to threaten the very life of the former as serious factors in research. On the other hand, the maintenance of a common scientific life seems to be of the highest importance in view of desirable interactions and special emphasis on what is most fundamental.

The history of mathematics has taught us that some subjects which were apparently far apart and which were long developed separately were later seen to have most important common elements. The discovery of those common elements and their development has led to marked advances in the separate fields themselves. By way of illustration I need only refer to the fields of Algebra and geometry so happily welded through the work of Descartes, Fermat and many others. In modern times the theory of groups and invariants has exhibited many important connections between subjects which had been supposed to be widely separated. The same tendency has, of course, manifested itself in other sciences and may be assumed to become more dominant as knowledge advances.

A pertinent difference between the mathematical investigators and investigators in other sciences is that the former are compelled to stay with their problems until a solution is reached which can be proved to be in accord with deductions from certain definite assumptions, while the latter enjoy much greater freedom in regard to the stage to which they may pursue a problem before announcing results. Hence these may hope for success in dealing with much more difficult questions than the mathematician could reasonably hope to solve at the present time. The limitations thus imposed upon the mathematician are compensated, at least in part, by the finality of his results as regards questions of rigor. Mathematical results *can* never be disproved, other accepted results *have* never been disproved. With respect to simplicity and style, the mathematical developments are seldom final, and in many cases, they appear to admit endless variations.

As instances of final mathematical results may be cited the useful tables which when once computed serve all succeeding generations. Such finality may be said to be a goal of all scientific endeavor, since the results enrich countless ages by increasing their capacity for accomplishments. In fact, such tables may be regarded as typical illustrations of the mathematical contributions to the advancement of knowledge even if they constitute a very minor portion of these contributions. The fact that mathematical results have increased the capacity of the world for doing things may be emphasized by noting, in particular, that in recent years prime numbers have been found which could not have been proved to be prime by the method employed by Eratosthenes, if the entire human race had been working in an organized manner on this single problem since the days of the ancient Greeks.

The present seems to be an especially appropriate time to consider the interrelations of scientific research in view of the rapidly growing public appreciation of the value of such research. Several decades of comparative peace immediately preceding the present great and deplorable conflict were unusually rich in great scientific triumphs. As well-known instances we may cite wireless telegraphy and the construction of the great Panama Canal, which became possible by our advanced knowledge in regard to sanitation. The world-wide health activities under the auspices of the Rockefeller Foundation and the activities of our agricultural colleges in directing attention to advantages resulting from scientific methods of farming are strong forces working towards a popular appreciation of science. Since the great European war began it has become evident through the new elements introduced by the submarines and other scientific devices that the very existence of a great nation may depend upon the scientific attainment of its people, and hence the question of scientific research has taken a prominent place among those of national policy. It is perhaps significant that our National Academy was founded in the midst of the Civil War.

Scientific research is as old as civilization and has often been protected by kings in a patronizing manner, bnt it is a new experience in the history of the world to see kings turn to scientific research for protection. For centuries governments have recognized the value of science and have provided with growing liberality for her development, but now they are calling to her to save them from destruction. They have noticed that in spite of many excellencies in other directions the ignorance of causes may entail their destruction as separate nations. This

new attitude towards our field of work may at first tend to gratify us, but a second thought reveals the fact that it is fraught with grave dangers. Kings in government and finance are interested in the dead results of science instead of in the great living and growing organism itself, whose growth seems to have just begun and whose development has always been more keenly inspired by love of truth than by hope of gain.

Is there not a danger that the sudden recognition of the great political importance of certain types of research will have somewhat the same effect on science as the discovery of gold in California and in Australia about the middle of the preceding century had on the development of the regions concerned? People flocked from one mining camp to the other and often neglected duties which are essential for the harmonious development of the resources of a country. Hence there seems to be a special need at present to urge our colleagues to remain at their posts of duty, notwithstanding glowing reports of chances to amass scientific fortunes quickly in certain newly discovered gold fields. The get-rich-quickly schemes in science should be scrutinized as carefully as similar schemes relating to the accumulation of money.

The remaining at one's post of duty in scientific research does not imply a lack of support in the solution of pressing problems or a lack of vacation trips and acquaintance with other fields of work. In fact, such support and acquaintance are highly desirable. It is, however, a question whether the nomadic scientific life, which seems to have become fashionable during the last few decades, at least in mathematics, is the one which will in the long run bring the best results. Science is not primarily a grazing country. Large tracts are suited for agriculture and mining. What is new is not necessarily good and what is good is not necessarily new, and prophesies in regard to the great importance of certain new developments have not always been fulfilled. On the other hand, it should be remembered that reasonable hope and optimism are essential for progress, and that we need prospectors as well as miners in the scientific world.

It should be noted that the miner needs some of the qualifications of the prospector since he is apt to meet with new situations and needs to take advantage of the available by-products. In fact, while he is mining for gold he may strike deposits of copper which are richer than the gold deposits which he was primarily seeking. Some of the richest mathematical discoveries were made while the investigator was looking primarily for other results, and even problems which have not been

solved at all up to the present have been the source of very useful developments. I understand that similar conditions hold in other fields of scientific effort and these facts point to the great importance of freedom on the part of the investigator, and, incidentally to the danger of too much organization in scientific research.

As a very recent instance of an unexpected mathematical by-product, I may be pardoned for referring to a somewhat trivial case which has, however, the important property that it can be understood by all. It is well known that the theory of substitution groups was developed for the purpose of clarifying the theory of algebraic equations and not for the purpose of adding to the enjoyment of parties engaged in playing games of cards. In fact, the study of such an advanced mathematical theory as that of substitution groups might appear to involve concepts, which are at the opposite pole from those entering the minds of people seeking recreation at card tournaments.

Notwithstanding this apparent wide separation, I was pleased to be able to say recently to a friend, who desired to have each one of a large party play once and only once with each of the others during a series of successive games, that an arraugement of the players meeting this condition could be determined directly by means of substitutions of certain transitive groups. This should perhaps have been expected, since a transitive substitution group is an ideal republic treating all its etters in exactly the same way. On the contrary, an operation group may have elements enjoying special privileges and hence it has more extensive contact in the actual world of thought.

A little study of the stated problem revealed the interesting fact that when the number of tables is any power of 2 the substitutions of a well-known type of substitution groups and its group of isomorphisms exhibit directly how the players can be arranged so that each one will play once and only once with, and twice and only twice against, each of the others in a certain series of games. To make myself perfectly clear, I may say that if 8 tables, or 32 players, are involved, one can write directly by means of a certain regular substitution group of order 32 a set of possible arrangements so that in 31 successive games each one of these 32 players would play once and only once with each of the others and twice and only twice against each of them. This was, however, not the first solution of the general problem in question. In fact, about twenty years ago Professor E. H. Moore published a different solution of it in Volume 18 of the *American Journal of Mathematics* under the title "Tactical Memoranda."

I have referred to this matter here mainly for the purpose of emphasizing the fact that intellectual penetration is often attended by the most unexpected by-products, but I should also be pleased to have people know that certain kinds of recreation can easily be enriched by making use of results which the mathematician developed for a totally different purpose. Science should and does enrich both work and play. More than a thousand years ago the Hindu astronomer Brahmagupta said:

As the sun obscures the stars, so does the proficient eclipse the glory of other astronomers in an assembly of people by the recital of algebraic problems, and still more by their solutions.*

The playful question, Where do the finger nails find so much dark dirt to put under them? may serve to arouse a thoughtful attitude on the part of the boy who has been taught to keep his hands clean. In fact, our play and recreation are perhaps as fundamentally affected by questions of science as our serious work and the victrolas and moving pictures should have a marked influence on the popular attitude towards science in view of the fact that they reach so many people. If it is true that the greatest service which science is rendering the human race is the reduction of superstition, it is clear that the efficiency of science depends largely upon its popularity.

The hypothesis that space and the operations of nature are discontinuous clearly excludes the hypothesis that they are continuous, but it is interesting to note that the mathematics relating to the discontinuous does not exclude that relating to the continuous. On the contrary, there are the most helpful interrelations between these two types of mathematics. Such a subject as number theory, relating decidedly to discrete quantities, has been greatly extended by analytic methods relating to continuous quantities, and, on the other hand, processes relating to the study of continuous functions are largely based upon those relating to the discontinuous.

This may perhaps tend to show that even if our hypotheses in regard to the continuity of space and the operations of nature have to be largely modified, as seems now probable, the mathematical methods of attack may require less modification than might at first appear to be necessary. The language which mathematics has provided for science includes not only concepts relating to the continuous and the discontinuous, but fortunately it also shows relations between these concepts and these relations become more pronounced with its development.

* H. T. Colebrooke, "Algebra with Arithmetic and Mensuration from the Sanskrit," by Brahmagupta and Bhascara, 1817, p. 379.

In view of the age of this language and its contact with various sciences it may be readily understood why mathematical history occupies a prominent place in the history of science. In fact, the history of science constitutes one of the fields where scientists may find common interests most fully represented, even if the past is too rich in events to be studied completely. It may therefore be appropriate on this occasion to refer to a few recent developments relating to the history of mathematics, especially since the interest in the history of science has increased rapidly during recent decades, as is partly evidenced by the efforts that are now being made to establish an institute of historical scientific research in our land.

One of the most interesting questions relating to the early history of mathematics is the use of positional values of numbers and the closely connected use of a symbol for zero. Until a decade or two ago it was commonly assumed by mathematical historians that the use of zero as a positional number symbol originated in India, and this view has not yet been entirely abandoned, notwithstanding the fact that the Babylonians employed numbers with positional value and a symbol which seems to have fulfilled the main function of our zero several centuries before the Christian era. On the other hand, the first definite evidence of the use of zero among the Hindus falls in the second half of the first millennium of this era.

In view of these facts it is extremely interesting to note the early use of zero, in connection with numbers having positional value, by the Maya, a people inhabiting the Atlantic coast plains of southern Mexico and northern Central America. One of the worthy alumni of your university recently referred to this matter in the columns of SCIENCE in the following words:

Special interest attaches to the occurrence of zero-symbols and the principle of local value among the inhabitants of the flat lands of Central America, at a period as early as the beginning of the Christian era, if not much earlier. It would seem that in this invention, the Maya in Central America possessed priorty over Asiatic people by a margin of five or six centuries.

If further investigation will lead mathematical historians to agree that the zero as a symbol in a numerical notation with positional value was actually first used in America, according to the preserved records, it will effect a very fundamental change as regards interest in the early mathematical attainments of the American aborigines. Unfortunately these early mathematical attainments failed to become the source of extensive further developments on American soil. They exhibit clearly

that central concepts may be discovered independently and they direct attention to the danger in trying to establish one source for a particular concept in historical insvestigation. They also show that the small strip of country marked now by Boston has not always been the intellectual hub of America.

The history of some of the mathematical attainments of the Maya people has recently been made more easily accessible through the publication of "An Introduction to the Study of the Maya Hieroglyphs," prepared by S. G. Morley and published as Bullettin 57 of the Bureau of American Ethnology, Smithsonian Institution of Washington. On page 92 of this bulletin a dozen different symbols for zero are noted and on page 131 numbers varying from 21 to 12,489,781, and involving the use of zero, are represented in the Maya notation. It is of interest to note that the value of a unit in a higher position is always 20 times the value of a unit in the next lower position, except in the case of the third place, where its value is only 18 times that of the second place.

In historical research and elsewhere,the mathematician seeks cordial cooperation with other scientists, and he regrets that the confusion of tongues, resembling the experiences at the tower of Babel, is making it more and more difficult to understand each other. In the case of scientists this confusion is mainly due to a rapid growth of language in various directions. May we not hope that as many theories which were supposed to be distinct suddenly exhibited profound connections, so also this extensive language will tend towards unity and simplicity as we see more clearly the fundamental underlying principles. Science knows no bounds in method or in subject-matter and the artificial limitations set by man for his own convenience in making a start must break down before the onward march of truth. All science is a unit and all scientific investigaton should be inspired by their common interests.

[7] F. Cajori. SCIENCE, N. S., Vol. 44 (1916), p. 715.

Astronomical Notes.

Eclipses.

The first eclipse of the year 1918 will occur on June 8, when there will be a total eclipse of the Sun. The eclipse will be visible as a partial eclipse from the north-eastern part of Asia, the north polar regions and the whole of the north American continent. The track of totality runs from a point in the Bahamas Islands nearly to the East coast of China ; as it runs right across the centre of the United States the eclipse is likely to be well observed in spite of the war ; the remainder of the central track lies in the Pacific Ocean.

There will be a partial eclipse of the Moon on the night of June 23—24, the eclipse is a small one, magnitude 0·135 and is invisible in India.

R. A. S. Gold Medal.

The Gold Medal of the Royal Astronomical Society was awarded this year to Mr. J. Evershed, Director of the Solar Physics Observatory at Kodaikanal.

Astronomical Consequence of a Curvature of Space.

In Mon. Not. R. A. S. 1917 Nov. de Litter investigates the effect on astronomical phenomena of a curvature of space taken in conjunction with Einstein's theory.

It is quite clear that two dimensional beings living on the surface of a sphere of very large radius might conceive themselves to be resident on a plane, and a similar notion may be extended to space in three dimensions, but by extending our observations to a distance comparable with the radius of curvature, we may hope to detect the difference ; hence it is to astronomical observations that we must look for any effect of curvature of space.

The new solution of Einstein's equations leads to a form of space with constant positive curvature. Such space may be either

(I) Riemann's spherical space in which all straight lines starting from a point intersect again in the antipodal point, whose distance from the starting point is π. R (R=radius of curvature) along all lines, and this is the greatest possible distance between two points ; or

(II) Newcomb's elliptical space, in which any two lines have at most one point in common, and the largest possible distance between two points is $\frac{1}{2}\pi$. R.

In (I) All lines are of length $2\pi R$; in (II) all lines are of length πR

de Litter finds that (on the assumption of a curved space) no star can possibly have a parallax less than a/R where a is the distance between the Earth and Sun.

Secondly it is found that the lines in the spectra of very distant stars and nebulæ will be displaced towards the red, producing a spurious positive radial velocity; a similar result holds in rectangular space on Einstein's theory, but the displacement is considerably augmented in a curved space. In this way it may be possible to account for the very large velocities of spiral nebulæ, which are certainly very distant, should they turn out to be positive on the whole. de Litter makes a number of estimates of the value of R, the data of course is somewhat scanty, since the smallest parallaxes are naturally the most uncertain, and very few absolute parallaxes are known, while the number of spiral nebulæ of which we have reliable determinations of radial velocity is very small (de Litter uses only three). R comes out about 10^{13} Astronomical Units.

If space is curved we should see an image of the sun at the antipodal point, that is the point of the sky opposite to the sun, but opinions appear to differ as to whether this image would be as large and bright as the sun or whether it would appear as an extremely faint star.

R. J. Pocock.

SOLUTIONS

Question 329.

(M. Bhimasena Rao) :—If the pedal circle of I' with respect to a triangle ABC touches the nine-points-circle of ABC, show that the sum of the angles PAB, PBC, PCA is constant.

Additional Solution by the Proposer.

The following simple result concerning the rectangular hyperbola which may be proved easily by the anharmonic property of conics is here assumed :—

ABCPQ is a rect hyp, perpendiculars QD, QE, QF are dropped on the sides of ABC intersecting PA, PB, PC in A', B', C'.

Then $QD.\ QA' = QE.\ QB' = QF.\ QC'$ (1)

This is the converse of the theorem. 'If two triangles ABC and A'B'C' are conjugate with respect to a circle, they are in perspective; if P and Q be the centre of perspective and the centre of the circle respectively, the conics ABC PQ and A'B'C' PQ are rect hyp.

If P and Q be isogonal conjugate points and D'E'F' the pedal triangle of P

we have $PD'.\ QD = PE'.\ QE = PF'.\ QF$ (2)

From (1) and (2) we see that when ABC PQ is a rectangular hyp, QA', QB' and QC' are proportional to PD', PE' and PF', and being parallel respectively, the triangles A'B'C' and D'E'F' are homothetic. But we know that any inverse triangle of ABC—call it LMN—with respect to P is similar to the pedal triangle of P, the angle of similitude being the complement of $\hat{PAB} + \hat{PBC} + \hat{PCA}$. It follows therefore that A'B'C' and LMN are similar and being in perspective, one of the two alternatives arises — either they are homothetic, or ABCP is concyclic as also A'B'C' P. The second alternative being rejected since P is not on the circum circle of ABC, we see that A'B'C' and LMN are homothetic. Therefore D'E'F' which is the pedal triangle of P, A'B'C' and LMN are homothetic taken two by two, and the angle of similitude vanishes.

Hence PAB+PBC+PCA is equal to one right angle.

Corollary. When a triangle ABC and its inverse with respect to a point P are orthol ogic, the pedal circle of P with respect to either triangle touches the nine-point-circle of that triangle.

The above proof fails when P is on the circumcircle of ABC; but in this case the result follows more easily. For properties of the points of intersection of McCay's cubic and the circumcircle of ABC, see Mr. S. Narayanan's paper on Three special points' page 85, Vol. I of the Journal.

Question 427.

(S. RAMANUJAN) :—Express

$$(Au^2+Buy+Cy^2)\,(Ap^2+Bpq+Cq^2)$$

in the form $\qquad Av^2+Buv+Cv^2,$

and hence shew that if

$$(2x^2+3xy+5y^2)(2p^2+3pq+5q^2)=2u^2+3uv+5v^2,$$

one set of values of u and v, is

$$u=\tfrac{5}{7}\,(x+y)(p+q)-2xp,\ v=2qy-(x+y)(p+q).$$

Solution (1) *by* '*Zero*', (2) *by S. Narayan.*

(1) Let $\quad f(xy)\equiv Ax^2+Bxy+Cy^2\equiv A(x-\alpha y)(x-\beta y),$

then $\quad f(pq)\equiv Ap^2+Bpq+Cq^2\equiv A(p-\alpha q)(p-\beta q),$

where α and β are the roots of $A\lambda^2+B\lambda+C=0$

Now $\quad f(xy).f(p,q)\equiv A^2\{(x-\alpha y)(x-\beta y)(p-\alpha q)(p-\beta q)\}$

$$\equiv A^2\{px+\alpha^2qy-\alpha(py+qx)\}$$
$$\{px+\beta^2qy-\beta(py+qx)\}$$

But $\quad A\alpha^2+B\alpha+C=0,$ and $A\beta^2+B\beta+C=0.$

$\therefore \quad f(x,y)f(p,q)\equiv\{Apx-A\alpha(py+qx)-qy(B\alpha+C)\}\times$

$$\{Apx-A\beta(py+qx)-qy(B\beta+C)\}$$
$$\equiv A(u-\alpha v)(u-\beta v)$$
$$\equiv Au^2+Buv+Cv^2,$$

where $\quad Apx-Cqy=u\sqrt{A},\ A(py+qx)+Bqy\equiv v\sqrt{A}.$

A *general* solution can be obtained by the following artifice. Take (a, b) such that

$$f(a, b)=a \text{ square}$$
$$=c^2 \text{ (say)}.$$

Then $f(xy).f(pq).f(ab)=A^3(x-\alpha y)(x-\beta y)(p-\alpha q)(p-\beta q)$
$(a-\alpha b)(a-\beta b).$

By virtue of the quadratic $A\alpha^2+B\alpha+C=0$, the product $(x-\alpha y)$ $(p-\alpha q)\,(a-\alpha b)$ can be reduced to the form $(u-\alpha v)$, where u and v are rational functions of $(x\ y\ p\ q\ a\ b)$

Similarly $\quad (x-\beta y)\,(p-\beta q)\,(a-\beta b)=u-\beta v.$

$\therefore \quad f(x\,y).\,f(p\,q).\,f(a\,b)=A^3(u-\alpha v)\,(u-\beta v)$

$$=A^2(A\,u^2+Buv+Cv^2)$$

i.e., $\quad c^2.\,f(xy).\,f(p\ q)=A^2(A\,u^2+Buv+Cv^2)$

Hence $\quad f(x\,y).\,(f\,p\ \ q)=A\,u'^2+Bu'\,v'+Cv'^2,$

where $\quad u'=Au/c,\ v'=Av/c.$

The *general* solution of the indeterminate equation

$$f(a, b)=c^2$$

leads to a corresponding *general* solution of the proposed question.

In the particular case of A=2, B=3, C=5, one solution of

$$2a^2+3ab+5b^2=c^2$$

is $a=1$, $b=-1$, $c=2$, and we easily write a set of values of u and v different from those given by the proposer.

(2) $Ax^2+Bxy+Cy^2$ may be easily thrown into one of the three forms

$$(lx+ly)^2+(mx+ny^2,$$
$$(lx+my)^2+(nx+y)^2, (lx+my)^2+(x+ny)^2,$$

provided $B^2+4AC<0$.

Similarly $Ap^2+Bpq+Cq^2$ may be expressed.

And since $(a^2+b^2)\ (c^2+d^2)$

can be expressed as the sum of two squares in two ways, it follows that

$$(Ax^2+B\ xy+Cy^2)\ (Ap^2+Bpq+Cq^2)$$

can be expressed as the sum of two perfect squares. Hence expressing

$$Au^2+Buv+Cv^2$$

as the sum of two squares and comparing with the above form, the values of u and v are easily obtained. It is clear that there are 108 ways of doing this. The values of u, v given by the proposer for the particular question constitute one of these ways.

And
$$u=\frac{3}{2}px-\frac{7}{2}y-\frac{1}{8}(x+y)\ (p+q),$$
$$v=\frac{3}{2}qy+\frac{1}{2}qx-\frac{11}{8}(x+y)\ (p+q).$$

constitute another solution.

Question 508.

(S. P. Singaravelu Mudaliar):—If s_n stand for the sum of the reciprocals of the first n natural numbers, find the sum of the infinite series.

$$s_1-\frac{1}{2}\left(\tfrac{1}{2}\right)^2 s_2+\frac{1}{3}\left(\frac{1\cdot3}{2\cdot4}\right)^2 s_3-\ldots$$

Remarks by H. Br.

We are required to evaluate

$$S=\sum_{n=0}^{\infty}(-1)^n\ a_n^2\ \frac{s_{n+1}}{n+1}$$

where $a_n=\dfrac{1\cdot3\ldots\ldots(2n-1)}{2\cdot4\ldots\ldots(2n)}$, and $s_{n+1}=1+\dfrac{1}{2}+\ldots+\dfrac{1}{n+1}$

It is easy to obtain a variety of definite integrals, (simple, double, and triple) for the value of S, but I have not been able to find a formula giving its numerical value—

(1) As $$-\int_0^1 \log(1-x).\, x^n dx = \frac{s_{n+1}}{n+1},$$

$$S = -\int_0^1 dx\, \log(1-x).\left(\sum_0^\infty a_n^2 (-x)^n\right).$$

Now $$(1+x\sin^2\theta)^{-\frac{1}{2}} = \sum_0^\infty a_n(-x)^n \sin^{2n}\theta\,;$$

$$\therefore \quad \int_0^{2\pi} \frac{d\theta}{(1+x\sin^2\theta)^{\frac{1}{2}}} = \frac{\pi}{2}\sum_0^\infty a_n^2(-x)^n,$$

and $$S = -\frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta \int_0^1 \frac{\log(1-x).}{(1+x\sin^2\theta)^{\frac{1}{2}}} dx,$$

The integration with respect to x can be easily performed, but the result does not throw any fresh light on the subject.

(2) Again we may take $a_n^2 = \frac{4}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin^{2n}\theta\, \sec^{2n}\phi\; d\theta\; d\phi$, and we get

$$S = -\frac{4}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \frac{\log(1-x).\, dx\, d\theta\, d\phi}{1+x\sin^2\theta\cos^2\phi}.$$

(3) We have $$a_{n+1} = a_n \cdot \frac{2n+1}{2n+2} = a_n\left[1-\frac{1}{2(n+1)}\right]$$

$$\therefore \frac{a_n^2}{n+1} = 2(a_n^2 - a_n a_{n+1})$$

$$= \frac{8}{\pi^3}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin^{2n}\theta.\, \sin^{2n}\phi.\, \cos^2\phi\, d\phi$$

and $(-1)^n s_{n+1}$ is the coefficient of

$$+t \text{ in } \frac{(t-1)(t-2)(t-3)\ldots(t-n+1)}{(n+1)!}$$

$$\therefore \quad S = \text{coefficient of } t \text{ in } \frac{8}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{d\theta\, d\phi.\, \cos^2\phi}{\sin^2\theta\, \sin^2\phi} \sum_{n=0}^{\infty} \frac{(t-1)(t-2)\ldots(t-n-1)}{(n+1)!\ \sin^{2n+2}\theta\, \sin^{2n+2}\phi}$$

$$= \text{coefficient of } t \text{ in } \frac{8}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{d\theta\, d\phi.\, \cos^2\phi}{\sin^2\theta\, \sin^2\phi} \left[-1+(1+\sin^2\theta\, \cos^2\phi)^{t-1}\right]$$

$$= \frac{8}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{d\theta\, d\phi.\, \cos^2\phi}{\sin^2\theta.\, \sin^2\phi} \frac{\log(1+\sin^2\theta\, \sin^2\phi)}{1+\sin^2\theta\, \sin^2\phi}.$$

(4) Another from of the integral may be obtained by using the fact that

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{1-2r\cos\theta+r2}} = 2\pi\Sigma a_n^2\, r^{2n}.$$

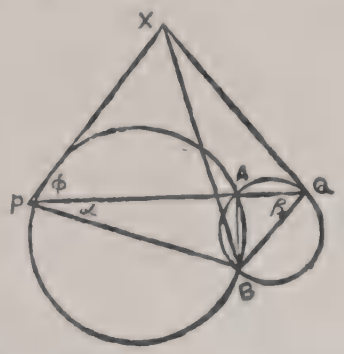
Question 591.

(A. H. KRISHNASWAMI AIYANGAR):—Two circles intersect at A and B; the tangents at the extremities of a double chord through A meet in X. Shew that XY perpendicular to BX envelopes a circle.

Solution by 'Zero'.

Let PAQ be the double chord, then

$$XPA + XQA = PBA + QBA = PBQ.$$



$\therefore$ PBQX is cyclic.

Now $BP = AB \sin(\phi+\alpha)/\sin\alpha$

$BQ = AB \sin(\phi+\alpha)/\sin\beta$

Also $BP.BQ = BX.BA$, since in the cyclic quadrilateral PBQX

$PBA = \phi = XBQ$.

Hence BX $= AB \sin^2(\phi+\alpha)/(\sin\alpha \sin\beta)$

$= AB \{(1-\cos(2\phi+2\alpha)\}/(2\sin\alpha\sin\beta)$

$\propto \{1-\cos(\pi+\theta-\beta+\alpha)\}$,

where $\theta = ABX = (2\phi-\pi+\alpha+\beta)$.

In other words, the locus of X is the cardioide

$$r = a\{(1+\cos(\theta+\alpha-\beta)\},$$

shewing that the envelope of XY is a circle.

Question 650.

(N. B. PANDYA) :—Circumscribe an ellipse about a given triangle so that incentre of ellipse may coincide with the incentre of the triangle.

Solution by C. Bhaskaraiya.

The areal co-ordinates of the in centre are

$$a : b : c.$$

If $Fyz + Gzx + Hxy = 0$ be the required ellipse, the polar w. r. t. if of $(a : b : c)$ ought to be identical with the line at infinity, $x+y+z=0$.

$$\therefore \quad cG+bH = Ha+cF = Ga+bF.$$

$$\therefore \quad F : G : H = a(a-b-c) : b(b-c-a) : c(c-a-b)$$

Hence the ellipse is

$$a(b+c-a).yz+b(c+a-b)zx+c(a+b-c)xy=0,$$

To trace the ellipse put $\frac{y}{b}=\frac{z}{c}$, and see where it cuts the bisector of the angle A; and so on.

Question 671.

(K. J. SANJANA, M.A.) :—The numbers from 1 to 2^n-1 being arranged on n cards on which the least numbers are respectively 2^0, 2^1 2^{n-1} as in Question 640, prove that the sum of the numbers on the r^{th} card is $2^{2n-2}+2^{n+r-s}-2^{n-s}$

Solution by N. Sankara Aiyar and V. Anantaraman.

The first number on the r^{th} card is 2^{r-1} and all numbers will be found on it which when divided by 2^r leave remainders greater than or equal to 2^{r-1}.

Hence the numbers on that card are

$$\{2^{r-1}+(2^{r-1}+1)+..+(2^{r-1}+2^{r-1}-1)\}+\{(2^r+2^{r-1})+(2^r+2^{r-1}+1)+.\}$$

$$+\{(2\cdot2^r+2^{r-1})+(2\cdot2^r+2^{r-1}+1)+ \quad ... \quad ... \quad \} \quad ... \quad ...$$

$$+\{(2^n-2^{r-1})+(2^n-2^{r-1}+1)+ \quad ... \quad +(2^n-1)\} \quad ... \quad ...$$

$$=2^{r-1}\times2^{r-1}(1+3+5+ \quad ... \quad +2^{n-s+1}-1)\} \quad ... \quad ...$$

$$+\frac{2^{n-r+1}}{2}(0+1+... \quad ... \quad +2^{r-1}-1).$$

$$= 2^{2r-2} \cdot \left(\frac{2^{n-r}+1}{2}\right)^2 + 2^{n-r} \cdot \frac{2^{r-1}}{2}(2^{r-1}-1)$$
$$= 2^{2r-2} \cdot 2^{2n-2r} + 2^{n-r} \cdot 2^{r-2}(2^{r-1}-1)$$
$$= 2^{2n-2} + 2^{n+r-3} - 2^{n-2}.$$

Question 682.

(S. RAMANUJAN) :—Show how to find the cube root of surds of the form $A+\sqrt[3]{B}$; and deduce that

$$\sqrt[3]{(\sqrt[3]{2}-1)} = \sqrt[3]{\frac{1}{9}} - \sqrt[3]{\frac{2}{9}} + \sqrt[3]{\frac{4}{9}}.$$

Solution by 'Zero.'

If α denote a cubic surd any expression in α can be written in the form $(a+b\alpha+c\alpha^2)$ and further

$$(a+b\alpha+c\alpha^2)^3 \equiv P+Q\alpha+R\alpha^2 \text{ (say)}.$$

Thus, when either Q or R is zero, we have a binomial cubic surd on the right side.

Hence we can write

$$\sqrt[3]{(A+\sqrt[3]{B})} \equiv r^{\frac{1}{3}}(a+b\alpha+c\alpha^2),$$

and proceed to find the values of a, b, c, α satisfying the following :

$$\left.\begin{aligned} rP &= A \\ Q &= 0 \\ rR\alpha^2 &= B^{\frac{1}{3}} \end{aligned}\right\} \text{ or } \qquad \begin{aligned} rP &= A \\ R &= 0 \\ rQ\alpha &= B^{\frac{1}{3}} \end{aligned}$$

In the particular case stated, $A=-1$, $B=2$;

$$\therefore \quad P \equiv a^3+b^3\alpha^3+c^3\alpha^6+6abc\alpha^3,$$
$$Q \equiv 3a^2b+3ac^2\alpha^3+3b^2c\alpha^3,$$
$$R \equiv 3ab^2+3a^2c+3bc^2\alpha^3,$$

and suitable values for a, b, c, r are seen to be

$$a=-b=c=1, \; r=\frac{1}{9}.$$

More generally if $AQ=BP$, we may put $a=c$ and $\alpha=\sqrt[3]{B}$, so that $R=0$ reduces to

$$a^2+b^3+ab\mathrm{B}=0$$

$$\therefore \quad b=-m\,a, \text{ if } B=\left(m+\frac{1}{m}\right);$$

and we may, without loss of generality, write $a=c=1$, $b=-m$.

The problem is thus solved, when B is of the form $\left(m+\frac{1}{m}\right)$,

and $3A\{B(1-m)-m\} = B\{B^2-B(m^3+6m)+1\}$

Question 725.

(K. B. Madhava) :—Shew that

$$\int^{\infty} \frac{x^{\alpha} dx}{1+x^{\beta} \sin^2 x} \text{ converges,}$$

but that

$$\int^{\infty} \frac{x^{\beta} dx}{1+x^{\alpha} \sin^2 x} \text{ diverges.}$$

Solution by S. R. Ranganathan.

The condition for the convergence of

$$\int^{\infty} \frac{x^{\beta}}{1+x^{\alpha} \sin^2 x}$$

are fully discussed in Bromwich : *Infinite Series*, App. III. Art. 166. It is proved there that the integral converges or diverges with the series

$$\sum' \left(n^{\beta} - \frac{\alpha}{2} \right),$$

i.e., according as $\alpha > 2(\beta+1)$ or $\alpha \leq 2(\beta+1)$.

Now taking $\beta = 3$, we get that

$$\int^{\infty} \frac{x^{3}}{1+x^{\alpha} \sin^2 x} dx$$

converges if $\alpha > 8$, and diverges if $\alpha \leq 8$.

Hence the results given in the question.

Question 800.

(S. Malhari Rao) :—Shew that the sum of all fractions which may be represented by a recurring decimal of the form $.a\,b\,\dot{c}\,\dot{d}$ is 50, provided

$$a+c=b+d=9.$$

Solution (1) *by G. L. Gupta, M.A. and S. V. Venkatachalayya.*

(i) For all values of a, b, c and d the recurring decimal $.a\,b\,\dot{c}\,\dot{d}$

$$= \frac{abcd-ab}{9900} = \frac{(1000a+100b+10c+d)-(10a+b)}{9900}$$

$$= \frac{990a+99b+10c+d}{9900}. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

But here $a+c=b+d=9.$

∴ (1) which may be written as

$$\frac{10(a+c)+(b+d)+98(10a+b)}{9900}$$

reduces to

$$\frac{99+98(10a+b)}{9900} = \frac{99+98(ab)}{9900} \quad \ldots \quad \ldots \quad (2)$$

But since each of a and b may have any of the values 0,1,2,3,...9 ab includes all integral numbers from 1 to 99, besides 0.

Hence the required sum

$$= \frac{1}{9900}\{(99\times 100)+98(1+2+3+\ldots+99)\}$$

$$= \frac{1}{9000}\left\{9900+\frac{98\times 90\times 100}{2}\right\} = 50.$$

Note:—There is a slight mistake in the printed question in which the decimal reads $(\cdot\dot{a}b\dot{c}d)$.

Solution by H. Br. and (2) *K. B. Madhava.*

(2) The recurring fraction $(.\dot{a}b\dot{c}d)$

$$= \frac{1000a+100b+10c+d}{9999}$$

$$= \frac{99.(10a+b)+10(a+c)+(b+d)}{9999}$$

$$= \frac{10a+b+1}{101}, \text{ as } a+c=b+d=9.$$

As a and b assume independently all values from 0 to 9 (inclusive). the numerator assumes (once only) every integral values from 1 to 100, The sum of the fractions is therefore

$$\frac{1+2+3+\ldots\ldots+100}{101} = 50.$$

Question 802.

(S. Krishnaswami Iyengar):—Prove that

(i) $$\sum_{1}^{\infty}\left\{\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\right\}^2\cdot\frac{1}{n} = \frac{4}{\sqrt{\pi}}\left(\frac{4}{\sqrt{\pi}}-\sqrt{\pi}\right),$$

(ii) $$\sum_{0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\cdot\frac{1}{(2m+2n+1)}\cdot\frac{1}{(m+n+1)}$$

$$= \pi\left\{\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}-\frac{2}{2m+1}\cdot\frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})}\right\}.$$

Solution by S. R. Ranganathan and S. V. Venkatachallayya.

(i) This is the same as Q 664 solved on p. 69 of Vol. VIII, J.I.M.S.

(ii) We have, for $0 \leq x \leq 1$,

$$(1-x)^{-\frac{1}{2}} = 1+\tfrac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2+\ldots$$

so that, $$\Gamma(\tfrac{1}{2})(1-x)^{-\frac{1}{2}} = \sum_{0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}x^n,$$

so that, $$\Gamma(\tfrac{1}{2})\int_0^1 x^m(1-x)^{-\frac{1}{2}}dx = \int_0^1 \sum_0^\infty \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}x^{m+n}dx$$

$$= \sum_0^\infty \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\int_0^1 x^{m+n}dx$$

$$= \sum \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\frac{1}{m+n+1};$$

i.e., $$\{\Gamma(\tfrac{1}{2})\}^2\frac{\Gamma(m+1)}{\Gamma(m+\frac{3}{2})} = \sum \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\frac{1}{m+n+1}.$$

In the above the change of order of integration and summation is permissible since the integral on the left is convergent and the terms in the series on the right are all positive (cf. Bromwich *Infinite Series* App. III, Art. 175. Theorem B.)

By treating similarly the expansion of $(1-x^2)^{-\frac{1}{2}}$ we get

$$\tfrac{1}{2}\{\Gamma(\tfrac{1}{2})\}^2\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)} = \sum_0^\infty \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\frac{1}{(2m+2n+1)}$$

Now,
$$\sum_0^\infty \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\frac{1}{(2m+2n+1)(m+n+1)}$$

$$= \sum_0^\infty \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\left\{\frac{2}{2m+2n+1} - \frac{1}{m+n+1}\right\}$$

$$= \{\Gamma(\tfrac{1}{2})\}^2\left\{\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)} - \frac{\Gamma(m+1)}{\Gamma(m+\frac{3}{2})}\right\}$$

$$= \pi\left\{\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)} - \frac{2}{(2m+1)}\frac{\Gamma(m+1)}{\Gamma(m+\frac{3}{2})}\right\}.$$

Question 810.

(T. P. Trivedi, M.A., L.L.B.) :—Find the values other than zero which satisfy the equations :—

$$x^2 = y-z\ ;\ y^2 = z-x\ ;\ z^2 = x-y.$$

Solution by H. Br. S. Gangadharan, K. Santanam, Kasturi Reddi, B. Sc., and F. H. V. Gulasekkaran, B. A.

We have

$$\left.\begin{aligned} x^2 &= y-z \\ y^2 &= z-x \\ z^2 &= x-y. \end{aligned}\right\} \quad \ldots \quad (A)$$

$$\left.\begin{aligned} &\therefore \quad x^2+y^2+z^2=0\,; \\ &\quad\; x^3+y^3+z^3=0\,; \\ \text{and } & x^4+y^4+z^4=\Sigma x^2(y-z)=-(y-z)(z-x)(x-y)=-x^2y^2z^2 \end{aligned}\right\} \quad B$$

[The system of equations B is composed of the system A together with the system derived from A by changing the signs of x, y, z.]

Let (x, y, z) be the roots of $t^3-pt^2+qt-r=0$. Then from (B),

$$\left.\begin{aligned} p^2 &= 2q\,; \\ pq &= 3r \\ \text{and} \quad r(p+r) &= 0. \end{aligned}\right\}$$

If $r=0$, we get $p=q=0$, or $x=y=z=0$.

If $r\neq 0$, we get $q=-3$, $r=-p$; $p^2=-6$.

Thus x, y, z are the roots of

$$t^3-pt^2-3t+p=0, \text{ where } p^2=-6.$$

Put $\mu=pt+2$, and we get

$$\mu^3+6\mu+16=0,$$

The roots of this equation are

$$\mu_1=\sqrt{2}\,\{(3-2\sqrt{2})^{\frac{1}{3}}-(3+2\sqrt{2})^{\frac{1}{3}}\}$$

$$\mu_2=\sqrt{2}\,\{(3-2\sqrt{2})^{\frac{1}{3}}\omega-(3+2\sqrt{2})^{\frac{1}{3}}\omega^2\}$$

$$\mu_3=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

From these we deduce the following types of values of x, y, z.

$$\left.\begin{aligned} 100x &= -1532\cdot \\ 100y &= -118-462\cdot \\ 100z &= \;\;118-462\cdot \end{aligned}\right\} \text{approximately.}$$

QUESTIONS FOR SOLUTION.

943. (HEMRAJ) :—If n be prime and

$$(x+1)(x+2)\ldots(x+n-1) = x^{n-1} + A_1 x^{n-2} + A_2 x^{n-3} + \ldots A_{n-1}$$

shew that all the odd A's except A_1 are divisible by n^2—

944. (SADANAND) :—Prove the identity

$$\phi(x) + S_1\phi'(x).\frac{x}{1!} - S_2\phi''(x).\frac{x^2}{3!} + \ldots\ldots$$

$$\equiv \phi(2x) - S_2\phi''(2x)\frac{x^2}{2!} + S_4\phi''^{1}(2x)\frac{x^4}{4!} - \ldots\ldots$$

where S_{2n} is the nth Eulerian number and S_{2n} is the n the" prepared Bernoullean number.

Deduce that $\log 2 = \frac{S_1}{1} - \frac{S_2}{2^2.1} - \frac{S_3}{3} + \frac{S_4}{2^5.2} + \frac{S_5}{5} \ldots\ldots$

945. (SADANAND) :—Fill up the vacant cells of the following magic square :—

46				16
	54		39	
		27		
	50		35	
73				43

946. (M. K. KEVALRAMANI) :—Show that there are forty points on the curve $x^2/a^2 + y^2/b^2 = 8x^2y^2$. Such that if tangents be drawn from them to the ellipse $x^2.a^2 + y^2/b^2 = 1$, the points of contact have got their eccentric angles in the ratio 1 : 2.

947. (M. K. KEVALRAMANI):—A triangle ABC inscribed in any lipse, touches a confocal ellipse at the points D, E, F respectively. Show that the ratio of the triangle DEF to ABC $= r/(2R)$, where r, R refer to the triangle ABC as usual.

948. (M. K. KEVALRAMANI):—A perfectly elastic particle, acted on by no forces is projected from the centre of a rectangle whose sides are $2a$ and $2b$ $(a>b)$ to strike the bigger side first and then goes on re-

bounding from its sides. If it ever pass through an angular point show that the direction of projection makes with the smaller side an angle $\tan^{-1}\left(\frac{a}{b}\frac{2n+1}{2m+1}\right)$ where m & n are integers, and if it ever strike any point of the smaller side other than the extremities, the angle must be of the form $\tan^{-1}\left(\frac{a}{b}\cdot\frac{2n+1}{2m}\right)$

949. (C. KRISHNAMACHARY):—With the usual notation for the numbers of Bernoulli and Euler, show that

$$2n = B_1\,2^2\,(2^2-1)\binom{2n}{2} - B_2\,2^4\,(2^4-1)\binom{2n}{4} + B_3\,2^6\,(2^6-1)\binom{2n}{6} - \quad + (-1)^{n-1}\,Bn\,2^{2n}\,(2^{2n}-1).$$

$$(-1)^n\,E_n\,(2n+1) = (-1)^n\,B_n\,2^{2n}\,(2^{2n}-1)\binom{2n+1}{1} + (-1)^{n-1}\,B_{n-1}\,2^{2n-2}\,(2^{2n-2}-1)\binom{2n+1}{3} + \quad -B_1\,2^2\,(2^2-1)\binom{2n+1}{2} + \quad (2n+1)$$

950. (C. KRISHNAMACHARY) :—Collect the co-efficients of x^n in the series—

$$\frac{1}{1-x} - \frac{x}{(1-x)^2} + \frac{1\cdot3}{2!}\,\frac{x^2}{(1-x)^3} - \frac{1\cdot3\cdot5}{3!}\,\frac{x^3}{(1-x)^4} + \quad \ldots \quad \ldots$$

and show that its sum is zero when n is odd, and $(-1)^m$

$$\frac{1\cdot3\cdot5\cdot(2m-1)}{2\cdot4\cdot6\cdot\quad 2m}$$ when n is $2m$.

951. (C KRISHNAMACHARY) :—Prove that

(1) $$\int_0^{\frac{\pi}{4}} (\log\tan x)^{2n}\,dx = \frac{E_n\,\pi^{2n+1}}{2^{2n+2}},$$

(2) $$\int_0^{\frac{\pi}{4}} \frac{(\log\tan x)^{2n-1}}{\cos x\,(\cos x+\sin x)}\,dx = -\frac{(2^{2n-1}-1)}{2n}B_n\pi^{2n},$$

(3) $$\int_0^{\frac{\pi}{4}} \frac{(\log\tan x)^{2n-1}}{\cos 2x}\,dx = -\frac{(2^{2n}-1)B_n\pi^{2n}}{4n},$$

(4) $$\int_0^{\frac{\pi}{4}} \frac{(\log\tan x)^{2n-1}}{\cos x\,(\cos x-\sin x)}\,dx = -\frac{2^{2n-1}B_n}{2n}\pi^{2n}.$$

952. (Communicated by Mr. Hemraj) :—The polar reciprocal of an equiangular spiral with respect to a rectangular hyperbola having its centre at the pole and touching the spiral is the curve itself.

953. (Communicated by Mr. Hemraj) :—Two of the common tangents to a circle S and a conic T meet in P and the other two in Q. Show that P and Q lie on the same confocal to T.

954. (K. B. Madhava) :—When asked his age De Morgan once humorously observed : "I am one of those whose age shall be in a certain year belonging to the century of their birth the square root of that year." For, being born in 1806 A. D., he was 43 years old in 1849, and $\sqrt{(1849)}=43$. Show that the same observation will be true in the case of those born in 9 particular years in all subsequent centuries.

955. (K. B. Madhava) :—The 5th, 13th, 17th, 29th and 37th roots (if rational) of a number end in the same digit as the number itself, the numbers when fractional being expressed in decimals. Prove this property and illustrate how it enables one to give out at sight the above rational roots of big numbers.

956. (K. B. Madhava) :—Find the years of the present century in which the month of February will have (i) no full moon, (ii) no new moon.

957. (Martyn M. Thomas, M. A.) :—From a flexible envelope in the form of a surface of revolution formed by the curve $s=f(y)$ revolving about the axis of x, the part between two meridians the planes of which are inclined to each other at an angle $\frac{2\pi}{m}$ is cut away, and the edges are then sewed together. Prove that the meridian curve of the new envelope will be $s=f\left(\frac{m\,y}{m-1}\right)$.

Hence show that, if a lune of angle $\frac{4\pi}{5}$ be cut off from an oblate spheroid, the minor axis of whose generating ellipse is c, and eccentricity $\frac{3}{5}$, the meridian curve of the new surface of revolution will be the curve of sines $y=\frac{3}{4}c\sin\frac{x}{c}$.

958. (Martyn M. Thomas, M. A.,) :—Establish the formula

$$\frac{d^n u}{dx^n}=x^{r-n}\frac{d^r}{dx^r}\left[x^q\,\frac{d^{q-r}}{dx^{q-r}}\left\{x^{p-r}\,\frac{d^{p-q}}{dx^{p-q}}\left(x^{n-q}\,\frac{d^{n-p}}{dx^{n-p}}\cdot\frac{u}{x^p}\right)\right\}\right],$$

where $n>p>q>r$.

959. (M. T. Naraniengar) :—The sides AB, AC of a triangle ABC are produced to D, E such that $BD=CE=BC$.

If the circle ADE cuts the straight line AH_1 in J, prove that

$$I_1J=\tfrac{1}{2}\,H_1.$$

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PROGRESS REPORT.

The committee regret to have to report the premature death at Chicago U. S, of Mr. B. K. Srinivasa Iyengar, M.A., who was an Associate of the Society since August 1910. The deceased was a brilliant graduate of the Madras University and held the post of Assistant Professor of Mathematics in the Central College, Bangalore. He went to America as a Mysore Government Scholar to study 'Systems of Education' some two years ago. In him the Society has lost an ardent lover of Mathematics and an enthusiastic member.

POONA.
31*st May* 1918.

D. D. KAPADIA,

Honorary Secretary.

On the addition Formulae for the Jacobian Functions E and Π

In the following pages I propose to deduce a few of the expressions for $Eu_1+Eu_2+Eu_3+Eu_4$ and those for $\Pi(u_1,a)+\Pi(u_2,a)+\Pi(u_3,a)+\Pi(u_4,a)$ in terms of the sn, cn, dn functions of u_1, u_2, u_3, u_4, when $u_1+u_2+u_3+u_4$ is congruent to zero.

§ I. Preliminary.

1. To begin with let us establish a few preliminary results which will be helpful to us in what follows.

For this purpose, let us consider the points of intersection of the fixed curve

$$y^2=x(1-x)(1-kx) \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

with any arbitrary chosen parabola

$$y=l+mx+nx^2 \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Now, if any point (x_1, y_1) be taken on the cubic (i), the equation in u

$$\operatorname{sn}^2(u, k)-x_1=0, \text{ where } k \text{ is constant,}$$

has two solutions $+u_1, -u_1$, and all other solutions are congruent to these two.

Again since $\frac{\partial}{\partial u}(\operatorname{sn}^2 u)=2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u$, we have

$$\left[\frac{\partial}{\partial u}(\operatorname{sn}^2 u)\right]^2=4y^2.$$

Let us choose u_1 to be the solution for which $\frac{\partial}{\partial u}(\operatorname{sn}^2 u)=+2y_1$; and let (x_r, y_r), where $r=1, 2, 3, 4$, be the points of intersection of (i) and (ii); so that x_1, x_2, x_3, x_4 are the roots of the equation

$$\mathrm{F}(x)=(l+mx+nx^2)^2-x(1-x)(1-k^2x)=0 \quad \ldots \quad \text{(iii)}$$

and

$$\mathrm{F}(x)=(x-x_1)(x-x_2)(x-x_3)(x-x_4). \quad \ldots \quad \ldots \quad \text{(iv)}$$

The variation ∂x_r in one of these abscissae due to the variation in position of the parabola (ii) is given by the equation

$$\mathrm{F}'(x_r)\partial x_r-2(l+m x_r+n x_r^2)(\partial l+x_r\partial m+x_r^2\partial n)=0.$$

Remembering that $l+m x_r+n x_r^2=y_r$, we obtain

$$\frac{\partial x_r}{y_r}=\frac{+2(\partial l+x_r\partial m+x_r^2 n)}{\mathrm{F}'(x_r)} \quad \ldots \quad \ldots \quad \text{(v)}$$

$$=\frac{f(x_r)}{\mathrm{F}'(x_r)} \text{ suppose.}$$

Then since $xf(x)$ is of lower degree than $F(x)$,

$$\frac{xf(x)}{F(x)} = \sum_{r=1}^{4} \frac{f(x_r)}{F'(x_r)} \cdot \frac{x_r}{x - x_r} \quad \ldots \quad \ldots \quad \text{(vi)}$$

provided x_1, x_2, x_3, x_4 are all unequal, and $F'(x_r) \neq 0$.

Putting $x = 0$, in (vi)

$$\sum_{r=1}^{4} \frac{f(x_r)}{F'(x_r)} = 0.$$

$$\therefore \quad \sum_{r=1}^{4} \frac{\partial x_r}{y_r} = 0.$$

$$\therefore \quad \sum_{r=1}^{4} \partial u_r = 0.$$

Hence the sum of the parameters of the point of intersection is a constant and independent of the position of the parabola and it may be easily shewn to be congruent to zero.

Hence so long as $u_1 + u_2 + u_3 + u_4 = 0$, we may deduce relations between symmetrical functions of s_r, c_r, d_r, (where s_r, c_r, d_r stand for sn u_r, cn u_r, dn u_r respectively) from the values of the symmetrical functions of the roots of the equation. (iii)

2. We have from equations (iii) and (iv) above,

$$\sum_{r=1}^{4} s^2{}_r = \sum_{r=1}^{4} x_r = \frac{k^2}{n^2} - 2\frac{m}{n}. \quad \ldots \quad \ldots \quad (\alpha)$$

$$s_1 s_2 s_3 s_4 = \sqrt{x_1 x_2 x_3 x_4} = \frac{l}{n}. \quad \ldots \quad \ldots \quad \ldots \quad (\beta)$$

$$c_1 c_2 c_3 c_4 = \sqrt{(1-x_1)(1-x_2)(1-x_3)(1-x_4)} = \frac{l+m+n}{n}. \quad \ldots \quad (\gamma)$$

$$d_1 d_2 d_3 d_4 = \sqrt{(1-k^2x_1)(1-k^2x_2)(1-k^2x_3)(1-k^2x_4)}$$

$$= 1 + k^2 \cdot \frac{m}{n} + k^4 \frac{l}{n}. \quad \ldots \quad \ldots \quad \ldots \quad (\delta)$$

Hence eliminating l, m, n between (β), (γ) and (δ), we obtain the well known identity of Gudermann

$$k'^2 - k^2 k'^2 s_1 s_2 s_3 s_4 + k^2 c_1 c_2 c_3 c_4 - d_1 d_2 d_3 d_4 = 0, \quad \ldots \quad (A)$$

where $k'^2 = 1 - k^2$.

Writing in the identity (A) successively, (u_3+K, u_4-K), (u_3-iK', u_4+iK'), $(u_3+K-iK', u_4-K+i'K)$ for (u_3, u_4) respectively, we obtain the following relations quoted in Whittaker and Watson's *Modern Analysis* (Revised Edition) from H. J. S. Smith, *Proceedings*, Lond. Math. Soc. (I) X

$$k^2(s_1s_2c_3c_4-c_1c_2s_3s_4)-d_1d_2+d_3d_4=0 \quad \ldots \quad \ldots \quad \text{(A1)}$$

$$k'^2(s_1s_2-s_3s_4)+d_1d_2c_3c_4-c_1c_2d_3d_4=0 \quad \ldots \quad \ldots \quad \text{(A2)}$$

$$s_1s_2d_3d_4-d_1d_2s_3s_4+c_3c_4-c_1c_2=0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(A3)}$$

3. Again, let us evaluate the functions

$$\sum_{r=1}^{4}\frac{c_rd_r}{s_r}, \quad \sum_{r=1}^{4}\frac{s_rd_r}{c_r}, \quad \sum_{r=1}^{4}\frac{s_rc_r}{d_r} \quad \text{in terms of } l, m, n.$$

We have

$$\sum_{r=1}^{4}\frac{c_rd_r}{s_r}=\sum_{r=1}^{4}\frac{y_r}{x_r}=\sum_{r=1}^{4}\frac{l+mx_r+nx_r^2}{x_r}$$

$$=\Sigma\frac{1}{x_r}+n\Sigma x_r+4m,$$

$$=\frac{1}{l}+\frac{k^2}{n}.$$

Again $$\sum\frac{s_rd_r}{c_r}=\sum\frac{y_r}{1-x_r}=\sum\frac{l+mx_r+nx_r^2}{1-x_r}.$$

This function is best evaluated by substituting p_r for $1-x_r$ and forming the equation whose roots are the four values of p_r. We can then by easy algebraical processes deduce that

$$\sum\frac{s_rd_r}{c_r}=\frac{k^2}{n}-\frac{k'^2}{l+m+n}.$$

Similarly from the equation whose roots are $1-kx_r$, we deduce that

$$\sum\frac{s_rc_r}{d_r}=\frac{lk^4+mk^2+n(1+k'^2)}{n(n+mk^2+lk^4)}.$$

From all the foregoing, we are in a position to deduce several relations. We shall select only such as will be useful for our purpose.

4. We have

$$\frac{1}{n}=\frac{1}{k}\sqrt{(\Sigma s_r^2-2s_1s_2s_3s_4+2c_1c_2c_3c_4-2)} \quad \ldots \quad \ldots \quad \text{(B1)}$$

$$=\sqrt{(2-\Sigma d_r^2+2d_1d_2d_3d_4-2k^4s_1s_2s_3s_4)} \quad \ldots \quad \ldots \quad \text{(B2)}$$

$$=\frac{-1}{k^2+(s_1s_2s_3s_4)^{-1}}\sum\frac{c_rd_r}{s_r} \quad \ldots \quad \ldots \quad \ldots \quad \text{(B3)}$$

$$=\frac{1}{k^2-k'^2(c_1c_2c_3c_4)^{-1}}\sum\frac{s_rd_r}{c_r} \quad \ldots \quad \ldots \quad \ldots \quad \text{(B4)}$$

$$=\frac{1}{1+k'^2(d_1d_2d_3d_4)^{-1}}\sum\frac{s_rc_r}{d_r} \quad \ldots \quad \ldots \quad \ldots \quad \text{(B5)}$$

Further, writing $u_r+\mathrm{K}$ for u_r in B3 and in B4, and $u_r-i\,\mathrm{K}'$ for u_r in B5, we obtain

$$\frac{1}{n}=\frac{-1}{(c_1c_2c_3c_4)^{-1}+k^2(d_1d_2d_3d_4)^{-1}\sum\frac{s_r}{c_rd_r}} \quad \ldots \quad \ldots \quad \text{(B6)}$$

$$=\frac{1}{k^2k'^2(d_1d_2d_3d_4)^{-1}-(s_1s_2s_3s_4)^{-1}\sum\frac{c_r}{s_rd_r}} \quad \ldots \quad \ldots \quad \text{(B7)}$$

$$=\frac{k^2s_1s_2s_3s_4+c_1c_2c_3c_4\sum\frac{d_r}{s_rc_r}}{k'^2s_1s_2s_3s_4+c_1c_2c_3c_4\sum\frac{d_r}{s_rc_r}} \quad \ldots \quad \ldots \quad \ldots \quad \text{(B8)}$$

the summation extending to the suffixes 1, 2, 3, 4.

By eliminating x between the equations (i) and (ii) of § I, we obtain the equation whose roots are y_1, y_2, y_3, y_4. Evaluating from this equation $\sum_{r=1}^{4}\frac{1}{y_r}$, we can easily prove

$$\frac{1}{n}=-\left\{(s_1s_2s_3s_4)^{-1}+(c_1c_2c_3c_4)^{-1}+k^4(d_1d_2d_3d_4)^{-1}\right\}^{-1}\sum_{r=1}^{4}\frac{1}{s_rc_rd_r} \qquad \text{(B9)}$$

§ II. $\Sigma\,\mathrm{E}(u_r, k)$ when $u_1+u_2+u_3+u_4=0$.

5. We have $\mathrm{E}(u_r, k)=\int(\mathrm{dn}^2u_r)\,\partial u_r$

$$=u_r-k^2\int(\mathrm{sn}^2u_r)\partial u_r$$

$$=u_r-\frac{1}{2}k^2\int\frac{x_r\,\partial x_r}{y_r}.$$

$$\therefore \quad \sum_{r=1}^{4}\mathrm{E}(u_r, k)=-\frac{1}{2}k^2\sum_{r=1}^{4}\int\frac{x_r\,\partial x_r}{y_r}. \quad \ldots \quad \ldots \quad (\Sigma)$$

Now going back to our equation (v) of § I

$$x_r\frac{\delta x_r}{y_r}=\frac{+2x_r\,(\partial l+x_r\partial m+x_r^2\,\partial n)}{\mathrm{F}'(x_r)}.$$

By the method of partial fractions, we can easily prove that

$$\sum_{r=1}^{4}\frac{x_r}{\mathrm{F}'(x_r)}=0, \quad \sum_{r=1}^{4}\frac{x_r^2}{\mathrm{F}'(x_r)}=0 \quad \text{and} \quad \sum_{r=1}^{4}\frac{x_r^3}{\mathrm{F}'(x_r)}=+\frac{1}{n^3}.$$

Hence $$\sum_{r=1}^{4}\frac{x_r\,\partial x_r}{y_r}=\frac{1}{n^3}\partial n,$$

$$\therefore \quad \sum_{r=1}^{4}\int\frac{x_r\,\partial x_r}{y_r}=-\frac{2}{n}.$$

$$\therefore \text{ from } (\Sigma), \quad \sum_{r=1}^{4} \mathrm{E}(u_r, k) = k^2\left(\frac{1}{n}\right) \quad \ldots \quad \ldots \quad \ldots \quad (\mathrm{C})$$

Hence $\sum_{r=1}^{4} \mathrm{E}(u_r, k)$ has for its values in terms of s_r, c_r, d_r, each of the expressions in B 1—B 9 in § I multiplied by k^2.

6. $\mathrm{E}(u_1,k)+\mathrm{E}(u_2, k)+\mathrm{E}(u_3, k)$ when $u_1+u_2+u_3=0$.

In the above, putting $u_4=0$, we deduce that when $u_1+u_2+u_3=0$, $\mathrm{E}(u_1, k)+\mathrm{E}(u_2, k)+\mathrm{E}(u_3, k)$

$$= k\sqrt{s_1^2+s_2^2+s_3^2+2c_1c_2c_3-2} \quad \ldots \quad \ldots \quad \ldots \quad (\mathrm{D1})$$

$$= k\sqrt{1-c_1^2-c_2^2-c_3^2+2c_1c_2c_3} \quad \ldots \quad \ldots \quad \ldots \quad (\mathrm{D2})$$

$$= \sqrt{(1-d_1^2-d_2^2-d_3^2+2d_1d_2d_3)} \quad \ldots \quad \ldots \quad \ldots \quad (\mathrm{D3})$$

$$= -k^2s_1s_2s_3 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (\mathrm{D4})$$

$$= \frac{k^2}{k^2-k'^2(c_1c_2c_3)^{-1}}\left\{\frac{s_1d_1}{c_1}+\frac{s_2d_2}{c_2}+\frac{s_3d_3}{c_3}\right\} \quad \ldots \quad \ldots \quad (\mathrm{D5})$$

$$= \frac{k^2}{1+k'^2(d_1d_2d_3)^{-1}}\left\{\frac{s_1c_1}{d_1}+\frac{s^2c_2}{d_2}+\frac{s_3c_3}{d_3}\right\} \quad \ldots \quad \ldots \quad (\mathrm{D6})$$

§ III. $\Sigma\Pi(u_r, k, a)$ when $u_1+u_2+u_3+u_4=0$.

7. Now, since

$$\Pi(u_r, k, a) = \int \frac{k^2 \mathrm{sn}a\, \mathrm{cn}a\, \mathrm{dn}a\, \mathrm{sn}^2 u_r}{1-k^2\mathrm{sn}\, a\, \mathrm{sn}^2 u_r}\partial u_r$$

$$\therefore \quad \Pi(u_r, k, a)+\frac{\mathrm{cn}\, a\, \mathrm{dn}\, a}{\mathrm{sn}\, a}.u_r = \frac{\mathrm{cn}a\, \mathrm{dn}\, a}{\mathrm{sn}\, a}\int \frac{\partial u_r}{1-k^2\mathrm{sn}^2 a\, \mathrm{sn}^2 u_r}.$$

Then putting $\mathrm{sn}^2 u_r = x_r$; $\frac{1}{k^2\mathrm{sn}^2 a} = \lambda$; $\frac{\mathrm{cn}\, a\, \mathrm{dn}\, a}{k^2\mathrm{sn}^3 a} = \mu$; and remembering that $\sum_{r=1}^{4} u_r = 0$, we easily deduce that

$$\sum_{r=1}^{4} \Pi(u_r, k, a) = \tfrac{1}{2}\mu \sum_{r=1}^{4} \int \left\{\frac{1}{\lambda-x_r}\frac{\partial x_r}{y_r}\right\}. \quad \ldots \quad (\omega)$$

Again going back to equation (v) of § I,

$$\frac{\partial x_r}{y_r} = \frac{+2(\delta l+x_r\delta m+x_r^2\delta n)}{\mathrm{F}'(x_r)} = \frac{f(x_r)}{\mathrm{F}'(x_r)} \text{ (suppose).}$$

$$\sum_{r=1}^{4} \frac{1}{\lambda-x_r}\frac{\partial x_r}{y_r} = \sum_{r=1}^{4} \frac{f(x_r)}{(\lambda-x_r)\mathrm{F}'(x_r)}.$$

Again from (iv) of § 1,

$$F(\lambda) = n^2(\lambda - x_1)(\lambda - x_2)(\lambda - x_3)(\lambda - x_4)$$

$$\therefore \quad \sum \frac{f(\lambda)}{F(\lambda)} = \sum_{r=1}^{4} \frac{f(x_r)}{(\lambda - x_r) F'(x_r)}$$

since, $f(\lambda)$ is of lower degree than $F(\lambda)$,

$$\text{Hence} \quad \sum_{r=1}^{4} \frac{1}{\lambda - x_r} \frac{\partial x_r}{y_r} = \frac{f(\lambda)}{F(\lambda)} = \frac{+2\partial (l + m\lambda + n\lambda^2)}{(l + m\lambda + n\lambda^2)^2 - \mu^2},$$

$$= \frac{1}{\mu} \left[\frac{\partial (l + m\lambda + n\lambda^2)}{l + m\lambda + n\lambda^2 - \mu} - \frac{\partial (l + m\lambda + n\lambda^2)}{l + m\lambda + n\lambda^2 + \mu} \right]$$

$$= \frac{1}{\mu} \partial \log \frac{l + m\lambda + n\lambda^2 - \mu}{l + m\lambda + n\lambda^2 + \mu}$$

$$\therefore \quad \mu \sum_{r=1}^{4} \int \frac{1}{\lambda - x_r} \cdot \frac{\partial x_r}{y_r} = \log \frac{l + m\lambda + n\lambda^2 - \mu}{l + m\lambda + n\lambda^2 + \mu}.$$

$$\therefore \quad \sum_{r=1}^{4} \Pi(u_r, k, a) = \frac{1}{2} \log \frac{l + m\lambda + n\lambda^2 - \mu}{l + m\lambda + n\lambda^2 + \mu}.$$

$$= \tfrac{1}{2} \log \Phi, \text{ say.} \qquad \ldots \qquad \ldots \quad (\mathrm{H})$$

After substituting for λ and μ,

$$\Phi = \frac{1 + \frac{m}{n} k^2 \mathrm{sn}^2 a + \frac{l}{n} k^4 \mathrm{sn}^4 a - \frac{k^2}{n} \mathrm{sn}a\, \mathrm{cn}a\, \mathrm{dn}a}{1 + \frac{m}{n} k^2 \mathrm{sn}^2 a + \frac{l}{n} k^4 \mathrm{sn}^4 a + \frac{k^2}{n} \mathrm{sn}a\, \mathrm{cn}a\, \mathrm{dn}a}$$

Hence after substituting from (β), (γ) of § I and from (C) of § II respectively,

$$\Phi = \frac{1 + \mathrm{P}\, \mathrm{sn}^2 a + \mathrm{Q}\, \mathrm{sn}^4 a - \mathrm{R}\, \mathrm{sn}a\, \mathrm{cn}a\, \mathrm{dn}a}{1 + \mathrm{P}\, \mathrm{sn}^2 a + \mathrm{Q}\, \mathrm{sn}^4 a + \mathrm{R}\, \mathrm{sn}a\, \mathrm{cn}a\, \mathrm{dn}a},$$

where

$$\mathrm{P} = k^2(c_1c_2c_3c_4 - s_1s_2s_3s_4 - 1) = (d_1d_2d_3d_4 - k^4 s_1s_2s_3s_4 - 1)$$

$$= \frac{1}{k'^2}(d_1d_2d_3d_4 - k^4 c_1c_2c_3c_4 - 1 + k^4),$$

$$\mathrm{Q} = k^4 s_1 s_2 s_3 s_4,$$

$$\mathrm{R} = \sum_{r=1}^{4} \mathrm{E}(u_r, k),$$

which is already expressed in terms of s_r, c_r, d_r in § II.

8. Now putting $u_4=0$, we have from above, when $u_1+u_2+u_3=0$,

$$\Pi(u_1, k, a)+\Pi(u_2, k, a)+\Pi(u_3, k, a)$$

$$=\tfrac{1}{2}\log\frac{1+\mathrm{U}\,\mathrm{sn}^2a-\mathrm{W}\,\mathrm{sn}a\,\mathrm{cn}a\,\mathrm{dn}a}{1+\mathrm{U}\,\mathrm{sn}^2a+\mathrm{W}\,\mathrm{sn}a\,\mathrm{cn}a\,\mathrm{dn}a}=\tfrac{1}{2}\log\Omega\ (\text{say}) \qquad \dots \quad (\mathrm{K})$$

where $\mathrm{U}=k^2(c_1c_2c_3-1)=d_1d_2d_3-1$

$$=k^2[s_1s_2c_3d_3-s_3^2]$$

$\mathrm{W}=\mathrm{E}(u_1, k)+\mathrm{E}(u_2, k)+\mathrm{E}(u_3, k)$

= the expressions D1—D6 in § II.

$=-k^2s_1s_2s_3$ in particualr.

With these values

$$\Omega=\frac{\mathrm{dn}^2a+k^2\mathrm{sn}^2a\ c_1c_2c_3+k^2\mathrm{sn}a\,\mathrm{cn}a\,\mathrm{dn}a.s_1s_2s_3}{\mathrm{dn}^2a+k^2\mathrm{sn}^2a\ c_1c_2c_3-k^2\mathrm{sn}a\,\mathrm{cn}a\,\mathrm{dn}a.s_1s_2s_3} \qquad \dots \quad (\mathrm{K1})$$

$$=\frac{\mathrm{cn}^2a+\mathrm{sn}^2a\ d_1d_2d_3+k^2\mathrm{sn}a\,\mathrm{cn}a\,\mathrm{dn}a\ s_1s_2s_3}{\mathrm{cn}^2a+\mathrm{sn}^2a\ d_1d_2d_3-k^2\mathrm{sn}a\,\mathrm{cn}a\,\mathrm{dn}a\ s_1s_2s_3} \qquad \dots \quad (\mathrm{K2})$$

expressions easily remembered.

9. It will be interesting to deduce from the above the expressions for Ω given in Dixon's *Elliptic Functions*.

Taking $\mathrm{U}=k^2(s_1s_2c_3d_3-s_3^2)$

and $\mathrm{W}=-k^2s_1s_2s_3$,

the expression (K) above reduces to

$$\Omega=\frac{(1-k^2\mathrm{sn}^2a\mathrm{sn}^2u_3)+k^2\mathrm{sn}a.s_1s_2(\mathrm{sn}a.c_3d_3+s_3.\mathrm{cn}a\ \mathrm{dn}a)}{(1-k^2\mathrm{sn}^2a\mathrm{sn}^2u_3)+k^2\mathrm{sn}a.s_1s_2(\mathrm{sn}a.c_3d_3-s_3.\mathrm{cn}a\ \mathrm{dn}a)}$$

dividing the numerator and denominator by $1-k^2\mathrm{sn}^2a\mathrm{sn}^2u_3$, we obtain

$$\Omega=\frac{1+k^2\mathrm{sn}a.s_1s_2\mathrm{sn}(a+u_3)}{1+k^2\mathrm{sn}a.s_1s_2\mathrm{sn}(a-u_3)}. \qquad \dots \quad \dots \quad \dots \quad (\mathrm{M1})$$

10. Again from the expressions for $\mathrm{sn}\,u_1\,\mathrm{sn}\,u_2$, $\mathrm{sn}\,a\,\mathrm{sn}\,(u_1+u_2-a)$ obtained by substituting in turn $\mathrm{A}=u_1$, $\mathrm{B}=u_2$; $\mathrm{A}=u_1+u_2-a$, $\mathrm{B}=a$ in the formula

$$\mathrm{sn}\mathrm{A}\ \mathrm{sn}\mathrm{B}=\frac{\mathrm{sn}^2\tfrac{1}{2}(\mathrm{A}+\mathrm{B})-\mathrm{sn}^2\tfrac{1}{2}(\mathrm{A}-\mathrm{B})}{1-k^2\mathrm{sn}^2\tfrac{1}{2}(\mathrm{A}+\mathrm{B})\mathrm{sn}^2\tfrac{1}{2}(\mathrm{A}-\mathrm{B})}, \qquad \dots \quad \dots \quad (\mathrm{N})$$

we have $1-k^2\mathrm{sn}\,u_1\,\mathrm{sn}\,u_2\,\mathrm{sn}(u_1+u_2-a)=$

$$\left\{\frac{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2)}{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1-u_2)}\right\}\left\{\frac{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1-u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2-2a)}{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2-2a)}\right.$$

Writing $-a$ for a in this equation, we have a second equation, which divided by the first gives

$$\Omega=\left\{\frac{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1-u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2-2a)}{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1-u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2+2a)}\right\}\times$$

$$\left\{\frac{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2+2a)}{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2-2a)}\right\}. \qquad (\mathrm{M2})$$

Again from the formula (N) above,

$$1 \pm k \operatorname{sn} A \operatorname{sn} B = \frac{\{1 \pm k \operatorname{sn}^2 \tfrac{1}{2}(A+B)\}\{1 \mp k \operatorname{sn}^2 \tfrac{1}{2}(A-B)\}}{1-k^2 \operatorname{sn}^2 \tfrac{1}{2}(A+B) \operatorname{sn}^2 \tfrac{1}{2}(A-B)}.$$

$$\therefore \quad [1-k^2 \operatorname{sn}^2 \tfrac{1}{2}(A+B) \operatorname{sn}^2 \tfrac{1}{2}(A-B)]^2 = \frac{\{1-k^2 \operatorname{sn}^4 \tfrac{1}{2}(A+B)\}\{1-k^2 sn^4 \tfrac{1}{2}(A-B)\}}{(1-k^2 \operatorname{sn}^2 A \operatorname{sn}^2 B)}.$$

In this equation writing in turn

$$A = u_1 + a,\ B = u_2 + a;$$
$$A = u_1 - a,\ B = u_2 - a;$$
$$A = u_1 + u_2 - a,\ B = a;$$
$$A = u_1 + u_2 + a,\ B = a;$$

we get expressions for the square of each of the factors of the numerator and denominator of the expression M2.

Hence we have after cancelling like factors in the numerator and denominator,

$$\Omega^2 = \frac{\{1-k^2 \operatorname{sn}^2 (u_1+a) \operatorname{sn}^2 (u_2+a)\}}{\{1-k^2 \operatorname{sn}^2 (u_1-a) \operatorname{sn}^2 (u_2-a)\}} \times \frac{\{1-k^2 \operatorname{sn}^2 a \operatorname{sn}^2 (u_1+u_2-a)\}}{\{1-k^2 \operatorname{sn}^2 a \operatorname{sn}^2 (u_1+u_2+a)\}}. \quad \ldots \quad (M2)$$

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SHORT NOTES.

Pascal Hexagon.

The concurrence of the Pascal lines of a hexagon inscribed in a conic, is proved here by the properties of the Homographic Function. *Reference*: The Note at the end of Salmon's *Conics* and my previous paper on "the Homographic Function as an Operator." (J. I. M. S, vol. VIII, p. 202),

I. *Introductory*: The properties connected with Pascal's Hexagon deal with cases in which three correspondences have their double points in involution, No simple necessary conditions for this can be given, but the following theorems give sufficient conditions and are applicable in many cases.

Theorem I. *If S is a given correspondence and S_1 a variable correspondence with given double points, then the double points of $S_1 S$ belong to a fixed involution.*

Let I represent the involution determined by the double points of S_1 and S. Then S_1 can be decomposed into the product of a variable involution I_1 containing its double points and the involution I (J.I.M.S. vol. VIII, p. 202)

Thus $S_1 S = I_1 I \cdot S$.

Again since I contains the double points of S, IS is a fixed involution I′ containing the double points of S—(*Ibid*)

Hence $$S_1 S = I_1 I_1.$$

The double points of $S_1 S$ are the double points of $I_1 I'$ *i.e.*, the common pair of I_1 and I′. Thus the double points S_1 S belong to the fixed involution I′.

Cor. In particular if Ω be a correspondence of period three, the double points of S, Ω S, Ω^2 S, belong to an involution.

Theorem II. *If I_1, I_2, I_3, be three involutions having a common pair, the double points of $I_1 S$, $I_2 S$, $I_3 S$, belong to an involution.*

Denote the common pair by i, and the double pair of S by s. Then by the theorem quoted above $I_1 I_2 I_3$ can each be split up into the product of the involution I containing i and s and three correspondences having p as their double pair. Hence putting $IS = S'$, the theorem is reduced to Theorem I.

Theorem III. *If the product of three correspondences is an involution in whichever order they are taken, then their double points belong to an involution.*

If $S_1S_2S_3$ and $S_2S_1S_3$ are involutions, the product of S_1, S_2, S_3 in any order is an involution. (This follows from the theorem that the parameter of the product of a number of correspondences depends not on the actual but only on the cyclical order of the product. (*vide*: the paper on one-one correspondence above referred to)

The condition that the correspondence $\frac{ax+b}{cx+d}$ may be an involution, is $a+d=0$.

Hence forming the products $S_1S_2S_3$, $S_2S_1S_3$ by the ordinary rules of multiplication of matrices, we have, for the condition that these may be involutions, the equations

$$(a_1a_2+b_1c_2)a_3+(a_1b_2+b_1d_2)c_3+(c_1a_2+d_1c_2)b_3+(c_1b_2+d_1d_2)d_3=0.$$

and $(a_1a_2+b_2c_1)a_3+(a_2b_1+b_2d_1)c_3+(c_2a_1+d_2c_1)b_3+(c_2b_1+d_1d_2)d_3=0,$

Subtracting we find

$$\begin{vmatrix} c_1 & d_1-a_1 & b_1 \\ c_2 & d_2-a_2 & b_2 \\ c_3 & d_3-a_3 & b_3 \end{vmatrix} =0,$$

which shews that the double points of S_1, S_2, S_3 belong to an involution.

II. *Properties of three Points*:

Three points pqr determine two periodic correspondences

$$\begin{vmatrix} pqr \\ qrp \end{vmatrix} \text{ and } \begin{vmatrix} prq \\ rqp \end{vmatrix}.$$

If Ω denote one of these, the other is Ω^2. The double points of Ω will be called the *cyclic centres* of pqr. (These must be distingnished from the so-called harmonic centres. The cyclic centres are defined by a purely descriptive process while the harmonic centres, as the name *does not* indicate, are defined by a metrical property.)

(1). From the definition of the cyclic centres as the double points of the correspondence $\begin{vmatrix} pqr \\ qrp \end{vmatrix}$, it follows that each of the involutions (pp,qr) (qq,rp), (rr,pq) contain the cyclic centres.

(2) *Geometrical Interpretation.*

Let PQR be points on the fundamental conic corresponding to pqr. Let the tangents at PQR meet the opposite sides in P′,Q′,R′, Then P′ represents the double pair of the involution (pp,qr). Hence (1) proves that P′Q′R′ are collinear. The intersections of P′Q′R′ with the conic correspond to the cyclic centres of pqr.

(3) Theorem IV. *If a correspondence carries* (pqr) *into* $(p'q'r')$, *it carries the cyclic centres of* (pqr) *into the cyclic centres of* $p'q'r'$.

Let $S=\begin{vmatrix} p & q & r \\ p' & q' & r' \end{vmatrix}$, $\Omega=\begin{vmatrix} q & r & p \\ p & q & r \end{vmatrix}$, $\Omega^1=\begin{vmatrix} p'q'r' \\ q'r'p' \end{vmatrix}$.

Then we see that $\Omega'S\Omega=S$, identically.

Hence if x be a double point of Ω

$$S(x)=\Omega' S\Omega(x)=\Omega' S(x)\ ;$$

so that S (x) is a double point of Ω'; which proves the theorem, since the double points of Ω and Ω' are the cyclic centres of (pqr) and $(p'q'r')$ respectively.

III. *Six Points*: Six points $pqr\ p'q'r'$ can be divided into two groups of three, in 10 ways. From each division we can get six correspondences by keeping one group in a fixed order and permuting the other. Hence we get sixty correspondences (a correspondence and its inverse not being regarded as distinct).

If the six points are represented as the vertices of a hexagon inscribed in the fundamental conic, the double points of these sixty correspondences are the intersections with the conics of the sixty Pascal lines.

(1) We first shew that the double points of the six correspondences which can be derived from the same division, belong three and three to two involutions.

Consider for example, the division $(pqr)\ (p'q'r')$.

$$\text{Let } S_1=\begin{vmatrix} p & q & r \\ p' & q' & r' \end{vmatrix},\ S_2=\begin{vmatrix} p & q & r \\ q' & r' & p' \end{vmatrix},\ S_3=\begin{vmatrix} p & q & r \\ r' & p' & q' \end{vmatrix}.$$

$$S_1'=\begin{vmatrix} p & q & r \\ p' & r' & q' \end{vmatrix},\ S_2'=\begin{vmatrix} p & q & r \\ r' & q' & p' \end{vmatrix},\ S_3'=\begin{vmatrix} p & q & r \\ q' & p' & r' \end{vmatrix}.$$

If Ω, Ω^2 be the two periodic correspondences which carry the group $(p'q'r')$ into itself, we have evidently

$$S_1=S_1,\ S_2=\Omega S_1,\ S_3=\Omega^2 S_1,$$
$$S_1'=S_1',\ S_2'=\Omega^2 S_1',\ S_3=\Omega S_1'.$$

Hence, by Th. I, Cor., the double points of $S_1S_2S_3$ belong to an involution I and the double points of S_1', S_2', S_3' belong to another involution I'.

(2) *Each of the involutions I, I' contains the double points of the other*

Let (t_1t_2),$(t_1't_2')$ be the cyclic centres of (pqr)and$(p'q'r')$ respectively. Every one of the six correspondences we are considering carries the group (t_1t_2) into the group $(t_1't_2')$ (Th. IV.) Hence if i, i'' represent the involutions. $(t_1t_1',\ t_2t_2')$ and $(t_1t_2',t_1't_2)$ respectively, the double points of every one of the six correspondences, must belong either to i or to i''.

Hence the double points of two at least of the three correspondences S_1, S_2, S_3 must belong to one of the involutions i, i'' say i and therefore the double points of the third also must belong to i (for we have proved that the double points of S_1,S_2,S_3 belong to an involution).

Hence I must be the same as i.

Similarly it is seen that I' is the same as i''.

Now the product of i and i'' is an involution, viz. $(t_1t_2, t_1't_2')$. Hence (Th. II, Cor. in the paper mentioned above) i and i'' each contain the double points of the other. This is seen otherwise also, for i, i'' are represented in the plane by two sides of the harmonic triangle of the quadrangle $t_1t_2t_1't_2'$.

Thus the involutions determined by the double points of S_1, S_2, S_3 and S_1', S_2', S_3' each contains the double points of the other.

Hence we see that *the sixty Pascal lines meet three by three in* 20 *points* (*Steiner's points which form ten pairs of conjugate points w.r.t the conic, each conjugate pair corresponding to a division of the six points.* (The latter part is mentioned but not proved in Salmon).

(3) *Kirkman's Points.*

Consider the correspondences

$$S_1 = \begin{vmatrix} p & qr' \\ q'p'r \end{vmatrix}, \quad S_2 = \begin{vmatrix} q & rp' \\ r'q'p \end{vmatrix}, \quad S_3 = \begin{vmatrix} rp & q' \\ p'r'q \end{vmatrix}.$$

We have

$$S_1S_2S_3(q') = r$$
$$S_1S_2S_3(r) = q';$$

hence $S_1S_2S_3$ is an involution.

Again $S_1S_3S_2(p') = r$

$$S_1S_3S_2(r) = p';$$

hence $S_1S_3S_2$ is an involution.

Thus every product of S_1, S_2 and S_3 is an involution and therefore the double points of S_1, S_2 and S_3 belong to an involution. (Theorem III.)

Thus the Pascal lines corresponding to S_1, S_2, S_3 meet in the point (Kirkman's point) which represents the double pair of this involution. In the present case S_2, S_3 are got from S_1 by permuting cyclically two points in the upper group of S_1 and the non-corresponding point in the lower group and also permuting cyclically the corresponding other points in the opposite sense.

There are three ways in which this double permutation can be made.

Thus three types of correspondences S_2S_3 can be obtained from S_1.

Hence every Pascal line contains 3 Kirkman points (and one Steiner's point.)

There are therefore $\frac{60 \times 3}{3}$ or 60 Kirkman points.

R. VYTHYNATHASWAMY.

Note on the Gamma Function.

Gauss' Formula.

For n a positive integer $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx = (n-1)!$

$$\Gamma(a) = \int_0^\infty e^{-x}\, x^{a-1}\, dx = \frac{1}{a(a+1)\ldots(a+n)} \int_0^\infty x^{a+n}\, e^{-x}\, dx, \text{ where } a>0$$

$$= \frac{1\cdot2\cdot3\ldots n}{a(a+1)\ldots(a+n)} \cdot \frac{\int_0^\infty x^{a+n}\, e^{-x}\, dx}{\int_0^\infty x^n\, e^{-x}\, dx} = \frac{1\cdot2\cdot3\ldots n}{a(a+1)\ldots(a+n)}\, n^a \cdot \frac{\int_0^\infty x^{a+n}\, e^{-nx}\, dx}{\int_0^\infty x^n\, e^{-nx}\, dx}$$

It can now be shown that the ratio of the two integrals tends to 1 as n tends to infinity.

Suppose a to be between r and $r+1$, where r is a positive integer, then

$$\int_1^\infty x^{r+n}\, e^{-nx}\, dx < \int_1^\infty x^{a+n}\, e^{-nx}\, dx < \int_1^\infty x^{r+1+n}\, e^{-nx} dx; \qquad \ldots \quad (1)$$

also $\int_0^1 x^{a+n} e^{-nx}\, dx$ lies between 0 and $\int_0^1 x^{a+n}\, dx$, or $\frac{1}{a+n+1}$,

whence we have

$$\int_0^1 x^{a+n}\, e^{-nx}\, dx = \frac{\theta}{a+n+1}, \text{ where } 0<\theta<1.$$

Hence from the inequalities (1) we can write

$$\frac{\frac{\theta}{a+n+1} - \frac{\theta'}{r+n+1} + \int_0^\infty x^{r+n}\, e^{-nx}\, dx}{\int_0^\infty x^n\, e^{-nx}\, dx} < \frac{\int_0^\infty x^{a+n}\, e^{-nx}\, dx}{\int_0^\infty x^n\, e^{-nx}\, dx}$$

$$< \frac{\frac{\theta}{a+n+1} - \frac{\theta''}{r+n+2} + \int_0^\infty x^{r+1+n}\, e^{-nx}\, dx}{\int_0^\infty x^n\, e^{-nx}\, dx}.$$

Consider $\dfrac{\int_0^\infty x^{r+n}\, e^{-nx}\, dx}{\int_0^\infty x^n\, e^{-nx}\, dx} \quad \dfrac{1}{n^r} \quad \dfrac{1\cdot2\cdot3\ldots(r+n)}{1\cdot2\cdot3\ldots n}$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{r}{n}\right);$$

since there are only a finite number of factors the limit as n tends to infinity of the ratio of the integrals tends to 1, and hence

$$\Gamma(a)=\lim_{n\to\infty}\frac{1\cdot2\cdot3\ldots n}{a(a+1)\ldots(a+n)}n^a.$$

2. *The relation between the Beta and Gamma Functions.*

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}=\lim_{\mu\to\infty}\frac{1\cdot2\cdot3\ldots\mu}{n(n+1)\ldots(n+\mu)}\cdot\frac{(m+n)\ldots(m+n+\mu)}{m(m+1)\ldots(m+\mu)}.$$

$$B(m,n)=\int_0^\infty\frac{x^{n-1}}{(1+x)^{m+n}}dx=\frac{m+n}{n}\int_0^\infty\frac{x^n\,dx}{(1+x)^{m+n+1}}$$

$$=\frac{(m+n)(m+n+1)\ldots(m+n+\mu)}{n(n+1)\quad\ldots\quad(n+\mu)}\int_0^\infty\frac{x^{n+\mu}}{(1+x)^{m+n+\mu}}\,d$$

$$=\frac{(m+n)(m+n+1)\ldots(m+n+\mu)}{n(n+1)\ldots(n+\mu)}\int_0^1 x^{n+\mu}(1-x)^{m-1}dx$$

$$=\frac{(m+n)(m+n+1)\ldots(m+n+\mu)}{n(n+1)\ldots(n+\mu)}\cdot\frac{1.2.3\ldots\mu}{m(m+1)\ldots(m+\mu)}\times\frac{\int_0^1 x^{n+\mu}(1-x)^{m-1}dx}{\int_0^1 x^{\mu}(1-x)^{m-1}dx}.$$

If n lies between r and $r+1$

$$\frac{\int_0^1 x^{r+\mu}(1-x)^{m-1}d}{\int_0^1 x^{\mu}(1-x)^{m-1}dx}>\frac{\int_0^1 x^{\mu+n}(1-x)^{m+1}dx}{\int_0^1 x^{\mu}(1-x)^{m-1}dx}$$

$$>\frac{\int_0^1 x^{\mu+r+1}(1-x)^{m-1}dx}{\int_0^1 x^{\mu}(1-x)^{m-1}dx};$$

and

$$\operatorname*{Lt}_{\mu\to\infty}\frac{\int_0^1 x^{r+\mu}(1-x)^{m-1}dx}{\int_0^1 x^{\mu}(1-x)^{m-1}dx}=\operatorname*{Lt}_{\mu\to\infty}\frac{1.2.3\ldots(\mu+r)}{m(m+1)\ldots(m+\mu+r)}$$

$$\times\frac{m\ m+1)\ldots(m+\mu)}{1.2.3\ldots\mu}=\operatorname*{Lt}_{\mu\to\infty}\frac{(\mu+1)\ldots(\mu+r)}{(m+\mu+1)\ldots(m+\mu+r)}=1.$$

A. C. L. Wilkinson.

A Diophantine Problem.

1. If $ax^2-y^2=1$ is a conic passing through the *rational* points (p, q), (p', q'), then we can write

$$\left.\begin{aligned} pa^{\frac{1}{2}}&=\cosh u,\ q=\sinh u\ ;\\ p'a^{\frac{1}{2}}&=\cosh v,\ q'=\sinh v.\end{aligned}\right\}$$

$$\therefore\quad \left.\begin{aligned} \sinh(u\pm v)&=qp'a^{\frac{1}{2}}\pm pq'a^{\frac{1}{2}},\\ \cosh(u\pm v)&=pp'a\pm qq'.\end{aligned}\right\}$$

Hence, the *rational* points $(qp'\pm pq')$, $(app'\pm qq')$ lie on the conjugate hyperbola $ax^2-y^2=-1$.

The *rational* points may, therefore, be readily constructed as the ends of chords conjugate to the radius to (p, q) or (p', q'); and the process can be continued indefinitely.

Ex. (1). $2x^2-y^2=1$.

Here $(1, 1)$ is an obvious *rational* point. Therefore, if (p, q) is any other such, the points $(q\pm p)$,$(2p\pm q)$ lie on the conjugate $y^2-2x^2=1$; and so on.

Ex. (2). $2x^2-3y^2=1$.

Here, if (p, q) is a point, $\{(5p\pm6q), (4p\pm5q)\}$ is also a point on it.

2. If $ax^2+y^2=1$, we may put $\left.\begin{aligned} pa^{\frac{1}{2}}&=\cos\theta,\ q=\sin\theta,\\ p'a^{\frac{1}{2}}&=\cos\phi,\ q'=\sin\phi,\end{aligned}\right\}0$;

and deduce $\left.\begin{aligned} \sin(\theta+\phi)&=(qp'+pq')a^{\frac{1}{2}},\\ \cos(\theta+\phi)&=(app'-qq').\end{aligned}\right\}$

Thus $\{(qp'+pq'),(a\,pp'-qq')\}$ is also a point on the ellipse $ax^2+y^2=1$

3. When the equation is $ax^2+by^2=1$, put

$$pa^{\frac{1}{2}}=\cos\theta,\ qb^{\frac{1}{2}}=\sin\theta;$$

and suppose

$$p'=\cos\phi,\ q'\ (ab)^{\frac{1}{2}}=\sin\phi,$$

so that (p', q')lies on

$$x^2+aby^2=1.$$

Then

$$\cos(\theta+\phi)=(pp'-bqq')a^{\frac{1}{2}},$$

$$\sin(\theta+\phi)=(qp'+pq'a)b^{\frac{1}{2}};$$

and the points $\{\pm(qp'+apq'), \pm(pp'-bqq')\}$ will also lie on $ax^2+by^2=1$, being the ends of a chord conjugate to the radius to (p, q).

A similar method applies to $ax^2-by^2=1$.

Example : $2x^2-3y^2=1$.

Take the auxiliary conic $x^2-6y^2=1$, which is satisfied by (5, 2).

Hence if (p, q) is a point on $2x^2-3y^2=1$,

$$\{(5p+6q), (5q-4p)\},$$

also lies on the same, and so on.

M. T. Naraniengar.

Note on Q. 831.

(ENQUIRER) :—Find by elementary methods the invariant relation expressing the condition that hexagons may be inscribed in the conic $S=0$ which are also circumscribed to the conic $S'=0$.

[*N.B.*—The result quoted in Salmon's *Conic Sections*, 6th Edition p. 343, *foot-note*, contains an extraneous factor corresponding to degenerate hexagons. The result is given in Halphen, *Functions Elliptiques*, Tome II.]

Solution by A. C. L. Wilkinson.

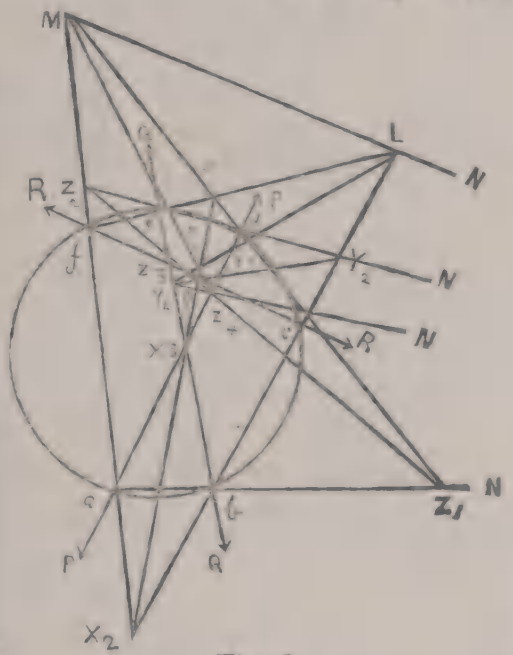


Fig. 1.

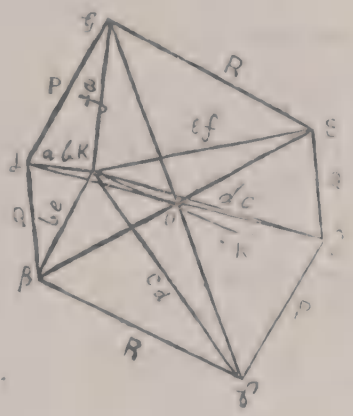


Fig. 2.

§ 1. Consider a jointed framework *abcdef* inscribed in a circle and kept in equilibrium by six equal and opposite forces P, Q, R acting along *ad*, *be*, *cf*. (fig. 1)

The conditions for equilibrium give

$$\frac{P}{\sin a \sin d}=\frac{T_{ab}}{\sin e \sin d}=\frac{T_{af}}{\sin c \sin d}=\frac{Q}{\sin b \sin e}=\frac{T_{bc}}{\sin e \sin f}=\text{etc.}$$

which are symmetrical and thus the frame—work is in equilibrium. Consider the force diagram for any point K (fig. 2) where $K\alpha$, $K\beta$ are parallel to *ab*, *bc*.....

Then *abcdef* is a funicular corresponding to K.

Now the hexagon $\alpha\beta\gamma\delta\varepsilon\zeta$ is symmetrical about O; thus considering K′, a point such that O is the middle point of KK′, then $K'\alpha$ is parallel to $K\delta$, etc.

Thus *defabc* is another funicular and the corresponding sides of these two funiculars intersect on a straight line parallel to KK′.

Hence *ab*, *de* ; *bc*, *ef* ; *cd*, *fa* intersect on a straight line NLM, the Pascal line of the hexagon *abcdef*, and this line is parallel to KOK′.

§ 2. Consider any funicular corresponding to O, the centre of the force diagram ; in general it consists of a hexagon inscribed in the triangle $X_3Y_3Z_3$, formed by the lines *ad*, *be*, *cf*, whose sides meet the corresponding sides of the funiculars *abcdef*, *defabc* in points lying on two straight lines parallel to LMN or KOK′.

Consider a straight line NY_4Z_4 parallel to $\alpha O\delta$ and complete the funicular $Z_4Y_4X_4Z_4'Y_4'X_4'$; since corresponding sides of this funicular and *abcdef* intersect on a straight line parallel to OK, that is, parallel to LMN, and L is the point of intersection of Z_4Y_4 and *ab*, therefore LMN is the line of intersection of the corresponding sides. Hence Y_4X_4 passes through L, X_4Z_4' through M and $Z_4'Y_4'$ through N. But $Z_4'Y_4'$ is parallel to $O\delta$ and hence to Y_4Z_4. Thus the funicular degenerates to the triangle $X_4Y_4Z_4$ inscribed in the triangle $X_3Y_3Z_3$, whose sides pass respectively through L, M, N.

§ 3. The pencil at K consisting of $K\beta$, $K\varepsilon$, KO and the line through K parallel to $\beta\varepsilon$ is harmonic ; therefore L*cb*, L*ef*, LMN, LX_4Y_4 form a harmonic pencil.

Produce *ab*, *cd*, *ef* to form the triangle $X_1Y_1Z_1$ and *bc*, *de*, *fa* to form the triangle $X_2Y_2Z_2$.

It follows from the harmonic pencils at L and M that $X_1X_4X_2$ are collinear.

§4. *The points* X_1, X_2, X_3, X_4 *are collinear.*

Consider the forces, P, Q acting at *a*, *b* ; their resultant passes through X_2 for they can be replaced by T_{ba}, T_{fa} and T_{ab}, T_{db} and thus X_3X_2 is the resultant.

Similarly by considering P, Q acting at *d*, *e*, X_3X_1 is the line of action of the resultant.

Again P at Z_4 can be replaced by forces along Z_4Y_4, X_4Z_4 represented by $O\alpha$, δO ; and Q at Y_4 can be replaced by forces along Y_4Z_4 and X_4Y_4 represented by αO, $O\beta$; whence X_4 is also a point on the resultant.

Similarly $Y_1Y_2Y_3Y_4$ and $Z_1Z_2Z_3Z_4$ are collinear.

Let X_3X_4, Y_3Y_4 meet in *g* ; then the moments about *g* for P, Q at *a*, *b* is zero and that about *g* for P, R at *d*, *c* is zero and the P forces are

equal and opposite, hence the sum of the moments about g for Q, R at b, c is zero.

Hence g also lies in Z_2Z_4.

Now $X_1X_2X_3$ is the Pascal line of *afe bcd*, $Y_1Y_2Y_3$ the Pascal line of *abc fed* and $Z_1Z_2Z_3$ of *afcdeb*. Thus g is the Steiner point of the Pascal lines X_1X_2, Y_1Y_2, Z_1Z_2.

The point g is $\begin{pmatrix} af, & cd, & be \\ bc, & fe, & ad \\ de, & ab, & cf \end{pmatrix}$ in the usual notation.

§5. The four triangles $X_1Y_1Z_1$, $X_2Y_2Z_2$, $X_3Y_3Z_3$, $X_4Y_4Z_4$ are copolar with a common pole at g.

The three axes of homology of any three triangles which are copolar with a common pole are known to be concurrent (a very simple proof by projecting the line joining the common pole to the intersection of two of the axes of homology to infinity). The axes of homology are seen to be

(12) the Pascal line LMN, (13)the Pascal line $\begin{pmatrix} ab, & cd, & ef \\ cf, & be, & ad \end{pmatrix}$

(14) LMN, (23) the Pascal line $\begin{pmatrix} ad, & cf, & be \\ bc, & de, & af \end{pmatrix}$, (24) LMN and the axis of mology (34).

Hence the 3 Pascal lines

$$\begin{pmatrix} af, & bc, & de \\ cd, & fe, & ab \end{pmatrix}, \quad \begin{pmatrix} af, & bc, & de \\ be, & ad, & cf \end{pmatrix}, \quad \begin{pmatrix} cd, & fe, & ab \\ be, & ad, & cf \end{pmatrix}$$

are concurrent and the axis of homology of the triangles $X_3Y_3Z_3$, $X_4Y_4Z_4$ also passes through this point of concurrence of the 3 Pascal lines.

Thus the axis of homology of the triangles $X_3Y_3Z_3$, $X_4Y_4Z_4$ meets LMN in a second g point, and we shall show that these two g points are conjugate with respect to the conic. (Staudt's Theorem).

§ 6. *The axis of homology of the triangles $X_3Y_3Z_3$, $X_4Y_4Z_4$ and the Pascal line LMN are harmonic conjugates with respect to the Pascal lines*

$$\begin{pmatrix} ab, & cd, & ef \\ cf, & be, & ad \end{pmatrix} \text{ and } \begin{pmatrix} ab, & cf, & be \\ bc, & de, & af \end{pmatrix}.$$

Project LMN to infinity so that we obtain fig. 3, where *ab*, *de* ; *bc ef* ; *cd*, *af* are respectively parallel. Then X_4, Y_4, Z_4 are the middle points of X_1X_2, Y_1Y_2, Z_1Z_2. Also X_2Z_2, X_4Z_4, X_1Z_1 are three parallel straight lines and hence they meet X_3Z_3 in three points U,V,W where V is the middle point of UW, and this establishes the above theorem. Also since X_4 is

the middle point of cf, we see that in fig. 1 the point where ad meets LMN is fourth harmonic with respect to Z_4 of a and d.

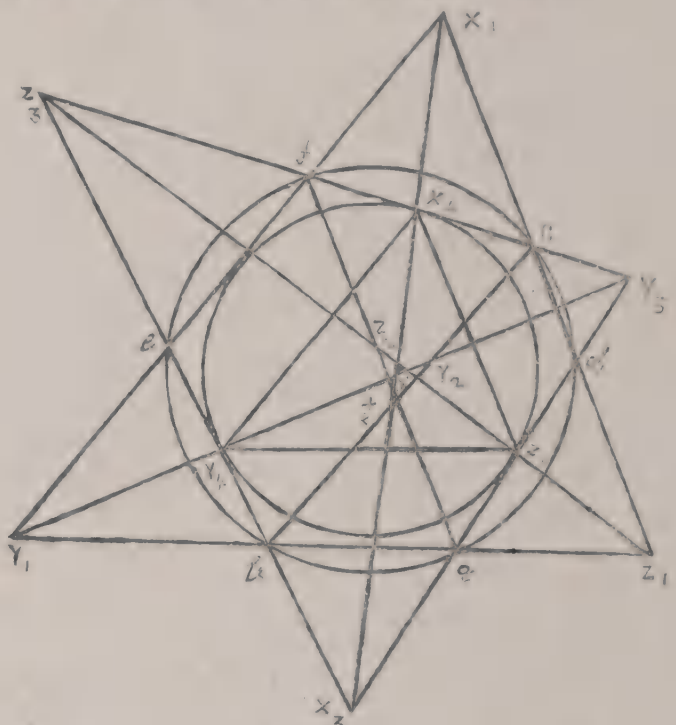


Fig. 3.

§ 7. Taking the conic of fig. 3 as a circle, since $ad=be=cf$ and Z_4, Y_4, X_4 are their middle points, we see that $X_4Y_4Z_4$ lie on a circle inscribed in the triangle $X_3Y_3Z_3$ and concentric with the circle $abcdef$.

Hence in fig. 1. the conics $abcdef$ and $X_4Y_4Z_4$ touching the sides of $X_3Y_3Z_3$ have double contact, the Pascal line LMN being the chord of contact; and more generally if we take any other hexagon $a'b'c'd'e'f'$ whose verties lie on the lines of action of the forces P, Q, R and whose corresponding sides $a'b'$, $d'e'$ pass through N, etc., the conics $a'b'c'd'e'f'$ all have double contact at the same points with the Pascal line LMN.

In fig. 3, g is the symmedian point of the triangle $X_4Y_4Z_4$ and its polar with respect to the circle $X_4Y_4Z_4$ is the axis of homology of $X_4Y_4Z_4$ and $X_3Y_3Z_3$; hence the polar of g with respect to $adcdef$ is parallel to the axis of homology of $X_4Y_4Z_4$, $X_3Y_3Z_3$ and thus passes through the point of intersection of LMN (the line at infinity) and the axis of homology of $X_4Y_4Z_4$, $X_3Y_3Z_3$; this was shown in § 5 to be the intersection of the three Pascal lines, *viz.* the reciprocal g point. We thus have a proof of Staudt's, Theorem that the two reciprocal g points are conjugate with respect to the fundamental conic.

A. C. L. Wilkinson.

Astronomical Notes.

A new minor planet.

Among the recent discoveries of minor planets is one of some importance. This planet (as yet unnamed) has a perihelion distance of 1·182 astronomical units, very little greater than that of Eros, so that on certain favourable occasions it may approach to within about seventeen million miles of the Earth, and will therefore be useful for determining the solar parallax. The eccentricity of the planet's orbit is very great so that its aphelion distance is 3·879. Its period is almost exactly four years. The planet is of the tenth magnitude and its diameter about four miles. The planet was discovered by Wolf who recently announced the discovery of a minor planet attended by a satellite which moved through eight degrees in a little over half an hour—a discovery of so startling a nature that one awaits its confirmation with some curiosity.

In connection with minor planets another point of interest occurs to me. It has been known for many years that at distances from the Sun at which the motion would be commensurate with that of Jupiter, there are no minor planets, any planets which may have originally existed in these positions having been diverted therefrom by the perturbing action of Jupiter. A recent investigation of the orbits of the many hundreds of planets now known fully confirms these facts, gaps being found at the ratios 2/1, 7/3, 5/2, 8/3, 3/1. It is however found that for values of the daily motion less than 500″ this avoidance of commensurability does not hold and planets are found with ratios 1/1, 4/3, 3/2.

The planets with ratio 1/1 are a special case, forming with Jupiter and the Sun an equilateral triangle and illustrating Lagrange's particular solution of the problem of three bodies; there are four such planets known, usually referred to as the Trojan group since they are named after the heroes of the Trojan war: Hector, Patroclus, &c.

R. J. Pocock.

SOLUTIONS.

Question 64.

(A. C. L. Wilkinson, M.A.):—Prove that no solution in positive integers exists of the equations:

$$x^2-y^2=y^2-z^2=z^2-u^2.$$

Solution by H. Br.

1. *Lemma.* The equations

$$\alpha^2=\beta^2+\delta^2=\gamma^2+9\delta^2 \quad \ldots \quad (1)$$

have no solution in integers, if $\delta \neq 0$.

Proof: (i) A commun factor f of any two of the quantities is a factor of the other two also. [This is easily seen except possibly in the case where $f=2$ and is a factor of β, γ. In this case

$$\left(\frac{\beta}{2}\right)^2-\left(\frac{\gamma}{2}\right)^2=2\delta^2.$$

$\therefore$ $\frac{\beta}{2}, \frac{\gamma}{2}$ are both odd or both even; hence the difference of their squares is a multiple of 4; *i.e.*, δ is even].

We may therefore without loss of generality take α, β, γ, δ prime to each other. As the sum of two odd squares is not a square δ is even.

(ii) By the ordinary rules, we must have

$$m^2+n^2=\alpha=M^2+N^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$2mn=\delta\ ;\ 2\,M\,N=3\,\delta$$

or

$$3\,m\,n=MN \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where m n are co-primes and n is even; and M, N are co-primes and N is even.

From (3) we get

either

$$\left.\begin{array}{ll} m=a\,b, & M=3\,a\,c \\ n=c\,d & N=b\,d \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

or

$$\left.\begin{array}{ll} m=a\,b & M=a\,c \\ n=c\,d & N=3\,b\,d \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

a, b, c, d being prime to each other, and d being even. We take the two cases separately.

(iii) From (2) and (4),

$$a^2\,b^2+c^2\,d^2=9\,a^2\,c^2+b^2\,d^2$$

or

$$b^2\,(d^2-a^2)=c^2\,(d^2-9a^2)$$

giving

$$\frac{d^2-a^2}{c^2}=\frac{d^2-9a^2}{b^2}=\text{an integer}=v.$$

$\therefore$ $8\,a^2$ is divisible by c^2-b^2 which must therefore be the common factor of a^2 and d^2.

As a, b, c, d are prime to each other and b and c are odd, v can only be $+1$ or -1.

If $v=1$, $d^2=a^2+c^2$, which is impossible as d is even and, a c are odd.

If $v=-1$, $9a^2=b^2+d^2$ which is also impossible as b, d are not both multiples of 3.

Hence the relations (4) do not hold.

(iv) From (2) and (5) we get

$$a^2\,b^2+c^2\,d^2=a^2\,c^2+9b^2\,d^2$$

or
$$\frac{a^2-d^2}{b^2}=\frac{a^2-9d^2}{c^2}=\pm 1.$$

The lower sign gives $9\,d^2=a^2+c^2$ which is impossible as a, c are not both multiples of 3.

The upper sign gives

$$a^2=b^2+d^2=c^2+9\,d^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

an equation of the same form as (1). Also $a=\frac{m}{b}<m^2+n^2<\alpha$.

(v) Proceeding in this manner we get a series of equations with diminishing values of α, and ultimately arrive at the equations

$$1=b_o^2+d_o^2=c_o^2+9\,d_o^2$$

which have no solution (if $d_o \neq 0$).

2. To prove that the system of equations

$$x^2-y^2=y^2-z^2=z^2-u^2$$

has no integral solution except the trivial one

$$x^2=y^2=z^2=u^2.$$

(i) Clearly x, y, z, u may be taken to be prime to each other without loss of generality $x^2=2\,y^2-z^2$

We may solve this by putting

$$x+z=X,\ x-z=Y,$$

or we may apply known results in the theory of quadratic forms, and noting that

$$x^2=2\,y^2-z^2=(2\,y+z)^2-2\,(y+z)^2$$

we may at once put down

$$\left.\begin{aligned} \pm\, x &= A^2-2\,B^2 \\ 2\,y+z &= A^2+2\,B^2 \\ y+z &= 2\,A\,B \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

where A is odd, and A and B are prime to each other.

Similarly

$$\left.\begin{aligned} \pm\, u &= C^2-2\,D^2 \\ y+2\,z &= C^2+2\,D^2 \\ y+z &= 2\,C\,D \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

C being odd, and C and D prime to each other.

Eliminating y and z

$$A^2+C^2+2B^2+2D^2=6AB=6CD \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

As AB=CD, we put

$$\left.\begin{aligned} A&=pr,\ B=qs \\ C&=ps,\ D=qr \end{aligned}\right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad (10)$$

Then p, q, r, s are prime to each other and p, r, s are odd

(ii) Substituting in (9)

$$p^2r^2+p^2s^2+2q^2s^2+2q^2r^2=6pqrs$$

or

$$(p^2+2q^2)(r^2+s^2)=6pqrs$$

Now r^2+s^2 has no factor common with r, s and p^2+2q^2 has no factors common with p, q. Also r^2+s^2 is even, but is not a multiple of 3

$\therefore$

$$\left.\begin{aligned} p^2+2q^2&=3rs \\ 2pq&=r^2+s^2 \end{aligned}\right\}$$

or

$$\left.\begin{aligned} 2(p+q)(2q+p)&=3(r+s)^2 \\ 2(p-q)(2q-p)&=3(r-s)^2 \end{aligned}\right\} \quad \dots \quad \dots \quad (11)$$

It follows that $r+s$ may be put in the form $2\alpha\beta$, where α, β are prime to each other, and $r-s$ may be put in the form $2\gamma\delta$ where γ, δ are prime to each other. As $p^2-q^2=-3r^2+3rs=$ a multiple of 3, and as p and q are not both multiples of 3, one of these quantities $p+q$, $p-q$ is a multiple of 3 but not the other. Two cases arise

(1)

$$\left.\begin{aligned} p+q&=6\alpha^2, \\ 2p+q&=\beta^2 \\ 2q-p&=3\gamma^2 \\ p-q&=2\delta^2. \end{aligned}\right\} \quad \dots \quad \dots \quad \dots \quad (12)$$

Eliminating p and q, we get

$$9\alpha^2=\beta^2+\delta^2,$$

or β, δ are multiples of 3, implying r, s multiples of 3, which is not the case as r, s are co-prime's. $\therefore$ (12) is inadmissible.

(2) In the second case

$$\left.\begin{aligned} p+q&=2\alpha^2 \\ 2q+p&=3\beta^2 \\ 2q-p&=\gamma^2 \\ p-q&=6\delta^2. \end{aligned}\right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad (13)$$

Eliminating pq, we get

$$\left.\begin{aligned} \alpha^2&=\beta^2+\delta^2 \\ &=\gamma^2+9\delta^2 \end{aligned}\right\}$$

This has no integral solutions unless $\delta=0$

If $\delta=0$, $r=s$

$$A=C,\ B=D$$

$$y=z,$$

and finally

$$x^2=y^2=z^2=u^2.$$

Question 378.

(J. C. SWAMINARAYAN) :—Having given that

$$\{\Sigma(b+c)x^2-2\Sigma(fyz)\}^2=4(x^2+y^2+z^2)[\Sigma(bc-f^2)x^2+2\Sigma(gh-af)yz],$$

prove that $\dfrac{bz^2+cy^2-2fyz}{y^2+z^2}=\dfrac{cx^2+az^2-2gzx}{z^2+x^2}=\dfrac{ay^2+bx^2-2hxy}{x^2+y^2}$

Remarks by H. Br.

The given expression may be written

$$[(a+b+c)(x^2+y^2+z^2)-(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)]^2$$
$$=4(x^2+y^2+z^2)(Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy).$$

Take (x, y, z) to be co-ordinates of a point, the axes being supposed to be rectangular. As $a+b+c$ is an invariant of the quadric $ax^2+\ldots$ it is clear that the above expression retains its form if the axes are rotated in any way. Consequently, the conclusions (whatever they be) to be drawn from the data will also retain their form unaltered if the axes are rotated in any way. But the conclusion that is asserted in the question does not satisfy the test.

To verify: Suppose the quadric referred to its principal axes, so that $f=g=h=0$.

$$[(b+c)x^2+(c+a)y^2+(a+b)z^2]^2=4(x^2+y^2+z^2)(bcx^2+cay^2+abz^2).$$

Expanding and simplyfying we get

$$\Sigma(b-c)^2x^4-2\Sigma(a-b)(c-a)y^2z^2=0.$$

Which implies

$$\sqrt{(b-c)}.x\pm\sqrt{(c-a)}.y\pm\sqrt{(a-b)}.z=0. \quad \ldots \quad \ldots \quad (1)$$

We are required to prove that

$$\frac{bz^2+cy^2}{y^2+z^2}=\frac{cx^2+az^2}{x^2+z^2}=\frac{ay^2+bx^2}{x^2+y^2}$$

i.e.

$$\left.\begin{aligned} z^2[(b-c)x^2+(c-a)y^2-(a-b)z^2]&=0\\ y^2[(b-c)x^2-(c-a)y^2+(a-b)z^2]&=0\\ x^2[-(b-c)x^2+(c-a)y^2+a-b)z^2]&=0\end{aligned}\right\} \quad \ldots \quad \ldots \quad (2)$$

The set of relations (2) is not a necessary consequence of (1).

Question 723.

(S. RAMANUJAN) :—If $[x]$ denote the greatest integer in x and n is a positive integer, show that

(i) $\left[\frac{n}{3}\right]+\left[\frac{n+2}{6}\right]+\left[\frac{n+4}{6}\right]=\left[\frac{n}{2}\right]+\left[\frac{n+3}{6}\right]$,

(ii) $[\frac{1}{2}+\sqrt{(n+\frac{1}{2})}]=[\frac{1}{2}+\sqrt{(n+\frac{1}{1})}]$,

(iii) $[\sqrt{n}+\sqrt{(n+1)}]=[\sqrt{(4n+2)}]$,

Solution by H. Br.

(i) Put $\phi(n)=\left[\frac{n}{3}\right]+\left[\frac{n+2}{6}\right]+\left[\frac{n+4}{6}\right]-\left[\frac{n}{2}\right]-\left[\frac{n+3}{6}\right]$ ()

Let $\left[\frac{n}{6}\right]=k$, so that $n=6k+m$, where $m<6$

$$\left[\frac{n}{3}\right]=\left[2k+\frac{m}{3}\right]=2k+\left[\frac{m}{3}\right], \text{ \&c.}$$

$\therefore$ $\phi(n)=\phi(m)$, where $m=0, 1, 2, 3, 4,$ or 5.

It is easily verified that $\phi(m)=0$ and therefore also $\phi(n)=0$.

(ii) Put $n=k^2+a$, where $a<2k+1$. Then $[\sqrt{n}\,]=k$.

Let
$$n+\tfrac{1}{2}=k^2+a+\tfrac{1}{2}=(k+\zeta)^2,$$
$$n+\tfrac{1}{4}=k^2+a+\tfrac{1}{4}=(k+\eta)^2.$$

We are required to prove $[\frac{1}{2}+\sqrt{n+\frac{1}{2}}]=[\frac{1}{2}+\sqrt{n+\frac{1}{4}}]$

or, in effect, $[\frac{1}{2}+\zeta]=[\frac{1}{2}+\eta]$.

If $a<k$, $0<\eta<\zeta<\frac{1}{2}$, and $[\frac{1}{2}+\zeta]=[\frac{1}{2}+\eta]=0$.

If $a\geq k$, $\frac{1}{2}<\eta<\zeta<1$, and $[\frac{1}{2}+\zeta]=[\frac{1}{2}+\eta]=1$.

(iii) As before let $n=k^2+a$, where $a<2k+1$.

Put
$$\left.\begin{aligned} n&=k^2+a=(k+\xi)^2,\\ n+\tfrac{1}{2}&=k^2+a+\tfrac{1}{2}=(k+\eta)^2,\\ n+1&=k^2+a+1=(k+\zeta)^2. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1)$$

We are required to prove

$$[\sqrt{n}+\sqrt{n+1}]=[\sqrt{4n+2}]$$

or in effect $[\xi+\zeta]=[2\eta]$.

If $a<k$, we have $0\leq\xi<\eta<\zeta<\frac{1}{2}$, and $[\xi+\zeta]=[2\eta]=0$

If $a>k$, we have $\frac{1}{2}<\xi<\eta<\zeta\leq 1$, and $[\xi+\zeta]=[2\eta]=1$.

If $a=k$, $\xi<\frac{1}{2}$, and η, ζ are $>\frac{1}{2}$, but <1. Thus $[2\eta]=1$. We have to show that $[\xi+\zeta]=1$. If $\xi=\frac{1}{2}-x$, and $\zeta=\frac{1}{2}+z$, we have to show that $z>x$.

Substituting for ξ and ζ in (1), we get

$$\tfrac{3}{4}=z\,(2k+1)+z^2,$$

and
$$-\tfrac{1}{4}=-x\,(2k+1)+x^2$$

$\therefore$ $(z-x)(2k+1)=\frac{1}{2}-(x^2+z^2)$.

As x and z are both less than $\frac{1}{2}$; x^2+z^2 is $<2(\frac{1}{2})^2<\frac{1}{2}$.

$\therefore$ $(z-x)(2k+1)$ is positive, or $z-x$ is positive.

Hence in every case $[\xi+\zeta]=[2\eta]$.

Question 836.

(Martyn M. Thomas, M.A.):—At a given instant, the same star is observed to be at the horizon of one place (lat. ϕ_1, long. L_1); on the prime vertical of another place (lat. ϕ_2, long. L_2) and at the zenith of a third place whose latitude is unknown, and long. L_3. Prove that

$$\cos(L_3-L_2)\tan\phi_1+\cos(L_3-L_1)\cot\phi_2=0,$$

and find the latitude of the third place.

Solution by R. J. Pocock, S Muthukrishnan, K. R. Rama Iyer, A. K. Anantanarayanan, R. D. Karve and C. Krishnamachary.

Let the hour angles of the star be h_1 h_2 in the first two cases; then

$$h_1 = L_s - L_1 ; \quad h_2 = L_s - L_2 ;$$

and if δ is the star's declination $\delta = \phi_s$.

Also
$$\tan \delta . \tan \phi_1 = -\cos h_1,$$
$$\tan \delta . \cot \phi_2 = \cos h_2.$$

Eliminating δ, h_1, h_2, we have

$$\cos (L_s - L_1) \cot \phi_1 + \cos (L_s - L_2) \tan \phi_2 = 0,$$

or
$$\cos (L_s - L_2) \tan \phi_1 + \cos (L_s - L_1) \cot \phi_2 = 0 ;$$

and
$$\tan \phi_s = \tan \delta = \cos (L_s - L_2) \tan \phi_2$$
$$= -\cos (L_s - L_1) \cot \phi_2.$$

Question 843.

(Lakshmishanker N. Bhatt):—From any point C tangents CD and CE are drawn to a circle, and from any fixed point O on the circle, chords OD, OE are drawn. If from any other variable point O′ on the circle, chords O′D, O′E are drawn cutting the former chords in P, Q and the tangents in R, S; then prove that (i) the line PQ always passes through C and (ii) the line RS always touches a conic.

Solution (1) *by Hemraj;* (2) *by V. M. Gaitonde, S. R. Ranganathan, and C. Bhaskaraiya, B.A. (Hon.)*

(1) Reciprocating with respect to O, we have :—

TP, TQ are tangents to a parabola at P, Q; and a variable tangent cuts them in R, S. Show that the lines through R and S parallel to TQ and TP respectively meet on PQ, and (ii) the locus of the point of intersection of the diagonals PS, RQ of the quadrilateral PQSR is a conic.

Take TP, TQ as axes. The parabola is

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1,$$

and a variable tangent is $\frac{x}{f} + \frac{y}{g} = 1$ with the condition $\frac{f}{a} + \frac{g}{b} = 1$, ... (1)

(i)
$$PQ \equiv \frac{x}{a} + \frac{y}{b} - 1 = 0$$

and the point of intersection of lines through R, S parallel to TQ, TP is evidently (f, g). It lies on PQ, if

$$\frac{f}{a}+\frac{g}{b}-1=0, \text{ which is (1)}$$

[For a geometrical proof, see Lachlan's *Modern Pure Geometry* Art 272].

(ii) $$PS\equiv\frac{x}{a}+\frac{y}{g}-1=0 \quad \ldots \quad \ldots \quad (2)$$

$$QR\equiv\frac{x}{f}+\frac{y}{b}-1=0. \quad \ldots \quad \ldots \quad (3)$$

Eliminating f, g between (1), (2), (3) we have the locus required, *viz.*
$x^2/a^2+xy/ab+y^2/b^2-2x/a-2y/b+1=0.$

It has double contact with the parabola at P and Q. Hence the theorem.

(2) (i) Let OO′ and DE meet in K. Then by the quadrangle construction for a polar, PQ is the polar of K. Also since K lies on DE the polar of C, the polar of K always passes through C.

Thus PQ always passes through C.

(ii) It can easily be shown that R and S generate homographic ranges on CE, CD respectively and that the ranges are not in perspective. Therefore RS touches a conic touching the fixed lines CD, CE.

Additional Solutions by S. V. Venkatachala Iyer and H. R. Kapadia.

Question 846.

(S. Malhari Rao, B.A):—Solve in positive integers $10^x+95^x=101^y$

Solution (1) *by H. Br.* (2) *by S. V. Venkatachala Aiyar and N. B. Mitra.*

(1) $$10^x+95^x\equiv 0 \pmod{101}$$

Multiply both sides by $(-17)^x$, and simplify. We get

$$32^x+1\equiv 0$$

or $$2^{5x}+1\equiv 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

As 2 is not a quadratic residue of 101, $2^{50}+1\equiv 0$.

Henceif x_o is the lowest positive value of x satisfying (1), 50 must be an odd multiple of 5 x_o.

$$\therefore \quad x_o=2 \text{ or } 10.$$

$x_o=2$, gives $2^{5x}\equiv 2^{10}\equiv 1024 \pm -1$; $x_o=10$ is obviously admissible.

The general value of x is any odd multiple of x_0 ,

or $$x=20k+10 ;$$

the value of y can be deduced when x is known.

(2) Forming the residues to the modulus 101 of the successive powers of 10^x, we have

$$10, -1, -10, 1, \ldots\ldots$$

(the same set recurring after this),

Similarly the residues of 95^x are

$$-6, 36, -14, -17, +1, \ldots\ldots$$

(the same set recurring).

In order that $10^x + 95^x$ may be divisible by 101, the sum of their residues must be zero. This is obviously the case when x has any of the values 10, 30, 50,..... ; for, when x has a value of the form $2+4n$, the residue in the first case is -1, and when x has a value of the form $5m$, the residue in the second case is $+1$;

Thus the values of x having been found, the corresponding values of y can be easily calculated.

Question 861.

(Communicated by Mr. J. H. GRACE, F. R. S.) :—A heterogeneous medium is such that the path of every ray through it is a plane curve. Find the index of refraction at any point.

Solution by Balak Ram.

If μ be the index of refraction at any point, the differential equations of a ray are

$$\frac{d}{ds}\left(\mu \frac{dx}{ds}\right) = \frac{\partial \mu}{\partial x},$$

and two similar equations. Denoting $\frac{d\theta}{ds}$ by θ', we have

$$\mu x'' = \frac{\partial \mu}{\partial x} - x'\mu',$$

and

$$\mu x''' = -2\mu' x'' - \frac{\partial \mu'}{\partial x} - x'\mu'', \text{ \&c.}$$

If the path is a plane curve, the torsion is zero. Simplifying the determinant factor of the value of the torsion, we get

$$\begin{vmatrix} x' & y' & z' \\ \frac{\partial \mu}{\partial x} & \frac{\partial \mu}{\partial y} & \frac{\partial \mu}{\partial z} \\ \frac{\partial \mu'}{\partial x} & \frac{\partial \mu'}{\partial y} & \frac{\partial \mu'}{\partial z} \end{vmatrix} = 0.$$

We must have relations of the form

$$\left.\begin{aligned} \frac{\partial \mu'}{\partial x} + u\frac{\partial \mu}{\partial x} &= vx', \\ \frac{\partial \mu'}{\partial y} + u\frac{\partial \mu}{\partial y} &= vy', \\ \frac{\partial \mu'}{\partial z} + u\frac{\partial \mu}{\partial z} &= vz'. \end{aligned}\right\}$$

The first equation is equivalent to

$$d\left(e^{\int u ds}\frac{\mu}{\partial\mu}\right)=ve^{\int u ds}dx.$$

This shows that the right hand side is a perfect differential.

Therefore $ve^{\int u ds}$ is independent of y and z. Similarly by considering the other equations, we find that it is independent of x, and is consequently a constant. Integrating, we get

$$\frac{\partial\mu}{\partial x}=v(x-x_0),$$

$$\frac{\partial\mu}{\partial y}=v(y-y_0),$$

$$\frac{\partial\mu}{\partial z}=v(z-z_0);$$

or $$d\mu=v\,[(x-x_0)dx+(y-y_0)dy+(z-z_0)dz].$$

Changing to polar co-ordinates, the new origin being taken to be at (x_0, y_0, z_0),

$$d\mu\equiv\frac{\partial\mu}{\partial r}dr+\frac{\partial\mu}{\partial\theta}d\theta+\frac{\partial\mu}{\partial\phi}d\phi=vrdr.$$

$$\therefore\qquad\frac{\partial\mu}{\partial\theta}=\frac{\partial\mu}{\partial\phi}=0,$$

or $$\mu=f(r)=F[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2],$$

where (x_0, y_0, z_0) is an arbitrary point and F is any function.

Question 874.

(S. Malhari Rao) :—Solve in primes

$$x+y=206,\ x+z=132,\ z+w=82.$$

Solution by H. Br. N. B. Mitra, C. Krishnamachary, K. B. Madhava, R. D. Karve, R. J. Pocock, G. Joshi and S. V. Venkatachala Aiyer.

As x, y, z, w are primes, and the value 2 is clearly inadmissible for any of them, each of them is of the form $6k\pm1$. From the given relations if follows that x and y are of the form $6k+1$, and z and w of the form $6k-1$. The values of w that make w and $82-w$ prime are

$$w=11,\ 23,\ 29,\ 41,\ 53,\ 59,\ 71,$$

$$z=82-w=71,\ 59,\ 53,\ 41,\ 29,\ 23,\ 11\ ;$$

of these $w=29, 53, 59$ give prime values of x and y, as follows :—

x	y	z	w
79,	127,	53,	29
103,	103,	29,	53
109,	97,	23,	59

Question 875.

(M. T. NARANIENGAR) :—P is any point on the circle of similitude of two circles A, B; and Q, R are the inverses of P with respect to A, B. Show that QR is bisected by the radical axis of A and B.

Solution (1) *by Hemraj, S. Muthukrishnan, and K. R. Rama Iyer,*

(2) *by Hemraj, M. K. Kewalramani, and S. V. Venkatachala Iyer.*

(1) Denote the centres of A and B by A and B, and the circle of similitude by C.

Since P is on C, $\frac{AP}{BP}=\frac{e}{e_1}$, where e, e_1 are the radii of A, B;

and since Q,R are the inverses of P *w.r.t.* A, B,

$$\therefore \quad \frac{AP}{e}=\frac{e}{AQ}, \text{ and } \frac{BP}{e_1}=\frac{e_1}{BR}$$

$$\therefore \quad \frac{e}{AQ}=\frac{e_1}{BR} \; i.e. \; \frac{AP}{BP}=\frac{AQ}{BR}$$

QR ∥ AB *i.e.* the radical axis of A, B is ⊥ QR. ... (1)

Evidently the circle PQR is orthogonal to A, B and C which is coaxal with A, B,

Thus the centre of the circle PQR is on the radical axis and on the right bisector of QR as well. Hence by (1) the radical axis bisects at right angles QR unless it is parallel to QR, *i.e.*, unless the centre of the circle PQR is at infinity, *i.e.*, unless P coincides with either centre of similitude; even in this case QR is bisected by the radical axis.

(2) Taking the line of centres and the radical axis of A, B as axes we have the equation of A, B in the form

$$x^2+y^2-2kx+\delta=0 \qquad x^2+y^2-2k_1x+\delta=0.$$

Hence the equation of C is

$$x^2+y^2-2\frac{kk_1+\delta}{k+k_1}x+\delta=0. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Now P (x_1, y_1) is any point on (1),

$$\therefore \quad x_1^2+y_1^2-2\frac{kk_1+\delta}{k+k_1}x_1+\delta=0; \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

and Q is the point of intersection of the polar of P *w.r.t.* A and the line through the centre of A perpendicular to the polar; *i.e.* the intersection of the lines

$$(x_1-k)x+y_1y-kx_1+\delta=0, \; y_1x-(x_1-k)\,y-y_1k=0.$$

Hence Q is $\left(\frac{k_1-k}{k+k_1-2x_1}x_1, \frac{k+k_1}{k+k_1-2x_1}y_1\right)$.

Similarly R is $\left(\frac{k-k_1}{k+k_1-2x_1}x_1, \frac{k+k_1}{k+k_1-2x_1}y_1\right)$.

Hence the esult follows at once.

Question 878.

(S. Narayana Aiyar, M.A.):—Show that

$$(1-k^2x^2)^m(1-x^2)^n = \sum_{r=0}^{\infty} \frac{(-x^2)^r}{\Gamma(r+1)} \cdot \frac{\Gamma(n+1)}{\Gamma(n-r+1)} F(-m,-r,n-r+1,k^2)$$

when F denotes the hypergeometric function.

Hence show that

$$\int (1-k^2x^2)^{-\frac{1}{2}}(1-x^2)^{-\frac{1}{2}}dx = \sum_{r=0}^{\infty} \frac{(-)^r x^{2r+1}}{\Gamma(r+1)(2r+1)} \cdot \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-r)} \times F(\tfrac{1}{2},-r,\tfrac{1}{2}-r,k^2).$$

Solution by K. B. Madhava, S. R. Ranganathan, C. Krishnamachary and Sudanand.

If neither m nor n is a positive integer,

$$(1-k^2x^2)^m = \sum_{r=0}^{\infty} \frac{\Gamma(m+1)}{\Gamma(m-r+1)\Gamma(r+1)} k^{2r}(-x^2)^r$$

and

$$(1-x^2)^n = \sum_{r=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-r+1)\Gamma(r+1)} (-x^2)^r.$$

Hence the coefficient of $(-x^2)^r$ in the product $(1-x^2)^n(1-k^2x^2)^m$ is

$$\sum_{s=0}^{s=r} \frac{\Gamma(n+1)\Gamma(m+1)}{\Gamma(n-r+1)\Gamma(r+1)} \cdot k^{2s} \frac{\Gamma(n-r+1)\Gamma(r+1)}{\Gamma(m-s+1)\Gamma(r-s+1)\Gamma(s+1)\Gamma(n-r+s+1)}$$

$$= \frac{\Gamma(n+1)}{\Gamma(n-r+1)\Gamma(r+1)} \sum_{s=0}^{s=r} \frac{\Gamma(m+1)}{\Gamma(m-s+1)} \cdot \frac{\Gamma(r+1)}{\Gamma(r-s+1)} \cdot \frac{\Gamma(n-r+1)}{\Gamma(n-r+s+1)} \cdot \frac{k^{2s}}{\Gamma(s+1)}$$

$$= \frac{\Gamma(n+1)}{\Gamma(n-r+1)\Gamma(r+1)} F(-m,-r,n-r+1,k^2),$$

where the hypergeometric series contains only a finite, *viz.* $r+1$ terms.

Since neither m nor n is a positive integer, the product can be summed for r fron 0 to ∞.

Hence

$$(1-k^2x^2)^m(1-x^2)^n = \sum_{r=0}^{\infty} \frac{(-x^2)^r}{\Gamma(r+1)} \cdot \frac{\Gamma(n+1)}{\Gamma(n-r+1)} F(-m,-r,n-r+1,k^2).$$

The second result follows by a simple integration, and putting $m=-\frac{1}{2}$ and $n=-\frac{1}{2}$.

Question 889.

(Hemraj).—Two parabolas have each double contact with a given ellipse of semi-axes a, b, and their axes parallel to its axes respectively PP′, QQ′, are the chords of contact, and ρ_1, ρ_2 the radii of curvature of the parabolas; and ρ, ρ', those of the ellipse at P, Q. If the distances of the chords of contact from the centre of the ellipse are in the ratio of its axes, prove that $\rho\,\rho_2 = \rho'\rho_1$. In particular if the distances are equal, then

$$a^2\,\rho\,\rho_2 = b^2\,\rho'\,\rho_1.$$

But if Q coincides with P, then

$$\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

Solution (1) *by K. Appukuttan Erady and K. B. Madhava;*

(2) *by C. Krishnamachary, S. V. Venkatachala Aiyer and Sadanand.*

(1) Let the equations to the chords P P′ and Q Q′ be $x-f=0$ and $y-g=0$ respectively.

Then the equations to the two parabolas are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1-\frac{1}{a^2}(x-f)^2=0, \quad \ldots \quad \ldots \quad \text{(i)}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1-\frac{1}{b^2}(y-g)^2=0. \quad \ldots \quad \ldots \quad \text{(ii)}$$

For the ellipse at P,

$$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2},$$

$$\frac{y}{b^2}\cdot\frac{dy}{dx} = -\frac{x}{a^2},$$

$$\frac{y}{b^2}\cdot\frac{d^2y}{dx^2} = -\frac{1}{a^2}-\frac{1}{b^2}\left(\frac{dy}{dx}\right)^2.$$

For the parabola (i) at P,

$$\frac{y^2}{b^2} = 1-\frac{2fx}{a^2}+\frac{f^2}{a^2},$$

$$\frac{y}{b^2}\cdot\frac{dy}{dx} = -\frac{f}{a^2},$$

$$\frac{y}{b^2}\cdot\frac{d^2y}{dx^2} = -\frac{1}{a^2}-\frac{1}{b^2}\left(\frac{dy}{dx}\right)^2.$$

Since the parabola (i) touches the ellipse at P, their curvatures are in the ratio of their $\frac{d^2y}{dx^2}$ at P.

Hence in the notation of the Question

$$\frac{\rho}{\rho_1} = \frac{\frac{1}{b^2}\left(\frac{dy}{dx}\right)^2}{\frac{1}{a^2}+\frac{1}{b^2}\left(\frac{dy}{dx}\right)^2} = \frac{\frac{f^2b^2}{a^4y^2}}{\frac{1}{a^2}+\frac{f^2b^2}{a^4y^2}} = \frac{\frac{b^2}{a^2}}{\frac{y^2}{b^2}+\frac{f^2}{a^2}} = \frac{f^2}{a^2}.$$

And it follows from symmetry that at R,

$$\frac{\rho'}{\rho_2}=\frac{g^2}{b^2}.$$

Hence if $\frac{f}{a}=\frac{g}{b}$,

$$\frac{\rho}{\rho_1}=\frac{\rho'}{\rho_2},$$

or $$\rho\rho_2=\rho'\rho_1;$$

and if $f=g$, $$\frac{a^2\rho}{\rho_1}=\frac{b^2\rho'}{\rho_2},$$

or $$a^2\rho\rho_2=b^2\rho'\rho_1.$$

Also if P coincides with Q,

$$\rho'=\rho \text{ and } \frac{f^2}{a^2}+\frac{g^2}{b^2}=1,$$

and consequently $$\frac{\rho}{\rho_1}+\frac{\rho}{\rho_2}=1,$$

i.e., $$\frac{1}{\rho}=\frac{1}{\rho_1}+\frac{1}{\rho_2}.$$

(2) Let the eqnation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

and P, Q, be the points whose excentric angles are α, β.

If P′ be α', the eqn. to the parabola with its axis parallel to the x—axis is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)-k\left\{\frac{x}{a}\cos\frac{\alpha+\alpha'}{2}+\frac{y}{b}\sin\frac{\alpha+\alpha'}{2}-\cos\frac{\alpha-\alpha'}{2}\right\}=0.$$

Since the axis of this parabola is parallel to the x—axis,

$$k=1, \text{ and } \alpha+\alpha'=0;$$

and the equation to the parabola is

$$\frac{y^2}{b^2}+\frac{2x}{a}\cos\alpha-(1+\cos^2\alpha)=0. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Similarly the equation to the parabola with Q Q′ as chord of contact and axis parallel to the y—axis is

$$\frac{x^2}{a^2}+\frac{2y}{b}\sin\alpha-(1+\sin^2\alpha)=0. \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Now $$\rho=\frac{(a^2\sin^2\alpha+b^2\cos^2\alpha)^{\frac{3}{2}}}{ab}, \quad \rho'=\frac{(a^2\sin^2\beta+b^2\cos^2\beta)^{\frac{3}{2}}}{ab}.$$

Also we can easily find ρ_1, ρ_2, the radii of curvature of the parabolas (1) and (2) at P, Q, and these are given by

$$\rho_1=\frac{(a^2\sin^2\alpha+b^2\cos^2\alpha)^{\frac{3}{2}}}{ab\cos^3\alpha},\ \rho_2=\frac{(a^2\sin^2\beta+b^2\cos^2\beta)^{\frac{3}{2}}}{ab\sin^3\beta}.$$

The perpendiculars from the origin on the chords of contact PP' QQ' are $a\cos\alpha$, $b\sin\beta$.

If these are in the ratio of the axes

$$\cos\alpha=\sin\beta\ ;\ \rho\rho_2=\rho'\rho_1.$$

If these are equal, $a\cos\alpha=b\sin\beta$,

$$a^2\rho\rho_2=b^2\rho'\rho_1.$$

But if P, Q coincide, $\alpha=\beta$, and the relation

$$\frac{1}{\rho}=\frac{1}{\rho_1}+\frac{1}{\rho_2}$$

easily follows.

Question 898.

(HEMRAJ):—A parabola has double contact with a given ellipse of semi-axes a, b and its axis parallel to either axis of the latter. Find the equation of the locus of the centre of curvature of the parabola corresponding to the point of contact; and show that

$a^2b^2\rho_1^2\rho_2^2=\{(\rho_2-\rho_1)a^2+\rho_1b^2\}^3$, or $\{(\rho_2-\rho_1)b^2+\rho_1a^2\}^3$

where ρ_1, ρ_2 are the radii of curvature of the curves at a point of contact.

Solution by K. B. Madhava and C. Krishnamachary.

We notice that

$$y^2=4A(x+C) \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

is a parabola with its axis parallel to the major axis of

$$1=\frac{x^2}{a^2}+\frac{y^2}{b^2} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

and having double contact with it, if

$$4AC\,b^2=b^4+4A^2a^2. \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

The points of contact of (1) and (2) are

$$-\frac{2Aa^2}{b^2},\ \pm\frac{1}{b}\sqrt{b^4-4A^2a^2}$$

where $\rho_1^2\,(\text{Ellipse})=\frac{1}{a^2b^2}(a^2-x^2e^2)=\frac{a^4}{b^{14}}(b^4-4A^2a^2e^2)^3$... (4)

and $\rho_2^2\,(\text{Parabola})=\frac{4}{A}(A+x+C)^3=\frac{1}{16A^4b^6}(b^4-4A^2a^2e^2)^3$; (5)

whence the required relation is got by eliminating A between (4) and (5) *i.e.* between (4), and $\rho_1 b^4 = 4 A^2 a^2 \rho_2$.

$$\therefore \qquad \rho_1^2 = \frac{a^4 b^{12}}{b^{14} \rho_2^3} (\rho_2 - \rho_1)^3,$$

i.e. $$a^2 b^2 \rho_1^2 \rho_2^3 = \{ a^2 (\rho_2 - \rho_1) + b^2 \rho_1 \}^3.$$

Similarly the other result (when the axis of the parabola is parallel to the minor axis of the ellipse) can be got by interchanging a and b.

To obtain the locus of the centre of curvature, we eliminate x, y, A and C from (1), (3) and

$$\xi = 2A + 2C + 3x \text{ and } \eta = -\frac{y^3}{4A^2}.$$

Question 904.

(R. J. Pocock, B.A., B. Sc.) :—Show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{2n}\phi}{\left(1-\frac{1}{2}\sin^2\phi\right)^{n+\frac{1}{2}}} d\phi = 2^{2n-2} \frac{\Gamma^2\left(\frac{2n+1}{4}\right)}{\Gamma\left(\frac{2n+1}{2}\right)}.$$

Solution (1) *by F. H. V. Gulasekharam and by the Proposer*; (2) *by A. C. L. Wilkinson and K. R. Rama Iyer.*

(1) Consider $$I = \int_0^\infty x^{2n} e^{-x^4} dx = \frac{1}{4}\Gamma\left(\frac{2n+1}{4}\right);$$

by substituting $x^4 = y$

whence $$2^{2n} I^2 = 2^{2n} \int_0^\infty \int_0^\infty x^{2n} y^{2n} e^{-x^4 - y^4} dx dy$$

$$= \int_0^\infty \int_0^{\frac{\pi}{2}} r^{4n} \sin^{2n} 2\theta \, e^{-r^4 (1 - \frac{1}{2}\sin^2 2\theta)} r \, dr \, d\theta$$

$$= \frac{1}{4} \int_0^\infty \int_0^{\pi} z^{2n} \sin^{2n}\phi \, e^{-z^2 (1 - \frac{1}{2}\sin^2\phi)} dz \, d\phi$$

$$= \frac{1}{2} \int_0^\infty \int_0^{\frac{\pi}{2}} \frac{\sin^{2n}\phi}{\left(1-\frac{1}{2}\sin^2\phi\right)^{\frac{2n+1}{2}}} t^{2n} e^{-t^2} dt \, d\phi$$

$$= \frac{1}{4} \Gamma\left(n+\frac{1}{2}\right) \int_0^{\frac{\pi}{2}} \frac{\sin^{2n}\phi}{\left(1-\frac{1}{2}\sin^2\phi\right)^{\frac{2n+1}{2}}} d\phi$$

This solution is suggested by a similar analysis, due to Forsyth given in G. H. Darwin's paper "On the Mechanical Conditions of a Swarm of Metorites" (Phil. Trans. Series A, Vol. 180 (1880) pp. 1—69 ; also Collected Papers, Vol. IV, pp. 362—431).

Putting $n=0$, we get $\Gamma\left(\frac{1}{4}\right)=2\pi^{\frac{1}{4}}\,\mathrm{F}^{\frac{1}{2}}$,

where F is the complete elliptic integral of the first kind, modulus $\frac{1}{\sqrt{2}}$. From $\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)=\pi\sqrt{2}$, and Legendres formula connecting the complete elliptic integrals of the first and second kind, which for modulus $\frac{1}{\sqrt{2}}$ becomes $2\,\mathrm{FE}-\mathrm{F}^2=\frac{1}{2}\pi$, we obtain

$$\Gamma\left(\frac{3}{4}\right)\cdot=\pi^{\frac{1}{4}}(2\,\mathrm{E}-\mathrm{F})^{\frac{1}{2}};$$

and we thus obtain the expressions for

$$\int_0^\infty e^{-x^4}dx,\quad \int_0^\infty x^2\,e^{-x^4}\,dx$$

given in the paper quoted above ; and more generally

$$\int_0^\infty x^{2n}\,e^{-x^4}\,dx=\frac{1}{4}\Gamma\left(\frac{2n+1}{4}\right)$$

can be reduced to one or other of these two expressions.

(2) Write $\sin x=\dfrac{\frac{1}{2}\sin^2\phi}{1-\frac{1}{2}\sin^2\phi}$, $\cos x=\dfrac{\cos\phi}{1-\frac{1}{2}\sin^2\phi}$; so that

$$\cos x\,dx=\frac{\sin\phi\cos\phi\,d\phi}{\left(1-\frac{1}{2}\sin^2\phi\right)^2},\quad dx=\sqrt{2\sin x}\,\frac{d\phi}{\left(1-\frac{1}{2}\sin^2\phi\right)^{\frac{1}{2}}}$$

$$\therefore\qquad \mathrm{I}=2^{n-\frac{1}{2}}\int_0^{\frac{1}{2}\pi}\sin x^{\,n-\frac{1}{2}}\,dx=2^{n-1}\,\frac{\Gamma^2\left(\frac{2n+1}{4}\right)}{\Gamma\left(\frac{2n+1}{2}\right)}.$$

Question 905.

(T. P. Bhaskara Sastri) :—Investigate the effect of precession and nutation on the position angle of a double star.

[See : Ball's *Spherical Astro* : p. 185, Ex. 16.]

Solution by C. Krishnamachary and the Proposer.

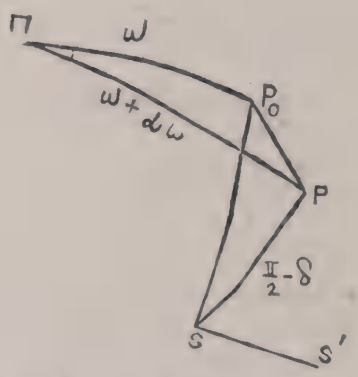
Let π be the pole of the ecliptic.

P_o, the celestial pole at epoch T_o.

P, the same at a subsequent epoch T.

S, the principal star and S′ his companion

Fig. 1.



also let the position angle at the first epoch be p_o, and p that at the second epoch.

$\angle$ P π P_o = the arc between the poles of the circles πP_o and π P $= k$.

From the triangle P_o S P,

$$\cos\delta \sin(p-p_o) = -\sin P P_o \sin \angle P P_o S.$$

But $\angle P P_o S = \angle P P_o \pi - \angle S P_o \pi = \angle P P_o \pi - (90^\circ + \alpha_o)$

$$= -(90^\circ - \angle P P_o \pi) - \alpha_o.$$

$$\therefore \cos\delta \sin(p-p_o) = \sin P P_o \sin P P_o \pi \sin \alpha_o + \sin P P_o \cos P P_o \pi \cos \alpha_o$$

From the triangle $P P_o \pi$,

$\sin P P_o \sin P P_o \pi = \sin(\omega + d\omega) \sin k$ and $\sin P P_o \cos P P_o \pi = \cos(\omega + d\omega) \sin \omega - \sin(\omega + d\omega) \cos \omega \cos k$.

Making these substitutions, we get

$$(p-p_o)\cos\delta = [k \sin \omega \sin \alpha - d\omega \cos \alpha],$$

neglecting small quantities of order higher than the first.

i.e. $$p - p_o = [n \sin \alpha - d\omega \cos \alpha] \sec \delta.$$

QUESTIONS FOR SOLUTION.

903. *Correction:* Insert at the end of the question the following:—"where the summation is extended only to odd values of n".

960. (V. Ramaswami Aiyar, M.A.):—Given a triangle ABC, if a point P and the feet of the perpendiculars from it on the sides and altitudes of ABC all lie on a Cartesian Oval, show the negative pedal of the oval with respect to P touches the nine points circle. Show that the pedals of the inscribed and escribed circles of a triangle, taken with respect to the orthocentre, are cases of such ovals, illustrating the property.

961. (H. Br.):—Show that

$$\left.\begin{aligned} x^2 &= y^2 + z^2 \\ t^2 &= y^2 + 4z^2 \end{aligned}\right\}$$

have no solutions in non-zero integers

962. (A. C. L. Wilkinson):—If two circles are such that hexagons can be inscribed in the one which are circumscribed to the other, prove that the sum of the products of the diagonals taken two at a time is constant and equal to

$$\frac{4\,R^2 r^2 \{ (R^2 - c^2)^2 + 4\,R^2 r^2 \}}{(R^2 - c^2)^3},$$

where R, r, are the radii of the two circles and c the distance between their centres.

963. (A. C. L. Wilkinson):—If two circles are such that quadrilaterals can be inscribed in the one which are circumscribed to the other, the ratio of the areas of these quadrilaterals is constant.

964. (H. Br.):—If $f(x, y)$ is a rational algebraic function of x and y, such that $f(-x, y) = (-)^{k+1} f(x, y)$,

(i) Show that

$$\int_0^\infty f(\sin 2\theta, \cos 2\theta) \frac{d\theta}{\theta^{k+1}}$$

$$= \frac{(-1)^k}{k!} \int_0^\infty f(\sin 2\theta, \cos 2\theta) \tan\theta . \frac{d^k}{d\theta^k} \cot\theta . \frac{d\theta}{\theta}$$

$$= \frac{(-1)^k}{k!} \int_0^{\frac{\pi}{2}} f(\sin 2\theta, \cos 2\theta). \frac{d^k}{d\theta^k} \cot \theta . d\theta$$

and

$$\text{(ii)} \int_0^{\infty} f(\sin\theta, \cos\theta) \frac{d\theta}{\theta^{k+1}} = \frac{(-1)^k}{k!} \int_0^{\frac{\pi}{2}} f(\sin\theta, \cos\theta) \frac{d^k \lambda}{d\theta^k} d\theta$$

where $\lambda = \cot\theta$ or $\operatorname{cosec}\theta$, according as

$$f(-x,-y) = f(x, y) \text{ or } -f(x, y).$$

[The integrals are assumed to be finite.]

965. (HEMRAJ) :—ABC is a triangle. A point J is taken on AI_1 produced, such that I_1J subtends at B or C an angle α where $\cot\alpha = 3\cot\frac{1}{2}C$ or $3\cot\frac{1}{2}B$. Show that there exists a circle through A and J which cuts AB and AC in D and E such that $BD = CE = BC$.

[Suggested by Q. 959.]

965. (A. C. L. WILKINSON) :—Prove that

$$\frac{1-2(2+k^2)s_1^2+5k^2s_1^4}{2s_1c_1d_1} + \frac{1-2(1+k^2)s_2^2+3k^2s_2^4}{2s_2c_2d_2} - \frac{1-2(1+2k^2)s_3^2+5k^2s_3^4}{2s_3c_3d_3}$$
$$= -3k^3 s_1 s_2 s_3$$

where $s_1 = \operatorname{sn}\left(\frac{K}{3}\right), s_2 = \operatorname{sn}\frac{iK'}{3}, s_3 = \operatorname{sn}\frac{K+iK'}{3}$ etc.

966 (A. C. L. WILKINSON) :—Prove that

$$E - 3E\left(\frac{K}{3}\right) = -\frac{k^2}{k'}\operatorname{cn}^2\frac{K}{3}.$$

967. (M. K. KEWALRAMANI, M.A) :—Prove that

$$1^{-5}+2^{-5}-4^{-5}-5^{-5}+7^{-5}+8^{-5}-10^{-5}-11^{-5}+\ldots\ldots = \frac{35\pi^5}{8748\sqrt{3}}.$$

968 (V. ANANTARAMAN) :—Sum the series

$$\frac{1}{3}-\frac{1}{7}+\frac{1}{11}-\frac{1}{15}+\frac{1}{19}-+\ldots\ldots$$

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PROGRESS REPORT.

The first Conference of our society, having proved a success to some considerable extent, there has been a strong feeling amongst our members to hold a second Conference either in Bombay or Poona. The Committee have carefully considered this matter, and now feel great pleasure in announcing to the General Body that *a Conference of our society will be held in Bombay about the beginning of January* 1919. The details will be published hereafter in the October issue of our Journal, but members desirous of attending the conference and of reading papers at the conference are requested to communicate their intentions to the undersigned, with short synopses of papers to be read.

2. The following gentlemen have been elected members of our Society

(1) *Lt. Robert Francis Mudie* I.C.S., I.A.R.O.—School of Musketry, Satara.

(2) *Mr. T. S. Krishna Murti*, B.A., L.T.—Assistant Inspector of schools, Madura.

(3) *Mr. P. T. Arumuhaswami*,—Mathematics Teacher, St. Michael's College, Mungalasthan, Batticaloa (Ceylon).

3. (i) The following book has been purchased for the Library—*Synopsis of Pure Mathematics*—G.S. Carr, M.A., published by Francis Hodgson, London 1886, £ 1—14*s*—0.

(ii) The following books have been received as presents to the Library :—

1. *Cours de Mecanique* (professe' a l'ecole' Polytechnique) Vol. III,—by L. Lecorm, published by Gauthier Villars, Paris, 1918.
2. *Tables Numeriqes Usuelles*—L. Zoretti, publhished by Gauthier Villars, Paris, 1917.
3. *Astrographic Catalogue*—Measures of Rectangular Coordinates and Diameters of 6 3 4 3 6 stars, images &c.—by R. J. Pocock, B.A., B.Sc.C. F.R.A.S. published by H. H. the Nizam's Government, 1918. Rs. 12 or 16*s* net.
4. *Madras University Calendar*—Parts I and II for 1918, and volume of Examination papers 1917.
5. *Bombay University Calendar*—Parts I and II for 1918—1919.

POONA,)
5th Aug. 1918.)

D. D. KAPADIA,
Honorary Jt. Secretary.

Properties of Equations in the Algebra of the Nodal Cubic.

By R. Vythynathaswamy.

[The present paper purposes to give a formal development of some general function—properties in the Algebra of the Nodal Cubic, on the basis of the results already established from geometrical considerations in my previous paper on the subject.

The following is a general summary of the paper.

A. Discussion of Functional Forms.

(1) The general functional form transformable into a successive product *i.e.* a product in which the multiplier at each stage is a single element instead of being itself a product. The successive product taken as the universal standard form.

(2) Classification of forms into pure and mixed forms with respect to a variable x. Reduction of the general mixed form to three canonical types. Consequent reduction of the general equation

$$F_1(x) = F_2(x)$$

to two fundamental types,

$$P_x = a \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$P_x = a\,(\mathrm{or}\ Q_x) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where P_x, Q_x are pure functions.

(3) Classification of forms into two types, the simplex and the complex. A form $f(x)$ is complex if $f(n)$ is indeterminate and simplex if $f(n)=n$ or n^1 (n,n^1 being the nodal elements) A complex form transformable into a simplex one by the rules of the Algebra.

The conditions for the simplexity of the form

$$x^{p_1} n^{p_2} n^{p_3} \ldots p_2 p_1)$$

are (i) the secondary indices $p_2\ p_3 \ldots p_n$ are alternately even and odd or odd and even and (ii) $p_1\ p_2$ are both odd or both even.

B. Function—Properties.

(1) The apparent degree of a function when in a complex form higher than its true degree; *e. g.* $x^2 x^2$, a complex form of the function x^4.

The degree of a simplex form, in other words the true degree of any function is never a multiple of three.

(2) Classification of functions into odd and even functions. $f(x)$ is odd if $f(n)=n^1$ and even, if $f(n)=n$. The odd-ness or even-ness of $f(x)$ directly determinable if the form $f(x)$ is simplex. Whether the form $f(x)$ is simplex or complex. $f(x)$ is odd or even according as the characteristic is positive or negative.

(3) Classification of equations in complex and simplex types according as they are or are not satisfied by the nodal points. The degree of a simplex equation of the first type never a multiple of 3, the degree of a simplex equation of the second type always a multiple of 3.

(4) The problem of solving the general equation reduced to the root-extraction of a given correspondence (which is merely the problem of root extraction of a number and the geometrical problem of drawing a ray making with three rays a pencil of given cross-ratio.

The n roots of an equation of the n th degree form a periodic group of the homographic correspondence of period n whose fixed points are the nodal points.]

§ 1. *Reciprocal points*

Two points on the Nodal Cubic connected by the relation $ab^2=a^2s$ will be said to be reciprocal. This relation is of the third degree in a and there are therefore three a's for any b. One of these three roots is however b itself. Thus there are two points reciprocal to a given point b (not including b itself). We may verify that these two pointe are $\omega_1(\omega_2 b)$ and $\omega_2(\omega_1 b)$ where ω_1, ω_2, ω_3 are the points of inflexion. It may be noted that $\omega_1(\omega_2 x)=\omega_2(\omega_3 x)=\omega_3(\omega_1 x)$. This follows from the theorem $\omega_1(\omega_2\omega_3 x)=\omega_2 x$, proved in para (*b*), 9 '*Nodal Cubic*,' J. I. M. S. August 17.

For

$$b(\omega_1(\omega_2 b))^2=b(\omega_1(\omega_2 b^2))=b^2(\omega_1(\omega_2 b)).$$

Thus the two points reciprocal to b are $\omega_1(\omega_2 b)$ and $\omega_2(\omega_1 b)$.

We notice also that these two points are themselves reciprocal.

For

$$\begin{aligned}\omega_1(\omega_2(\omega_1(\omega_2 b))) &= \omega_1(b(\omega_1\omega_2))\\ &= \omega_1(\omega_3 b)\\ &= \omega_2(\omega_1 b)\end{aligned}$$

which proves $\omega_1(\omega_2 b)$ is reciprocal to $\omega_2(\omega_1 b)$.

Note (1). If Ω denote the homographic correspondence having the nodal points for its fixed points and carrying ω_1 to ω_2 (so that $\Omega(x) = \omega_1(\omega_2 x)$), then Ω carries ω_2 to ω_3 and ω_3 to ω_1. Ω is therefore periodic, of period three. The two points reciprocal to b are then seen to be $\Omega(b)$ and $\Omega^2(b)$.

Note (2). The only self-reciprocal points are the nodal points.

Note (3). If x_1, x_2, x_3 form a mutually reciprocal set,

$$x_1^2 = x_2 x_3,\ x_2^2 = x_3 x_1,\ x_3^2 = x_1 x_2.$$

For
$$\begin{aligned} x_2 x_3 &= [\omega_1(\omega_2 x_1)][\omega_1(\omega_3 x_1)] \\ &= \omega_1(\omega_1 x_1^2) = x_1^2. \end{aligned}$$

Note (4). The points $\omega_1, \omega_2, \omega_3$ form a mutually reciprocal set, so also do the points, $-\omega_1, -\omega_2, -\omega_2$, which are the points of contact of tangents from the inflexional points.

Note (5). $x = \Omega(x^2).\Omega^2(x)$ identically.

For $\Omega(x^2) = \omega_1(\omega_2 x^2)$ and $\Omega^2(x) = \omega_1(\omega_3 x)$

$\therefore\ \Omega(x^2).\Omega^2 x = \omega_1^2(\omega_2\omega_3.(x.x^2))$ (by the theorem $ab.cd = ac.bd$).

$$= \omega_1(\omega_1(x.x^2) = x$$

Note (6). $[\Omega(x)]^{2n} = \Omega(x^2)$

From *Note* (5) $\Omega(x^2) = x.\Omega^2(x)$

$$= [\Omega(x)]^2 (\textit{Note}\ (3)).$$

By successive applications of this, the general result follows.

§ 2. The general equation of the third degree may be reduced to one of the two forms

$$x^2 = bx \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$
$$ax^2 = bx \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

b is evidently one root of equation (1). Further if x is one nodal point n, x^2 is the other nodal point n^1 and bx is also the other nodal point n^1, so that the two other roots of equation (1) are n, n^1 the two nodal points. Equation (1) is thus degenerate and need not be considered further.

Consider now equation (2). It can be written in the form

$$x^2 = S(x), \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

S denoting a homographic correspondence with the nodal points as fixed points.

The three roots of equation (2) *form a mutually reciprocal set.*

To prove this we shew that if x_1 is one root $\Omega(x_1)$ is another. We have

$$\begin{aligned}[\Omega(x_1)]^2 &= \Omega(x_1^2) \{ \text{I, Note (6)} \} \\ &= \Omega\, \mathrm{S}(x_1) \{ \therefore x_1 \text{ is a root} \} \\ &= \mathrm{S}\, \Omega(x_1);\end{aligned}$$

so that $\Omega(x_1)$ is another root. Thus the three roots of equation (2) are of the form x_1, $\Omega(x_1)$, $\Omega^2(x_1)$.

The equation (2) thus associates each homographic correspondence with nodal points as fixed points, with a reciprocal set. The identical correspondence is associated with the inflexional triad, and the fundamental involution with the triad $-\omega_1, -\omega_2, -\omega_3$.

§ 3. *Solution of the cubic equation.*

Let Ω_x denote a homographic correspondence, with the nodal points as fixed points, carrying x to a point of inflexion. Thus Ω_x is not definite, but may be any one of the correspondences, $\begin{vmatrix} x \\ \omega_1 \end{vmatrix}$, $\begin{vmatrix} x & x \\ \omega_2 & \omega_3 \end{vmatrix}$. But it is easy to see that in any case, Ω_{x^3} is definite and unique and that if Ω_x^3 is known, x is determined as one of the set of mutually reciprocal points.

$$(\Omega_x^3)^{-3}(\omega_1),\ \Omega(\Omega_x^3)^{-1}(\omega_1)\ \Omega^2(\Omega_x^3)^{-3}(\omega_1).$$

We have the important relation

$$\Omega_x^3 = \begin{vmatrix} x \\ x^2 \end{vmatrix}$$

For
$$\begin{aligned}\Omega_x^3(x) &= x(\omega(x(\omega(x(\omega x))))) \\ &= x^2.\end{aligned}$$

This property leads immediately to the solution of the equation

$$x^2 = \mathrm{S}(x).$$

For since $x^2 = \Omega_x^3(x)$ identically and Ω_x and S have the same fixed points, it follows that

$$\Omega_x^3 = \mathrm{S} = \text{a known correspondence}$$

whence x is determined as shewn above and the equation is solved.

§ 4. *Properties of successive products.*

Consider the successive product.

$$x_n(x_{n-1}(x_{n-2}(\ldots\ldots\ldots\ldots x_2\, x_1)))$$

We shall find it convenient to write this

$$(x_n\, x_{n-1}\ldots\ldots\ldots x_2\, x_1)$$

Let us call $x_1\ x_3\ x_5$...the odd elements and $x_2\ x_4$...the even elements of the product. The following results readily follow from (7) para (6) 'Nodal cubic'.

(1) All the even elements and the first odd element can be arranged among themselves in any order.

(2) All the odd elements except the first can be arranged among themselves in any order.

Theorem I. If $P_m = (p_m\ p_{m-1} \cdots p_2\ p_1)$ and $X_m = (p_1\ p_m\ p_{m-1} \cdots p_2 x)$, then $P_m X_m = x$ or $p_1^2 x$, according as m is odd or even.

For $P_m X_m = (p_m\ p_{m-1} \cdots p_2\ p_1) \times (p_1\ p_m\ p_{m-1} \cdots p_2\ x)$

$= (p_1\ p_{m-1}\ p_{m-2} \cdots p_2\ p_1) \times (p_m\ p_m\ p_{m-1} \cdots p_2\ x)$

$= (p_1\ p_{m-1} \cdots p_2\ p_1) \times (p_{m-1}\ p_{m-2} \cdots p_2\ x)$

$= (p_{m-1}\ p_{m-1}\ p_{m-2} \cdots p_2\ p_1) \times p_1\ p_{m-2} \cdots p_2\ x)$

$= (p_{m-2}\ p_{m-3} \cdots p_2 p_1) \times (p_1 p_{m-2} \cdots p_2 x)$

$= P_{m-2} \times X_{m-2}$

$= P_3 \times X_3$ if m is odd

$= (p_3 p_2 p_1) \times (p_1 p_3 p_2 x)$

$= p_2 \times p_3 x = x$;

and $= P_2 \times X_2$ if m is even

$= p_2 p_1 \times (p_1 p_2 x)$

$= p_1^2 x$.

Theorem II. The product of two successive products can be expressed as a successive product.

Let the two successive products be denoted by P_m, Q_n where,

$$P_m = (p_m p_{m-1} \cdots p_2 p_1),\ Q_n = (q_n q_{n-1} \cdots q_2 q_1).$$

We may suppose m to be odd. If it were even, we may write p_1 as $p_1 p_1^2$ and thus convert P_m into an odd product. We shew that

$$P_m . Q_n = R \text{ where } R = (p_1 p_m \cdots p_2 q_n \cdots q_1).$$

We have

$$P_m R = P_m (p_1 p_m \cdots p_2 q_n q_{n-1} \cdots q_1)$$

$$= P_m (p_1 p_m \cdots p_2 Q_n)$$

$$= P_m (P_m Q_n) \text{ (Th I)}$$

$$= Q_n$$

$\therefore$ $P_m Q_n = R =$ a successive product.

Corollary. It follows by repeated application of them thereto that the most general function of a variable x, can be expressed as a successive product the elements of which are powers of x and constants.

Def. A successive product the elements of which are powers of x is said to be *a pure product.* If constants also appear, the product is said to be *mixed.*

Theorem III The most general mixed product can be reduced to one of the forms, $a\,\mathrm{P}_x$, $a\,(b\mathrm{P}_x)$, $x^{2m}\,(c\mathrm{P}_x)$, where P_x is a pure product. consider first the mixed product C_x, where

$$\mathrm{C}_x = x_1(c_1(x_2(c_2 \ldots x_n(c_n \mathrm{X}))))$$

$x_1, x_2 \ldots \mathrm{X}$ being pure functions of x and c's constants.

This product $= (x_1 \mathrm{X} x_2 c_2 x_3 c_3 \ldots x_n c_n')$

where $c_n' = c_1 c_n$

$= (x_1 \mathrm{X} x_2 c_2 c_n' c_3 x_1 c_4 \ldots x_{n-1} c_{n-1} x_n')$

where $x_n' = x_n x_3$

$= (x_1 \mathrm{X} x_2 b x_4 c_4 \ldots x_{n-1} c_{n-1} x_n')$

where $c_2(c_n'(c_3 d)) = bd$

$= (x_1 \mathrm{X} \mathrm{C}_x')$

where C_x' is of the same form as C_x but with fewer elements

Repeating this process of reduction, we would have finally

$$\mathrm{C}_x = (y_1 y_2 y_3 \ldots y_n b y)$$

or

$$\mathrm{C}_x = (y_1 y_2 y_3 \ldots y_n a b y)$$

where the y's are pure functions of x.

In the second case, C_x at once reduces to the form $a(b\mathrm{P}_x)$ where P_x is a pure product. In the first case, C_x would reduce to one of the two forms $b\mathrm{P}_x$, or $y_1\,(b\mathrm{P}_x)$.

The latter form can be reduced to $x^{2m}(b\mathrm{P}_x)$ by interchanging y_1 and an adequate power of x occuring in P_x expressed as a successive product.

Consider now the general mixed product. The constants occuring in it can be all shifted to the left-hand side of the product, so that if x_r, x_s be the first non-constant odd and even elements, no odd element to the right of x_r and no even element to the right of x_s are constants. The product then reduces to one of the forms C_x, $a\,\mathrm{C}_x$, $a(b\mathrm{C}_x)$. Taking each of the three possible forms of C_x, it is easy to verify that the product must reduce to one of the three forms mentioned in the theorem.

§ 5. Let the correspondence $\begin{vmatrix} nn^1 x \\ nn^1 x^2 \end{vmatrix}$ or as we shall write it for convenience $\begin{vmatrix} x \\ x^2 \end{vmatrix}$, be denoted by S_x. Then $\begin{vmatrix} x^2 \\ x^4 \end{vmatrix}$ would be denoted by $\mathrm{S}x^2$

Now $S_x{}^2 S_{x^2}(y) = x(x^2(x(x^2x(x^2(x^4y)))$
$= x(x^2(x(x^4y)))$
$= x(y(x(x^4x^2)))$
$= y.$

$\therefore \quad S_x{}^2 S_{x^2} = 1$ So that $S_{x^2} = S_x{}^{-2}$.

$$\therefore \left| \begin{matrix} x \\ x^2 \end{matrix}^n \right| = \left| \begin{matrix} x \\ x^3 \end{matrix} \right| \cdot \left| \begin{matrix} x^2 \\ x^4 \end{matrix} \right| \cdots \left| \begin{matrix} x^2 \\ x^2 \end{matrix}^n \right|^{n-1}$$

$$= S_x . S_x{}^3 \ldots S_x{}^{2^{n-1}}$$

$$= S_x{}^{\frac{1-(-2)^n}{3}}$$

$$\therefore \left| \begin{matrix} x^2 \\ x^2 \end{matrix}^{m_2}{}^{m_1} \right| = S_x{}^{\frac{(-2)^{m_1} - (-2)^{m_2}}{3}}$$

Writing the successive product $x^{2^{mn}} (x^{2^{mn-1}} \ldots (x^{2^{m_2}} x^{2^{m_1}})))$
as $x^{2^{(m_n\, m_{n-1} \cdots m_1)}}$, we may shew more generally

$$\frac{1+(-)^n \{ (-2)^{m_1} + (-2)^{m_2} - (-2)^{m_3} + - \}}{3}$$

that $$L \equiv \left| \begin{matrix} x \\ x^{2^{(m_n\, m_{n-1} m_1)}} \end{matrix} \right| = S_x$$

Proof. Assume the theorem to be true for products of n elements. We shall prove it true for products of $(n+1)$ elements. We have

$$\left| \begin{matrix} x \\ x^{2^{(m_{n+1},\, m_n \cdots m_1)}} \end{matrix} \right| = L \times \left| \begin{matrix} x^{2^{(m_n\, m_{n-1} \cdots m_1)}} \\ x^{2^{(m_{n+1}\, m_n \cdots m_1)}} \end{matrix} \right|$$

$$= L \times \left| \begin{matrix} x^{2^{m_{n+1}}} \\ x_2{}^{(m_n\, m_{n-1}' \cdots m_2\, m_1')} \end{matrix} \right|$$

where $m_r' = m_r + 1$

(This follows because $\left| \begin{matrix} P \\ QP \end{matrix} \right| = \left| \begin{matrix} Q \\ P^2 \end{matrix} \right|$)

$$= L \times \left| \begin{matrix} x^{2^{m_{n+1}}} & 1 \\ x & \end{matrix} \right| \left| \begin{matrix} x \\ x^{2^{(m_n' \cdots m_1')}} \end{matrix} \right|$$

$$= L \times S_x{}^{\frac{(-2)^{m_{n+1}} - 1}{3}} \times S_x{}^{\frac{1+(-)^{n+1} 2 \times (-2)_1{}^{m_1} + (-2)^{m_2} - + \ldots)}{3}}$$

$$= S_x{}^{\frac{1 + (-)^{n+1} \left((-2)^{m_1} + (-2)^{m_2} - + \ldots \right)}{3}}$$

The theorem is thus true for products of $(n+1)$ elements.

Since the theorem has been proved true for simple powers, the induction shews that it is universally true.

Def. The expression

$$\frac{1+(-)^m\left((1-2)^{p_1}+(-2)^{p_2}-(-2)^{p_3}+(-2)^{p_4}\ldots\right)}{3}$$

may be called *the characteristic* of the successive product $x^{2(p_m p_{m-1}\ldots p_1)}$. Evidently the characteristic is an intger $+^{ve}$ or $-^{ve}$.

§ 6. A function of x which $=n$ when $x=n'$ will be said to be an even function. A function which $=n'$, when $x=n$ will be saia to be an odd function.

A function which does not become indeterminate when x is made equal to a nodal point will be said to be simplex.

(a) A pure successive product is an even or odd function according as its characteristic is $-^{ve}$ or $+^{ve}$.

To prove this we observe that if $f(x)$ is an odd function $S_x[f(x)]$ is also an odd function; for

$$\begin{aligned} S_x(f(x)) &= x(x^2 f(x)) \\ &= n(n'n') \text{ (putting } x=n) \\ &= n' \end{aligned}$$

Hence $S_x(f(x))$ is an odd function. In the same way it is seen that if $f(x)$ is an even function $S_x^{-1}(f(x))$ is also an even function.

Now if P_x be the successive product and k its characteristic.

We have $\begin{vmatrix} x \\ P_x \end{vmatrix} = S_x{}^k$ or $P_x = S_x{}^k(x) = S_x{}^{k-1}(x^2)$

since x is an even function and x^2 an odd function it follows at once that P_x is even or odd according as k is $-^{ve}$ or $+^{ve}$

(b) To find the necessary and sufficient condition that a pure product may be simplex.

Evidently the necessary and sufficient condition is that, in the course of the product an odd function should nowhere be multiplied by an even function or an even function by an odd function.

Let the product be $x^{2(p_m p_{m-1}\ldots p_2 p_1)}$

Firstly x^{2p_1} and x^{2p_2} should be both odd or both even functions.

Which shews that p_1 and p_2 should be both odd or both even secondly it should be even or odd according as $x^{2(p_{r-1}p_{r-2}\cdots p_2p_1)}$ is even or odd, for all values of r.

The characteristics are therefore of the same sign.

Therefore $(-)^r(-2)^{p_r}$ is of the same sign as $((-2)^{p_1}+(-2)^{p_2}\ldots \pm(-2)^{p-r1})$

This shews that $p_2p_3p_4\ldots$ are alternately odd and even.

Cor. The degree of a pure simplex product cannot be a multiple of three.

The degree of the product in (b) above is

$$(2)^{p_1}+(2)^{p_2}+2^{(p_3)}\ldots$$

Now $2^{p_1}=\pm1 \pmod 3$

$2^{p_2}=\pm1 \pmod 3$

$2^{p_3}=\mp1 \pmod 3$

$2^{p_4}=\pm1 \pmod 3$

The upper signs being taken if p_1 is even, the lower if p_1 is odd. (These equations follow by applying the conditions that the product may be simplex).

Adding up, we see that the degree can never be a multiple of three.

§ 7. *The General Equation.*

We are now in a position to discuss the general equation

$$F_1(x)=F_2(x) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

F_1, and F_2 can be reduced to successive products [IV] and further reduced to one of the three forms aP_x, $a(bP_x)$, x^{2m} (cP_x) where P_x is a pure product. It is easy to verify that the equation (1) can in all cases be reduced to one of the two types

$$P_x=a \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$P_x=a(bQ_x) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where P_x and Q_x are pure products.

An equation is said to be simplex, when the nodal points are not roots of it. Evidently the equation of the first type is simplex if P_x is simplex and the equation of the second type is simplex if P_x and Q_x are not both odd or both even. We prove from this the interesting result:— *The degree of a simplex equation of the first type is never a multiple of 3; the degree of a simplex equation of the second type is always a multiple of 3.*

The first part follows from § 6, Cor.

To prove the second part, we refer back to § 6, Cor.

The sign of the characteristic of the product $x^a (P_m P_{m-1} \cdots P_1)$ is $(-)^{p_1+m}$. The degree of the product is of the form $3q+(-)^{m-1}$ if p_1 is even and of the form $(3q+(-)^m)$ if p_1 is odd; i.e. of the form $(3q-(-)^{p_1+m})$ whether p_1 is odd or even. Hence the degree of a simplex product is of the form $(3q \mp 1)$ according as its characteristic is $+^{ve}$ or $-^{ve}$. Applying this result to the equation of the second type, we see that the degree of the one of the functions P_x, Q_x is of the form $3q+1$, and of the other, of the form $3q-1$, so that the degree of the equation itself is a multiple of 3.

Solution of the General Equation.

(i) *The equation of the first type* $P_x = b$.

Let k be the characteristic of P_x, so that

$$S_x^k = \left| \begin{matrix} x \\ P_x \end{matrix} \right| = \left| \begin{matrix} x \\ b \end{matrix} \right| = \left| \begin{matrix} x \\ \omega_1 \end{matrix} \right| \left| \begin{matrix} b \\ \omega_1 \end{matrix} \right| -1$$

Now $S_x = \left| \begin{matrix} x \\ \omega_1 \end{matrix} \right|^3$

$$\therefore \quad \left| \begin{matrix} x \\ \omega_1 \end{matrix} \right|^{3k-1} \left| \begin{matrix} b \\ \omega_1 \end{matrix} \right|^{-1} = \text{a known correspondence}$$

This determines x. We see incidentally that the degree of the equation supposed simplex is $3k-1$.

(ii) *The equation of the second type* $P_x = a(bQ_x) = S(Q_x)$

If k, k' are the characteristics of P_x, Q_x, we have

$$S = \left| \begin{matrix} Q_x \\ P_x \end{matrix} \right| = \left| \begin{matrix} x \\ P_x \end{matrix} \right| \left| \begin{matrix} x \\ Q_x \end{matrix} \right|^{-1}$$

$$= S_x^{k-k'} \qquad = \Omega_x^{3(k-k')}$$

which determines Ω_x^3 and $\therefore x$.

We see that the degree of the equation is the numerical value of $3(k-k')$.

§ 8. The last paragraph shews that the solution of any equation is reduced to the extraction of roots and to the geometrical problem of drawing a ray making with three other rays, a pencil of given cross ratio. It also shews that if a is one root of an equation of the n^{th}

degree, all the roots are given by the series, α, $\Omega_n(\alpha)$, $\Omega^2{}_n(\alpha)$..... $\Omega_n{}^{n-1}(\alpha)$, Ω_n being a primitive n^{th} root of the identical correspondence with the nodal points as fixed points.. This is merely another aspect of the fact that in this algebra every equation involves only one parameter. This somewhat strange result is shewn by the theory of linear systems of points to be necessitated by the symmetrical relation between the nodal points and the linear system formed by the roots of all equations of a particular degree. Now there are only two ways in which *two* points can be related in absolute symmetry to a linear system, viz. (1) the system may be a complete one and the two points may form the indeterminate pair, in which case the order of the system must be three; This happens in the case of the nodal points and the complete system of collinear triads; or (2) the system must be a focal system of the second degree involving a single parameter with the two points as foci. The linear system of roots of equations of a particular degree belongs to this latter category.

SHORT NOTES.

Some Fundamental Expansions.*

§ 1. Consider $\frac{\cos \lambda x}{\sin \rho x}$ as the limit, when m becomes infinite, of

$$f(m) = i \frac{\left(1+\frac{i\lambda x}{m}\right)^m + \left(1-\frac{i\lambda x}{m}\right)^m}{\left(1+\frac{i\rho x}{m}\right)^m - \left(1-\frac{i\rho x}{m}\right)^m},$$

and suppose that m reaches infinity through odd positive integral values. Decomposing $f(m)$ into partial fractions, the factors of the denominator are obtained by solving

$$\left(1+\frac{i\rho x}{m}\right)^m - \left(1-\frac{i\rho x}{m}\right)^m = 0.$$

This gives

$$\frac{1+\frac{i\rho x}{m}}{1-\frac{i\rho x}{m}} = 1^{\frac{1}{m}} = e^{\frac{2ik\pi}{m}} \quad \text{[where } k \text{ is zero or any integer]}$$

$$\therefore \quad i\rho x = \frac{e^{\frac{ik\pi}{m}} - e^{-\frac{ik\pi}{m}}}{e^{\frac{ik\pi}{m}} + e^{-\frac{ik\pi}{m}}} = i \tan \frac{k\pi}{m}, \text{ i.e. } x = \frac{m}{\rho} \tan \frac{k\pi}{m}.$$

Denoting this value of x by x_k the partial fractions are

$$\frac{1}{\rho x} + \Sigma \frac{\delta_k}{x - x_k},$$

where by the usual rules

$$\delta_k = \left[\left(1+\frac{i\lambda x_k}{m}\right)^m + \left(1-\frac{i\lambda x_k}{m}\right)^m \right] \div \left[\left(1+\frac{i\rho x_k}{m}\right)^{m-1} + \left(1-\frac{i\rho x_k}{m}\right)^{m-1} \right]$$

and k takes the values, $-\frac{m-1}{2}, -\frac{m-3}{2}, \ldots -1, +1, \ldots \frac{m-1}{2}$,

Also since $x_k = -x_{-k}$ and $\delta_k = \delta_{-k}$

* The four fundamental expansions contained in this Note can be obtained by a method similar to that given in Todhunter's *Integral Calculus*, § 29. The method here given is, perhaps, more rigid.

$$\therefore \qquad f(m)=\frac{1}{\rho x}+\sum{}' \frac{2x\delta_k}{x^2-x_k^2},$$

the summation extending from $k=1$ to $k=\frac{1}{2}(m-1)$.

Now $(1+i\lambda x_k/m)^m+(1-i\lambda x_k/m)^m$

$$=\left(1+i\lambda\tan\frac{k\pi}{m}\Big/\rho\right)^m+\left(1-i\lambda\tan\frac{k\pi}{m}\Big/\rho\right)^m$$

$$=(1+i\tan\theta_k)^m+(1-i\tan\theta_k)^m, \text{ putting } \tan\theta_k=\frac{\lambda}{\rho}\tan\frac{k\pi}{m},$$

$$=2\sec^m\theta_k\cos m\theta_k;$$

and $(1+i\rho x_k/m,)^{m-1}+(1-i\rho x_k/m)^{m-1}$

$$=\left(1+i\tan\frac{k\pi}{m}\right)^{m-1}+\left(1-i\tan\frac{k\pi}{m}\right)^{m-1}$$

$$=2\sec^{m-1}\frac{k\pi}{m}\cos(m-1)\frac{k\pi}{m}$$

$$=2(-)^k\sec^{m-1}\frac{k\pi}{m}\cos\frac{k\pi}{m}.$$

Hence $\delta_k=\dfrac{(-1)^k}{\rho}\left(\dfrac{\cos k\pi/m}{\cos\theta_k}\right)\dfrac{\cos m\theta_k}{\cos^3\dfrac{k\pi}{m}}$.

For any particular large value of m, put

$$f_1(m)=\frac{1}{\rho x}+\sum_{k=1}^{k=p}{}' \frac{2x\delta_k}{x^2-x_k^2}, \text{ and } f_2(m)=\sum{}' \frac{2x\delta_k}{x^2-x_k^2};$$

the second summation extending from $k=p+1$ to $k=\frac{1}{2}(m-1)$; then $f(m)=f_1(m)+f_2(m)$. Now let m become infinite; then since $f(m)$ and $f_1(m)$ have each a definite limit, $f_2(m)$ has also a definite limit; also $\underset{m=\infty}{\text{Lt}}\tan\dfrac{k\pi}{m}=\dfrac{k\pi}{m}$; and $\underset{m=\infty}{\text{Lt}}\ \delta_k=\dfrac{(-1)^k}{\rho}\cos\dfrac{\lambda k\pi}{\rho}$;

and since $\cos\dfrac{\lambda k\pi}{\rho}\neq 1$, we may suppose p to be taken so great that the numerical value of $f_2(m)$ is less than

$$\sum{}' \frac{2|x|}{|\rho|(k^2\pi^2/\rho^2-x^2)}$$

This last is a convergent series, and therefore when p is taken sufficiently large, $f_2(m)$ can be made less than a quantity as small as we please. Proceeding to the limit when p tends to m or ∞, we get the fundamental form

$$\frac{\cos\lambda x}{\sin\rho x}=\frac{1}{\rho x}+\frac{1}{\rho}\operatorname*{Lt}_{m=\infty}\sum_{k-1}^{k=m}{}'\frac{(-1)^k\cos\dfrac{\lambda k\pi}{\rho}}{(x^2-k^2\pi^2/\rho^2)}2x. \quad \ldots \quad \ldots \quad (A)$$

In particular if $\rho=1=\lambda$, we have

$$\cot x=\frac{1}{x}+\operatorname*{Lt}_{m=\infty}\sum_{k=1}^{k=m}{}'\frac{(-1)^k\cos k\pi}{(x^2-k^2\pi^2)}2x=\frac{1}{x}+\sum_{k=1}^{k=\infty}{}'\frac{2x}{x^2-k^2\pi^2},$$

as is well known; and by putting $\lambda=0$, $\rho=1$, we get

$$\operatorname{cosec} x=\frac{1}{x}+\sum_{k=1}^{k=\infty}{}'\frac{(-1)^k}{x^2-k^2\pi^2}2x.$$

We can similarly find out the expansions of the remaining three forms, namely

$$\frac{\sin\lambda x}{\sin\rho x},\ \frac{\cos\lambda x}{\cos\rho x},\ \frac{\sin\lambda x}{\cos\rho x}$$

by reasoning as for the first fundamental form: the following convergent expansions will be obtained:—

$$\frac{\sin\lambda x}{\sin\rho x}=\sum_{k=1}^{k=\infty}{}'\frac{(-1)^k 2\pi k\sin\dfrac{\lambda k\pi}{\rho}}{(\rho^2x^2-k^2\pi^2)},\ [\lambda<\rho],$$

$$=\operatorname*{Lt}_{m=\infty}\sum_{k=-m}^{k=+m}{}'\frac{(-1)^k\sin\dfrac{\lambda k\pi}{\rho}}{(\rho x-k\pi)}; \quad \ldots \quad \ldots \quad (B)$$

$$\frac{\cos\lambda x}{\cos\rho x}=\sum_{k=1}^{k=8}{}'\frac{(-1)\cos\dfrac{\lambda}{\rho}(2k-1)\dfrac{\pi}{2}}{x^2-(2k-1)\dfrac{\pi^2}{4\rho^2}}\cdot\frac{(2k-1)\pi}{\rho^2},\ [\lambda<\rho]$$

$$=\sum_{k=1}^{k=\infty}{}'\frac{(-1)^k\cos\dfrac{\lambda}{\rho}(2k-1)\dfrac{\pi}{2}}{\rho^2x^2-(2k-1)^2\dfrac{\pi^2}{4}}\cdot(2k-1)\pi; \quad \ldots \quad (C)$$

and

$$\frac{\sin\lambda x}{\cos\rho x}=\sum_{k=1}^{k=\infty}{}'\frac{(-1)^k\rho\sin\dfrac{\lambda}{\rho}(2k-1)\dfrac{\pi}{2}}{\rho^2x^2-(2k-1)^2\dfrac{\pi^2}{4}}2x,\ [\lambda<\rho]$$

$$=\operatorname*{Lt}_{m=\infty}\sum_{k=-m}^{k=+m}{}'\frac{(-1)^k\sin(2k-1)\dfrac{\lambda\pi}{2\rho}}{\rho x-(2k-1)\dfrac{\pi}{2}}\rho. \quad \ldots \quad \ldots \quad (D)$$

In (B) and (D), the value $k=0$ should not be included.

These results may also be obtained from (A) by observing

$$\frac{\sin\lambda x}{\mathrm{sn}x} = \frac{1}{\sin\lambda\pi}\left\{\cos\lambda\pi\,\frac{\cos\lambda x}{\sin x} + \frac{\cos\lambda(x+\pi)}{\sin(x+\pi)}\right\}$$

and using similar trigonometrical identities.

Taking the first fundamental form and changing λ to $i\lambda$, we get

$$\frac{\cosh\lambda x}{\sin\rho x} = \frac{1}{\rho x} + \frac{1}{\rho}\sum_{k=1}^{k=\infty}{}' \frac{(-1)^k \cosh\dfrac{k\lambda\pi}{\rho}}{(x^2 - k^2\pi^2/\rho^2)}\,2x. \quad \ldots \quad \ldots \quad (A')$$

Further changing ρ to $i\rho$, we get

$$\frac{\cos\lambda x}{\sinh\rho x} = \frac{1}{\rho x} + \frac{1}{\rho}\sum_{k=1}^{k=\infty}{}' \frac{(-1)^k \cosh\dfrac{\lambda k\pi}{\rho}}{(x^2 + k^2\pi^2/\rho^2)}\,2x. \quad \ldots \quad \ldots \quad (A'')$$

Lastly changing λ to $i\lambda$ and ρ to $i\rho$, we get

$$\frac{\cosh\lambda x}{\sinh\rho x} = \frac{1}{\rho x} + \frac{1}{\rho}\sum_{k=1}^{k=\infty}{}' \frac{(-1)^k \cos\dfrac{k\lambda\pi}{\rho}}{(x^2 + k^2\pi^2/\rho^2)}\,2x. \quad \ldots \quad \ldots \quad (A''')$$

Taking the second fundamental form (B) and making similar changes, we shall have—

$$\frac{\sinh\lambda x}{\sin\rho x} = \sum_{k=1}^{k=\infty}{}' \frac{(-1)^k 2\pi k \sinh\dfrac{\lambda k\pi}{\rho}}{\rho^2 x^2 - k^2\pi^2}, \quad \ldots \quad \ldots \quad (B')$$

$$\frac{\sin\lambda x}{\sinh\rho x} = \sum_{k=1}^{k=\infty}{}' \frac{(-1)^{k-1} 2\pi k \sinh\dfrac{\lambda k\pi}{\rho}}{\rho^2 x^2 + k^2\pi^2}, \quad \ldots \quad (B'')$$

and

$$\frac{\sinh\lambda x}{\sinh\rho x} = \sum_{k=1}^{k=\infty} \frac{(-1)^{k-1} 2\pi k \sin\dfrac{\lambda k\pi}{\rho}}{\rho^2 x^2 + k^2\pi^2}. \quad \ldots \quad \ldots \quad (B''')$$

Similarly from the third fundamental form (C) we get

$$\frac{\cosh\lambda x}{\cos\rho x} = \sum_{k=1}^{k=\infty} \frac{(-1)^k \cosh\dfrac{\lambda}{\rho}(2k-1)\dfrac{\pi}{2}}{x^2 - (2k-1)^2\dfrac{\pi}{4\rho^2}}(2k-1)\frac{\pi}{\rho^2}, \quad \ldots \quad (C')$$

and from the fourth fundamental form (D)

$$\frac{\sinh\lambda x}{\cos\rho x}=\sum_{k=1}^{k=\infty}(-1)^k\rho\sinh\frac{\lambda}{\rho}(2k-1)\frac{\pi}{2}\cdot\frac{2x}{\rho^2x^2-(2k-1)^2\frac{\pi^2}{4}} \quad \ldots \quad \ldots \quad (D')$$

§ 2. A large number of results may be obtained from these forms by various analytical methods.

For example :—

If $z=\sin\rho x, \frac{\sin\lambda x}{\sin\rho x}=\frac{1}{z}\sin\left(\frac{\lambda}{\rho}\sin^{-1}z\right)$; hence by the ordinary expansion we have

$$\frac{\sin\lambda x}{\sin\rho x}=\frac{\lambda}{\rho}\left\{1+\frac{1}{3!}\left(1-\frac{\lambda^2}{\rho^2}\right)\sin^2\rho x+\frac{1}{5!}\left(1-\frac{\lambda^2}{\rho^2}\right)\left(3^2-\frac{\lambda^2}{\rho^2}\right)\sin^4\rho x+\ldots\right\}$$

$$=\frac{2}{\pi}\sin\frac{\lambda\pi}{\rho}\left(1-\rho^2x^2/\pi^2\right)^{-1}-\frac{4}{2^2\pi}\sin\frac{2\lambda\pi}{\rho}$$

$(1-\rho^2x^2/2^2\pi^2)^{-1}+\ldots$ by result (B);

from which, by equating coefficents of different powers of x, we derive the following series

$$\sin\frac{\lambda\pi}{\rho}-\frac{1}{2}\sin\frac{2\lambda\pi}{\rho}+\frac{1}{3}\sin\frac{3\lambda\pi}{\rho}-\ldots=\frac{\pi}{2}\cdot\frac{\lambda}{\rho} \quad \ldots \quad \ldots \quad (1)$$

$$\frac{1}{1^3}\sin\frac{\lambda\pi}{\rho}-\frac{1}{2^3}\sin\frac{2\lambda\pi}{\rho}+\frac{1}{3^3}\sin\frac{3\lambda\pi}{\rho}-\ldots=\frac{\pi^3}{1^2}\left(\frac{\lambda}{\rho}-\frac{\lambda^3}{\rho^3}\right) \quad \ldots \quad (2)$$

$$\frac{1}{1^5}\sin\frac{\lambda\pi}{\rho}-\frac{1}{2^5}\sin\frac{2\lambda\pi}{\rho}+\frac{1}{3^5}\sin\frac{3\lambda\pi}{\rho}=\frac{\pi^5}{720}\frac{\lambda}{\rho}\left(1-\frac{\lambda^2}{\rho^2}\right)\left(1-\frac{3\lambda^2}{\rho^2}\right) \quad \ldots \quad (3)$$

$$\frac{1}{1^7}\sin\frac{\lambda\pi}{\rho}-\frac{1}{2^7}\sin\frac{2\lambda\pi}{\rho}+\frac{1}{3^7}\sin\frac{3\lambda\pi}{\rho}-\ldots=\frac{\pi^7}{2}\frac{\lambda}{\rho}\left(1-\frac{\lambda^2}{\rho^2}\right)\left\{\frac{2}{45}-\frac{1}{240}\left(3^2-\frac{\lambda^2}{\rho^2}\right)+\frac{1}{7!}\left(3^2-\frac{\lambda^2}{\rho^2}\right)\left(5^2-\frac{\lambda^2}{\rho^2}\right)\right\} \quad \ldots \quad \ldots \quad (4)$$

The series (1), (2), (3), (4), are all convergent.

[See : Chrystal. Chap XXVI, §9]

M. K. Kewalramani.

An Interesting Experiment.

[The following interesting experiment described in December 29, 1917 issue of "Tit-Bits" was successfully performed by the present writer, and is reported here in view of its scientific interest. The details of the experiment can be varied, at pleasure, of course.]

Take a good-sized bowl, fill it nearly with water, and place it upon the floor of a room which is not exposed to shaking or jarring from the street traffic. Sprinkle over the surface of the water a coating of lycopodium powder, which can be obtained at any hospital or chemists' shop. Then upon the surface of this coating of powder make with powdered charcoal a straight black line, say an inch or two in length. Having made this little mark with the charcoal powder on the surface of the contents of the bowl, lay down upon the floor close to the bowl a stick (or some other straight object) so that it will be exactly parallel to the mark. (If the line happens to be parallel to a crack in the floor, or to any stationary object in the room, this will serve as well.)

Leave the bowl undisturbed for a few hours, and then observe the position of the black mark, with reference to the object to which it was parallel. It will be found to have moved in the direction opposite to that of the movement of the earth on its axis. The earth is simply rotating and has carried the water and everything else in the bowl round with it, but the powder on the surface has been left behind a little. The line will always be found to have moved from east to west, which is a perfectly good proof that everything else has moved the other way.

K. B. M.

An interesting Cubic Connected with an Ellipse.

Basset in his *Treatise on Cubic and Quartic Curves* has discussed the different forms of the cubics obtained by inverting a conic with respect to its vertex. (vide : §§ 125, 146-7). The following interesting case has however been overlooked by him :

Let an ellipse be referred to the extremity of its minor axis as origin. Then its equation is written

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2y}{b}$$

Inverting with respect to the origin, we have the circular cubic

$$2y\frac{(x^2+y^2)}{b}=k^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right),$$

whose form is shown below.

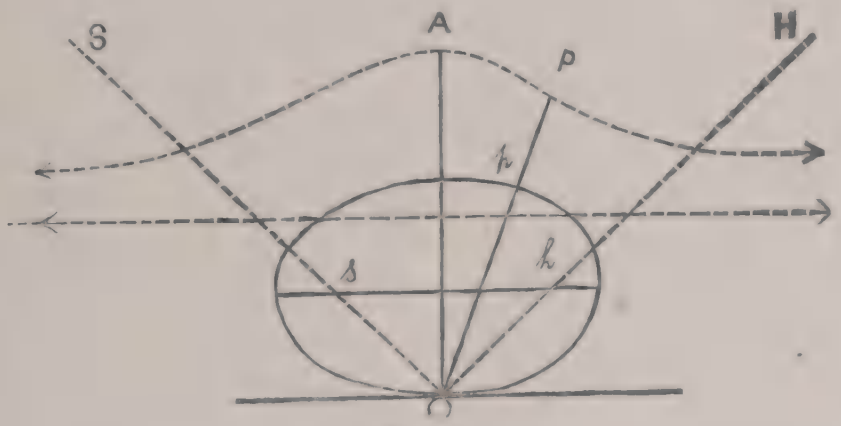


Fig. 1.

If S, H be the inverses of the foci s, h of the ellipse and P, p be any two corresponding points on the cubic and the ellipse, we readily obtain,

$$\frac{SP}{OS}+\frac{HP}{OH}=\frac{OP}{k^2}(sp+hp)$$
$$=2a.\frac{OP}{k^2}.$$

But $os=oh=a$, $OS\cdot os=k^2=OH.oh$.

$\therefore$ $OH=OS=k^2/a$.

Hence $SP+HP=2\cdot OP$,

In other words, the tri-polar equation of the cubic is

$$r_1+r_2=2r_3\,.$$

M. T. Naraniengar.

Degenerate Bicircular Quartics.

The equation of two inverse circles (the origin being the centre of inversion) is

$$(r^2+2cx+\delta^2\lambda)(r^2+2cx/\lambda+\delta^2/\lambda)=0.$$

Comparing this with equation (12) on p. 136 of Basset: *Cubic and Quartic Curves*, we infer that a bicircular quartic will degenerate in two inverse circles, if

$$4f = 2c\left(1+\frac{1}{\lambda}\right),\ g=0,$$

and
$$4f^2+2\delta^2-4a^2 = \frac{4c^2}{\lambda}+\delta^2\left(\lambda+\frac{1}{\lambda}\right),$$

$$2\delta^2-4b^2 = \delta^2\left(\lambda+\frac{1}{\lambda}\right).$$

Thus
$$f^2 = c^2/\lambda + (a^2-b^2)$$

$$= \frac{4f^2\lambda}{(1+\lambda)^2}+(a^2-b^2)\ ;$$

also
$$\delta^2\left(\lambda+\frac{1}{\lambda}+2\right) = 4(\delta^2-b^2).$$

$$\therefore \quad f^2\left(1-\frac{\delta^2}{\delta^2-b^2}\right) = a^2-b^2.$$

$$\therefore \quad f^2 = \frac{(a^2-b^2)(b^2-\delta^2)}{b^2}$$

$$\therefore \quad f = \pm\sqrt{\left\{\frac{(a^2-b^2)(b^2-\delta^2)}{b^2}\right\}}. \qquad \ldots \qquad \ldots \quad (1)$$

In other words, if the centre of inversion is the point $(f, 0)$ where f is determined by (1), the bicircular quartic reduces to two inverse circles.

Examples :

(i) Let the centre of inversion be the foot of the normal at P(ϕ) ; then

$$f = \frac{(a^2-b^2)}{a}\cos\phi\ ;$$

$$\therefore \quad \left(1-\frac{b^2}{a^2}\right)\cos\phi = \left(1-\frac{\delta^2}{b}\right) \text{ by (1).}$$

$$\therefore \quad \delta^2 = \frac{b^2}{a^2}(a^2\sin^2\phi+b^2\cos^2\phi) = \frac{b^2.\mathrm{CD}^2}{a^2},$$

$$= \mathrm{PG}^2.$$

Hence, we have the following result :

If from G a perpendicular GT be drawn on any tangent and a point R taken on this perpendicular such that $\mathrm{GT}^2-\mathrm{RT}^2=\mathrm{PG}^2$, then the locus of R is either of two circles.

(ii) Suppose $\phi=0$ in the above. Then $f=(a^2-b^2)/a$, and $\delta=b^2/a$.

For points outside the limits $\pm\ (a^2-b^2)/a$, the value of δ is real up to $\sqrt{(a^2-b^2)}$, and beyond that δ becomes imaginary, being the length of the imaginary normal from the point to the ellipse.

M. T. Naraniengar.

Astronomical Notes.

The New Star in Aquila.

The new star in Aquila appeared very suddenly on June 8 last when it was about the same magnitude as Altair, for about two days it increased in magnitude till it became nearly equal to Vega and then began to fade steadily. At first the star was white in colour, but about five days after its first appearance it became distinctly yellow, and a few days later became red. The spectrum about this stage showed a velocity of approach of about one-thousand miles per second, similar to that observed in Nova Persei.

The Nova appeared in the dark rift in the Milky way about one hour west of Altair, its approximate position being R.A. $18^h\ 44^m\ 47^s$ Decl. 0° 31′ N.

This is the fourth Nova which has been discovered in Aquila. In 389 A.D. a bright Nova appeared close to Altair, but vanished after three weeks In 1899 and again in 1905 photographic discoveries of Nova in Aquila were made at Harvard College Observatory. The position of the present Nova however differs greatly from any of the other three so that identity is not possible.

An interesting Problem.

The sixth and seventh satellites of Jupiter revolve about their primary at approximately the same distance and consequently in approximately the same period. The same is true of the eighth and ninth satellites, the sidereal periods of which are 739 days and 745 days respectively, a difference of less than one per cent. Now it appears that the perijories of the sixth and seventh satellites are separated by approximately 180°, while the same is found to be true of the eighth and ninth satellites and the question arises, is this purely accidental or is there any dynamical reason? for example, is this arrangement more stable than any other?

It may be mentioned that whereas the motion of the sixth and seventh satellites is direct that of the eighth and ninth is retrograde and it was shown a few years ago that at a considerable distance from the primary a retrograde satellite is more stable than a direct satellite having the same mean distance.

Relativity.

A great deal continues to be published on the subject of Relativity, both with respect to Einstein's recent theory and Sir Oliver Lodge's attempt to explain the motion of the perihelion of Mercury, theories, both of which have been previously referred to in this Journal. I cannot give all the references but the following paper may be found of interest by those interested in the subject.

Planetary motion in space—Time of any constant curvature according to the generalised principle of relativity; by L. Silberstein. *Mon. Not. R. A. S.* 1918 March, p. 363.

R. J. Pocock.

SOLUTIONS.

Question 612.

(N. Ganapathi Subba Aiyar):—A series of ellipses have a common focus (S), pass through a given point (P) and have their major axes of the same length Find their envelope.

Additional Solution by N. Durairajan.

Lemma: If two curves touch, their first positive pedals touch. Therefore if a number of curves have a common envelope the pedals of the curves have also a common envelope which is the the pedal of the original envelope.

In the present case, the pedals of the ellipses are circles of constant radii and also the circles on SP as diameter touches these circles. The envelope of these circles clearly consists of two circles of radii $2a+2r$, where a denotes the semi-major axis and r denotes SP.

Hence the envelope of the ellipses is the negative pedal of these two circles and consists of two conics having a common focus S.

Question 625.

(V. V. S. Narayan):—Show that the lines joining the vertices of a triangle to the corresponding vertices of its first Brocard triangle concur in a point Q whose trilinear co-ordinates are inversely proportional to those of the pole of $\Omega\,\Omega'$ with respect to the Brocard circle.

Solution by K. B. Madhava.

Let K be the symmedian point, and EF', FD', DE' be the antiparallels though it cutting the circle on SK (S is the circumcentre) as diameter at the points A', B', C' respectively. Then A'B'C' is said to be the first Brocard triangle and the circle as the Brocard circle.

Its equation in trilinears will be seen to be (by the method of Askwith *Coordinate Geometry*: p. 416. Ex. 4,).

$$\alpha^2+\beta^2+\gamma^2-\frac{a^2}{bc}\beta\gamma-\frac{b^2}{ca}\gamma\alpha-\frac{c^2}{ab}\alpha\beta=0.$$

Since the first Lemoine circle goes through EF' and is concentric with the Brocard circle through K, A'; KE=A'F'; *i.e.* AA' and AK

cut BC in points equidistant from its midpoint. Therefore by Ceva's theorem AK', BB', CC' concur and their point of intersection (Q) is the isotomic conjugate of K, *i.e.* its trilinear coordinates are

$$\left(\frac{1}{a^3}, \frac{1}{b^3}, \frac{1}{c^3}\right).$$

Moreover since the normal coordinates of

Ω are $\frac{c}{b}, \frac{a}{c}, \frac{b}{a}$ and of Ω^1 $\frac{b}{c}, \frac{c}{a}, \frac{a}{b}$ (2)

the equation of $\Omega\ \Omega^1$ is

$$\alpha\left(a^3-\frac{b^2c^2}{a}\right)+\beta\left(b^3-\frac{a^2c^2}{b}\right)+\gamma\left(c^3-\frac{b^2a^2}{c}\right)=0 \quad \ldots \quad (3)$$

and its pole with respect to the circle (1) is (a^3, b^3, c^3). Hence the result stated.

Question 734.

(J. C. SWAMINARAYAN) :—Solve the differential equation

$$(y^2-b^2)\frac{dy}{dx}+xy=\sqrt{b^2x^2+a^2y^2-a^2b^2}.$$

Solution by N. Durairajan.

Write $x=a\cos(\alpha\cdot v.)$

$y=b\sin(\alpha\cdot v.)$

Substituting in the equation and reducing we get

$$d\alpha+dv.\ \frac{\sqrt{v^2-1}}{v}=0.$$

The variables are separated and the equation may be considered as solved.

Question 753.

(S. RAMANUJAM) :—If

$$\phi(x)=\tfrac{1}{2}\log 2\pi x-x+\int_1^x\frac{[t]}{t}dt$$

where $[t]$ denotes the greatest integer in t, show that

$$\overline{\lim_{x\to\infty}}\ x\phi(x)=\frac{1}{24} \text{ and } \lim_{x\to\infty}\ x\phi(x)=-\frac{1}{12}.$$

Solution (1) *by N. Durairajan*; (2) *by K. B. Madhava.*

(1) We have

$$\phi(x)=\frac{1}{2}\log 2\pi x-x+\int_1^x\frac{[t]}{t}dt.$$

Expressing the integral as I, we have

$$I=\int_1^2 \frac{[t]}{t}\,dt+\int_2^3 \frac{[t]}{t}\,dt+\ldots+\int_{[x]}^{x} \frac{[t]}{t}\,dt.$$

$$=1\log\frac{2}{1}+2\log\frac{3}{2}+3\log\frac{4}{3}+\ldots+[x]\log\frac{x}{[x]}.$$

$$=\log\left[\frac{2}{1}\cdot\frac{3^2}{2^2}\cdot\frac{4^3}{3^3}\cdots\frac{x^{([x])}}{[x]^{([x])}}\right]$$

$$=\log\frac{x^{([x])}}{[x]\,!}$$

We have

$$\phi(x)=\log\frac{e^{-x}\sqrt{2\pi x}.\ x^{([x])}}{[x]\,!}$$

Now let $x=[x]+\delta=n+\delta$, and keeping δ constant make $n\to\infty$. Now

$$\lim_{n\to\infty}(x\phi(x))=\lim_{n\to\infty}(n+\delta)\log\frac{e^{-n-\delta}\sqrt{2\pi(n+\delta)}(n+\delta)^n}{n\,!}$$

δ being constant.

$$=\lim_{n\to\infty}(n+\delta)\log\frac{e^{-n}\sqrt{2\pi n}.n^n\left(1+\frac{\delta}{n}\right)^{n+\frac{1}{2}}e^{-\delta}}{n\,!}$$

$$=\lim_{n\to\infty}(n+\delta)\log e^{-\frac{1}{12n}-\theta}$$

$$+(n+\delta)\log\left\{e^{-\delta}\left(1+\frac{\delta}{n}\right)^{n+\frac{1}{2}}\right\}$$

by Stirling's theorem, where $\theta=0\left(\frac{1}{n^2}\right)$.

$$=(n+\delta)\left(-\frac{1}{12}-\theta\right)+(n+\delta)\left(\frac{\delta-\delta^2}{2n}+\frac{4\delta^3-3\delta^2}{12n^2}+\ldots\right)$$

$$=-\frac{1}{12}+\frac{1}{2}(\delta-\delta^2)-\frac{\delta}{12n}(1-6\delta+2\delta^2)+\ldots$$

In other words, the limit depends on δ. The maximum and minimum values of $\delta-\delta^2$ are when $\delta=\frac{1}{2}$ and 0, and equal to $\frac{1}{4}$ and 0 respectively. Hence $\lim x\phi(x)$ oscillates between the limits

$$-\frac{1}{12}+\frac{1}{2}\frac{1}{4} \quad \text{and} \quad -\frac{1}{12}+0 \ \textit{ie} \text{ between } \frac{1}{24} \text{ and } -\frac{1}{12},$$

(2) Differentiating we have

$$\frac{d\phi}{dx}=\frac{1}{2x}-1+\frac{n}{x},$$

which vanishes when $x=n+\frac{1}{2}$.

Also $\frac{d^2\phi}{dx^2}=-\frac{1+2n}{2x^2}=$ negative when $n\to\infty$

Hence we infer that

$$\overline{\lim_{x\to\infty}}\, x\phi(x)=\lim_{n\to\infty}\ (n+\tfrac{1}{2})\phi(n+\tfrac{1}{2}).$$

(Vide Solution to Q 740, page 220., Vol. VIII)

Question 783.

(S. Ramanujan) :—If $x=y^n-y^{n-1}$, and

$$J_n=\int_0^1 \log y\,\frac{dx}{x},$$

show that

(i) $J_0=\frac{\pi^2}{6}$; $J_{\frac{1}{2}}=\frac{\pi^2}{10}$; $J_1=\frac{\pi^2}{12}$; $J_2=\frac{\pi^2}{15}$.

(ii) $J_n+J_{\frac{1}{n}}=\frac{\pi^2}{6}$.

Solution by N. Durai Rajan and 'Zero'

(I) If $x=y^n-y^{n-1}$, it is obvious that for positive values of x, $y>1$ when positive. Also, y cannot be negative, since the question involves $\log y$. Thus, the values of y corresponding to the limits (0 , 1) of x are (1 , α), where

$$\alpha^n-\alpha^{n-1}-1=0 \quad \text{and } \alpha>1.$$

Again, if $x=y^{\frac{1}{n}}-y^{\frac{1}{n}-1}$, the similar limits for y are $(1,\beta)$, where

$$\beta^{\frac{1}{n}}-\beta^{\frac{1}{n}-1}-1=0$$

which may be written (after multiplication by $\beta^{1-\frac{1}{n}}$)

$$\beta-\beta^{1-\frac{1}{n}}-1=0$$

proving that $\beta=\alpha^n$, and $\frac{1}{\alpha}+\frac{1}{\beta}=1$. (1)

Now

$$J_n = \int_0^1 \log y \, d(\log x),$$

$$= [\log x \log y] - \int_1^\alpha \log x \, d(\log y),$$

$$J_n = -\int_1^\alpha \left\{ n \log y + \log\left(1 - \frac{1}{y}\right) \right\} \frac{dy}{y},$$

since the value of $(\log x \log y)$ between the limiting values of x and is easily seen to be zero.

Hence

$$J_n = -\left\{ \frac{n}{2}(\log y)^2 + \int \log\left(1 - \frac{1}{y}\right) \frac{dy}{y} \right\}_1^\alpha$$

$$= -\tfrac{1}{2}(\log \alpha \log \beta) - \int_1^\alpha \log\left(1 - \frac{1}{y}\right) \frac{dy}{y}. \quad \ldots \quad (2)$$

Similarly

$$J_{\frac{1}{n}} = -\frac{1}{2}(\log\alpha \log\beta) - \int_1^\beta \log\left(1 - \frac{1}{y}\right) \frac{dy}{y}. \quad \ldots \quad \ldots \quad (3)$$

To prove that $J_n + J_{\frac{1}{n}}$ is constant, we differentiate (2) and (3) with respect to α and β and remember that

$$\frac{d\alpha}{\alpha^2} + \frac{d\beta}{\beta^2} = 0 \text{ by virtue of (1.)}$$

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Thus

$$\frac{dJ_n}{d\alpha} = -\frac{1}{2}\left\{ \frac{\log\beta}{\alpha} + \frac{\log\alpha}{\beta}\frac{d\beta}{d\alpha} \right\} - \frac{1}{\alpha}\log\left(1 - \frac{1}{\alpha}\right)$$

$$= -\frac{1}{2}\frac{\log\beta}{\alpha} + \frac{\beta\log\alpha}{2\alpha^2} + \frac{\log\beta}{\alpha}$$

$$= \frac{(\alpha\log\beta + \beta\log\alpha)}{2\alpha^2};$$

and

$$\frac{dJ_{\frac{1}{n}}}{d\beta} = \frac{(\alpha\log\beta + \beta\log\alpha)}{2\beta^2}.$$

$$\therefore \quad \frac{d\left(J_n + J_{\frac{1}{n}}\right)}{d\alpha} = 0, \text{ since } \frac{d\alpha}{\alpha^2} + \frac{d\beta}{\beta^2} = 0.$$

Hence $J_n+J_{\frac{1}{n}}$ is a constant for all values of n.

In particular $J_1+J_1=2\left(\frac{\pi^2}{12}\right)=\frac{\pi^2}{6}$, by the first part. Finally, therefore, we have

$$J_n+J_{\frac{1}{n}}=\frac{\pi}{6},$$

II. Since $x=y^n-y^{n-1}=y^n\left(1-\frac{1}{y}\right)$ is transformed, by the substitution $\frac{1}{\eta}=1-\frac{1}{y}$, into

$$x^{\frac{1}{n}}=\left(1-\frac{1}{\eta}\right)\ \eta^{\frac{1}{n}},$$

we see that

$$J_{\frac{1}{n}}=\int_0^1\frac{d\xi}{\xi}\log\eta, \text{ where } \xi=x-\frac{1}{n},\ \frac{1}{\eta}=1-\frac{1}{y}.$$

Thus $J_{\frac{1}{n}}=\int_\infty^1\frac{1}{n}\frac{dx}{x}\log\left(1-\frac{1}{y}\right)$.

$$\therefore J_n+J_{\frac{1}{n}}=\int_0^1\frac{dx}{x}\log y+\int_\infty^1\frac{1}{n}\frac{dx}{x}\log\left(1-\frac{1}{y}\right)$$

$$=\int_0^\infty\frac{dx}{x}\log y+\int_\infty^1\left\{\frac{dx}{x}\log y+\frac{dx}{nx}\log\left(1-\frac{1}{y}\right)\right\}$$

$$=\int_1^\infty\left[n\frac{dy}{y}+d\log\left(1-\frac{1}{y}\right)\right]\log y+\int_\infty^1\frac{\log x}{n}d(\log x)$$

$$=\left[\frac{n}{2}(\log y)^2-\frac{1}{2n}(\log x)^2\right]_1^\infty+\int_1^\infty\log y\,d\log\left(1-\frac{1}{y}\right)$$

$$=\left[\log y\left(1-\frac{1}{y}\right)-\frac{1}{2n}\left(\log 1-\frac{1}{y}\right)^2-\log y\left(1-\frac{1}{y}\right)\right]_1^\infty$$

$$-\int_1^\infty\log\left(1-\frac{1}{y}\right)\frac{dy}{y}$$

$$=\frac{\pi^2}{6},$$

since the expression within the square brackets is zero.

Question 800.

(S. MALHARI RAO) :—Shew that the sum of all fractions which may be represented by a recurring decimal of the form $(.\dot{a}\ \dot{b}\ c\ d)$ is 50, provided $a+c=b+d=9$.

Further Remarks by H. B.

The error in printing the question on p. 199 of Vol. IX was fortunate; and so was the failure of the attempt on p. 251 of the same volume to correct this error. For, the solutions on p. 326 of the last issue of the Journal show that the second dot might be placed on a or on b without altering the proposition. On looking more closely into the matter I find that the second dot might also have been placed on c or nowhere at all. The moral credit for the following generalization thus belongs to the Printer's Devil.

If $(.ab...gh...\dot{n})$ is a recurring fraction (pure or mixed) its value is equal to

$$Aa+Bb+...+Nn$$

where the quantities A, B, ..,N are determined by the positions of the corresponding digits, but are independent of the value of those digits. As a general rule A, B,...N vary with the variations in the form of the recurring fraction, but for all such fractions

$$A+B+...+N=\frac{1}{9}$$

as is clear from the fact that the expression on the left is the value of the continued fraction when all the digits are equal to unity.

The number of the digits and the position of the dots being fixed, let us obtain all possible recurring fractions as under :—

(i) The digits in k assigned positions are given any value we please, independently of each other;

(ii) Each digit in any of the other positions is dependent on a digit in the first group in such a way that there is a complete 1–1 correspondence between the ten possible values of the two. Subject to this limitation, the law of dependence is arbitrary and not necessarily the same for any two pairs, and each of the independent digits may have any number (from zero upwards) of digits depending upon it.

The proposition is that the sum of all fractions is $\frac{1}{2}.10^{k}$

Proof :—The sum is

$$A\Sigma a + B\Sigma b + \ldots + N\Sigma n$$

where the summation extends over all admissible values of each digit.

But the 1–1 correspondence ensures that the summation for each of the two connected digits is the same. Clearly the summation for all the independent digits is the same, *i.e.*

$$10^{k-1}.(0+1+2+\ldots+9) = 10^{k-1} \times 45 = \frac{9}{2}.10^k.$$

Therefore the sum of all fractions is

$$\frac{9}{2}.10^k.(A+B+\ldots+N) = \frac{1}{2}10^k.$$

Whatever k be, the average value of the fractions is $\frac{1}{2}$ as it should be because the average value of *all* possible fractions is $\frac{1}{2}$, and we have taken a fair sample of *all* fractions.

Question 925.

(S. R. Ranganathan) :—If P, N, N_0, N_1, are the corresponding points on a curve, its evolute, the evolute of the evolute and the evolute of the α—evolute respectively, show that N_1 describes a circle on $N_0 N$ as diameter, as α varies from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$.

Solution by R. Srinivasan, M. A., M. R. A. S.

Let P, P^1, P^{11} be consecutive points on the curve, N, N^1 on the evolute ;

then $N \hat{P} N_1 = N \hat{P}^1 N_1 = \alpha$.

$\therefore$ $N \hat{N}_1 P_1 = N \hat{P} P^1 = 90°$ ultimately.

$\therefore$ $N_1 \hat{N} N_0 = 90° - N_1 \hat{N} P^1 = \alpha$.

Similarly $N_1 \hat{N}^1 N_0 = \alpha$ ultimately.

$\therefore$ $N_1 N N^1 N_0$ is cyclic.

$\therefore$ $N \hat{N}_1 N_0 = 180° - N \hat{N}^1 N_0 = 90°$ ultimately

$\therefore$ N_1 lies on the circle on $N N_0$ as diameter.

As α (*ie* $N_1 \hat{N} N_0$) varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, N_1 will describe the circle completely.

Question 866

(H. R. KAPADIA) :—Prove that

$$\int_{\infty} x^{-2n} \sum_{1}^{n}(A_r \cos a_r x)\, dx = \frac{\pi \cos n\pi}{(2n-1)} \sum_{1}^{n} (A_r a_r^{2n-1}).$$

Solution by K. B. Madhava, V. D. Gokhale, Sadanand, M, K. Kewalramani, S. V. Venkatachela Iyer, and K. R. Rama Iyer.

Since the integrand is even, it is equal to twice the value of

$$\int_0^{\infty} x^{-2n} \sum_{1}^{n} (A_r \cos a_r x)\, dx.$$

Integrating this by parts we are successively led to

$$\int_0^{\infty} \frac{1}{2n-1} \sum_{1}^{n} A_r a_r (\sin a_r x) \frac{dx}{x^{2n-1}}$$

and $$\int_0^{\infty} \frac{-1}{(2n-1)(2n-2)(2n-3)} \sum_{1}^{n} A_r a_r^3 \sin a_r x) \frac{dx}{x^{2n-3}}$$

and so on;

assuming $\Sigma A_r = 0$; $\Sigma A_r a_r^2 = 0$ up to

$$\int_0^{\infty} \frac{(-)^n}{(2n-1)!} \Sigma(A_r a_r^{2n-1}) \frac{\sin x}{x} dx$$

$$= \frac{\pi}{2} \cdot \frac{\cos n\pi}{(2n-1)!} \Sigma(A_r a_r^{2n-1}).$$

Hence the result.

Note: It is easy to see that the allied integral

$$\int_0^{\infty} x^{2n-1} \sum_{1}^{n} (A_r \cos a_r x)\, dx$$

leads us to $$\frac{\cos n\pi}{2n!} \int_0^{\infty} \Sigma(A_r a_r^{2n} \cos x) \frac{dx}{x}$$

with the conditions $\Sigma A_r = 0$; $\Sigma A_r a_r^2 = 0 \ldots \Sigma A_r a_r^{2n} = 0$

which gives finally $$\frac{\cos n\pi}{2n!} \Sigma(A_r a_r^{2n} \log a_r)$$

by Frullani's theorem. (cf. Bromwich, § 170).

Question 867.

(C. KRISHNAMACHARI) :—If the velocity of the Earth is instantaneously altered in the ratio of $\sqrt{2} : 1$, when it is at the extremity of the minor axis of its elliptic orbit, show that the new orbit subsequently described is a parabola whose latus-rectum is twice of the elliptic orbit.

Solution by S. V. Venkatachala Iyer, M. K. Kewalramani, Sadanand, S. Muthukrishnan and S. R. Ranganathan.

The velocity at any point of the orbit is given by

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right).$$

$\therefore$ At the given instant the velocity is given by

$$v^2 = \mu\left\{\frac{2}{a} - \frac{1}{a}\right\} = \frac{\mu}{a}.$$

$\therefore$ v_1^2, the square of the new velocity $= \frac{2\mu}{a}$

$\therefore$ the subsequent motion is given by

$$v^2 = \frac{2\mu}{r}$$

which shows that the subsequent path is a parabola.

The semi-latus rectum in the first case is given by $l = \frac{h^2}{\mu} = \frac{v^2p^2}{\mu}$

and in the second by $l_1 = \frac{p}{\mu}$. Since p, the perpendicular from the origin to the tangent to the path is the same in both cases

$$l : l_1 = v^2 : v_1^2 = \frac{\mu}{a} : \frac{2\mu}{a} = 1 : 2.$$

Question 888.

(HEMRAJ) :—ABCD is a tetrahedron, the edges DA, DB, DC are a, b, c the angles BDC, CDA, ADB, are λ, μ, ν; and 2 X is the sum of the areas of the four faces; and 2 Y the sum of the areas of the four faces less twice the area of the face opposite D; and V the volume. Show that

$$\frac{4X^2Y^2}{9V^2}\Delta\operatorname{cosec}^2\sigma - \Sigma a^2\sin^2(\sigma-\lambda) - \Sigma bc(\cos\lambda - \cos\mu\cos\nu)$$

where $2\sigma = \lambda + \mu + \nu$,

and $\Delta = 1 - \Sigma\cos^2\lambda + 2\cos\lambda\cos\mu\cos\nu$.

Solution by H. Br, K. Appukuttan Erady, H. R. Kapadia, K. B. Madhava and S. V. Venkatachala Iyer.

Let $\alpha, \beta, \gamma, \delta$ be twice the area of the faces. Then
$\alpha = bc \sin \lambda$, and $\beta = ca \sin \mu$, $\gamma = ab \sin \nu$.

By considering δ to be the resultant of forces α, β, γ, acting perpendicular to these faces we get

$$\begin{aligned}\delta^2 &= \alpha^2+\beta^2+\gamma^2+2\Sigma\beta\gamma\cos(\beta\gamma)\\ &= \alpha^2+\beta^2\gamma^2 2\Sigma a^2bc \sin\mu \sin\nu \cos(\beta\gamma)\\ &= \alpha^2+\beta^2+\gamma^2-2abc\Sigma a\cos\lambda-(\cos\mu\cos\nu)]\\ &= (\alpha+\beta+\gamma)^2-2abc\Sigma a(\cos\lambda-\cos\overline{\mu+\nu})\\ &= (\alpha+\beta+\gamma)^2-4abc\Sigma a\sin\sigma\sin(\sigma-\lambda)\end{aligned}$$

$$\therefore \quad 4XY = \tfrac{1}{4}[(\alpha+\beta+\gamma)^2-\delta^2] = abc\sin\sigma\Sigma a\sin(\sigma-\lambda)$$

or
$$\frac{16X^2Y^2}{a^2b^2c^2}\operatorname{cosec}^2\sigma = \Sigma a^2\sin^2(\sigma-\lambda)+2\Sigma bc\sin(\sigma-\mu)\,\delta\sin(\sigma-\nu)$$

$$= \Sigma a^2\sin^2(\sigma-\lambda)-bc(\cos\lambda-\cos\overline{\mu-\nu})$$

From this the given result follows with the help of the well-known formula

$$36V^2 = a^2b^2c^2(1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu)$$

Question 915.

(Chas. Saldanha) :—Given $(a+b)$, $(a+c)$ and A, give a geometrical construction for the ΔABC

Solution (1) *by R. Srinivasan*, (2) *by K. J. Sanjana and K. B. Madhava* (3) *by T. Krishna Rao and several others.*

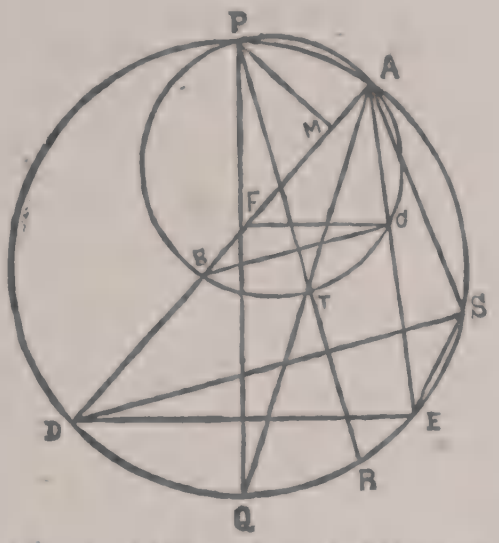
(1) Let ABC be a triangle and I_1 the excentre opposite A. Produce AB, AC to D and E such that BD = BC = CE.

Then $\Delta DBI \equiv \Delta CBI \equiv \Delta CEI$.

$$\therefore \quad \hat{DI_1E} = 3\hat{BI_1C} = 3\left(90^\circ-\frac{A}{2}\right).$$

Also $I_1 D = I_1 C$, ; $I E = I B$. :—

Hence the following construction.

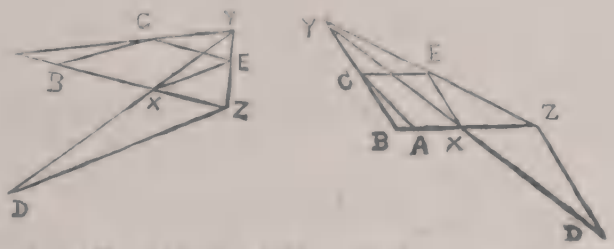


Construct a triangle ADE, such that $AD=a+c$, $AE=a+b$ and $DAE=A$. On DE describe an arc of a circle containing $3\left(90°-\frac{A}{2}\right)$. Let the bisector of the angle A meet this arc in I, with centre I, and radii I D and I E describe circles to cut AE, AD, in CB. Then ABC is the triangle required.

The proof is evident.

(2) More generally let two lengths AX, AY be given inclined at a known angle A; to draw a transversal BC across them so that

$$XB=BC=CY.$$



On AX take $AZ=AY$; let YX meet the circle with centre Z and radius ZA, in D. Join DZ, and draw XE parallel to it to meet YZ; draw EC prallel to AX to meet AY, and CB parallel to EX to meet AX or AX produced.

Then $$\frac{CE}{EY}=\frac{AZ}{ZY}=\frac{ZD}{ZY}=\frac{EX}{EY},$$ so that $CE=EX$.

$\therefore$ CEXB is a rhombus; and $CE=CY$, as $\angle CYE=\angle AZE=\angle CEY$

Thus XB = BC = CY and in the first figure ABC is the triangle required by the question. In the second figure where $AX < \frac{1}{2}AY$, this triangle does not geometrically exist and $AB + BX = AX$ *algebraically*.

(3) Construct a triangle ADE such that $AD = a + c$, $AE = a + b$ and $\angle DAE = A$. Draw the circum-circle of this triangle and PQ the diameter of the circle perpendicular to DE. Join AQ. Set off a chord QR equal to the difference of the given lengths. Join PR cutting AQ in T. Draw the circle PAT cutting AD, AE in BC respectively. Join BC. Then ABC is the required triangle.

Draw DS parallel BC to meet circle ADE in S and CF parallel to DE to meet AD in F. Join AS, SE. Draw PM perp. to AD.

$$\therefore \text{PR} \perp \text{DS and PQ} \perp \text{DE.}$$

$$\angle QPR = \angle EDS$$

i.e. chord $ES = QR = c - b$.

Since P is the mid point of segment DAE and PM perpendicular to AD

$$\therefore \quad AM = \tfrac{1}{2}(AD - AE)$$

Similarly $$AM = \tfrac{1}{2}(AB - AC)$$

i.e. $$AD - AE = AB - AC \text{ or } BD = CE.$$

In the triangles FBC and AES,

the triangles $$B = ADS = AES$$

and $$FCB = SDE = EAS$$

hence they are similar.

i.e. $$\frac{BC}{FB} = \frac{AE}{ES} = \frac{a+b}{c-b}.$$

Again since FC is parallel to DE

$$\frac{DF}{DA} = \frac{EC}{EA} = \frac{DF - EC}{DA - EA} = \frac{BF}{c-b}.$$

or $$\frac{EC}{BF} = \frac{EA}{c-b} = \frac{a+b}{c-b}.$$

$$\therefore \quad BC = EC = DB.$$

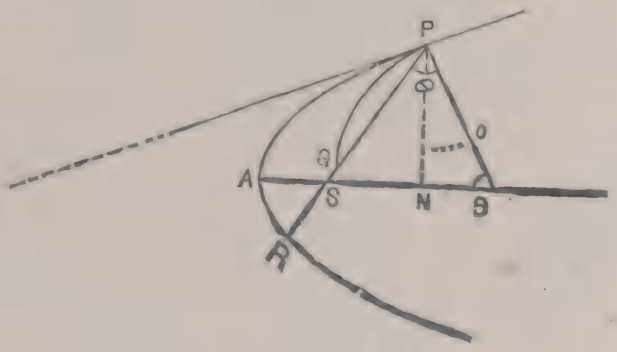
Question 916.

(N. P. Pandya) :—A circle touches a given parabola at P internally and is such that the focal distance of P is divided harmonically by the circle and the parabola. Find the locus of the centre of the circle.

Solution by Hemraj.

Let the equation of the parabola be $y^2=4ax$ and P $(at^2, 2at)$. If O, a point (x, y) on the normal at P, be the centre and P O $=r$, then

$$x=at^2+r\cos\theta,\ y=2at-r\sin\theta \quad \ldots \quad \ldots \quad (1)$$



where SPO $=\theta$ *i.e.*, $\tan\theta=t$.

Since $$(PQSR)=-1,\ \frac{2}{PS}=\frac{1}{PQ}+\frac{1}{PR}\ldots\ldots \quad \ldots \quad (2)$$

But PS $=a+at^2$, PQ $=2r\cos\theta=2(x-at^2)$ from (1)

$$PR=PS+SR=2a+a\left(t^2+\frac{1}{t^2}\right)$$

$\therefore$ On substitution in (2) we have

$$3at^4+2(3a-x)t^2+a-4x=0 \quad \ldots \quad \ldots \quad (3)$$

From (1) $$at^3+(2a-x)t-y=0 \quad \ldots \quad \ldots \quad (4)$$

Eliminating t between (3) O (4) we have the locus of O

$$(4x-a)(x^2-6ax+a^2)^2+y^2(2x^3-78ax^2+90a^2x-18a^3-27ay^2)=0.$$

Question 918.

(C. KRISHNAMACHARY) :—Show that if a and b are both less or greater than unity, and $a<b$, $u_n=\frac{1-a^n}{1-b^n}$ is a decreasing monotonic sequence. Deduce the inequalities,

(1) $$\frac{1-b^n}{1-b}.(1-a)>(1-a^n)>\frac{1-b^n}{1-b}\left(\frac{a}{b}\right)^{n-1}(1-a), \qquad \text{if } o<a<1$$

(2) $$\frac{b^n-1}{b-1}(a-1)>(a^n-1)>\frac{b^n-1}{b-1}\left(\frac{a}{b}\right)^{n-1}(a-1) \qquad \text{if } a>1.$$

Examine the inequalities for fractional values of n.

Solution by K. B. Madhava and S. R. Ranganathan.

In general, if a sequence $\left(\frac{a_n}{b_n}\right)$ is monotonic, the sequence $\left(\frac{\Sigma a_n}{\Sigma b_n}\right)$ is also monotonic and is varying in the same sense.

For, $\frac{\Sigma a_n}{\Sigma b_n} > \frac{\Sigma a_{n+1}}{\Sigma b_{n+1}}$

if $\frac{\Sigma a_n}{\Sigma b_n} > \frac{a_{n+1}}{b_{n+1}}$

i.e. if $\sum_{r=1}^{n}\left(a_r b_{n+1} - b_r a_{n+1}\right) > 0$ which is true, because each of the term is positive in cosequence of the sequence $\left(\frac{a_n}{b_n}\right)$ being non-increasing

Again if the sequence is non-increasing

$$\frac{a_1}{b_1} > \frac{\Sigma a_n}{\Sigma b_n} \text{ if } \sum_{r=2}^{n}\left(a_1 b_r - a_2 b_r\right) > 0.$$

Hence combining $\frac{a_1}{b_1} > \frac{\Sigma a_n}{\Sigma b_n} > \frac{a_{n+1}}{b_{n+1}}$. (A)

Take as $a_n = a^{n-1}(1-a)$ or $a^{n-1}(a-1)$ if $0<a<1$ or $a>1$

and $b_n = b^{n-1}(1-b)$ or $b^{n-1}(b-1)$ if $0<b<1$ or $b>1$

and then $a_n = \frac{\Sigma a_n}{\Sigma b_n}$ is a non-increasing monotonic sequence, and the inequalities (A) become (1) and (2) in the problem. (The second involves the condition $a<b$).

The inequalities may be verified for fractional values of n in the usual manner, by multiplying with the denominator, and thus proving it for an integral value of n. The reader may be referred to the examples in p. 21 of Bromwich : *Theory of Infinite Series* for interesting exercises of this type.

Question 920.

(H. R. Kapadia) :—Prove that an integral of

$$\frac{dx}{X^{\frac{1}{2}}} = \frac{dy}{Y^{\frac{1}{2}}}$$

is $$\left(\frac{X^{\frac{1}{2}} + Y^{\frac{1}{2}}}{y-x}\right)^2 = c + e(x+y) + f(x+y)^2$$

where $X = a + bx + cx^2 + ex^3 + fx^4$, and $Y = a + by + cy^2 + ey^3 + fy^4$

Solution by K. J. Sanjana, R. J. Pocock and K. B. Madhava and others.

This is Ex. 2 on p. 241 of Forsyth (Second Edn.).

If we assume $p=x+y$;

$$\frac{dx}{dt}=\frac{X^{\frac{1}{2}}}{y-x}\ ;\ \frac{dy}{dt}=\frac{Y^{\frac{1}{2}}}{y-x}\ ;$$

the solution proceeds exactly as in Forsyth p 240 with one or two slight changes which are obvious. We obtain finally

$$\left(\frac{dp}{dt}\right)^2=ep+fp^2+c$$

whence substituting for $\frac{dp}{dt}$ we get

$$\left[\frac{X^{\frac{1}{2}}+Y^{\frac{1}{2}}}{y-x}\right]^2=C+e(x+y)+f(x+y)^2$$

[*Vide* also, Boole : *Differential Equations.*, Ch. VI ; Roberts : *Integral Calculus* § § 87, 88 ; Dixon : *Elliptic Functions*, § 40.]

Question 923.

(M. K. Kewalramani) :—Supposing $\phi(x)$ vanishes when $x=0$, but neither it nor its differential coefficients vanish for any other value of x prove the following identity :—

$$x\phi(x)-\frac{x^2}{2!}\phi'(x)+\frac{x^3}{3!}\phi''(x)-\frac{x^4}{4!}\phi'''(x)+\ldots$$
$$\equiv A_1x\phi(x)+A_3x^3\phi''(x)+A_5x^5\phi''''(x)+\ldots$$

where

$$A_{2n+1}=2\frac{(2^{2n}-1)}{(2n)!}B_{2n-1},\ (B_{2n-1}\text{ being the }n^{th}\text{ Bernoullian.})$$

Also, show that

$$\frac{\pi}{2}A_1-\frac{\pi^3}{2^3}A_3+\frac{\pi^5}{2^5}A_5-\ldots=1$$

Remarks by Sadanand.

The identity is not true as it stands. We are required to prove that

$$\left(1-e^{xD}\right)\equiv\tan\left(\frac{1}{2}xD\right)$$

where D stands for the operative symbol $\frac{d}{dx}$, which is impossible.

The series

$$\frac{\pi}{2}A_1-\frac{\pi^3}{2^3}A_3+\ldots=-i\tan\left(i\frac{\pi}{4}\right)\neq 1;$$

but the series

$$\frac{\pi}{2}A_1+\frac{\pi^3}{2^3}A_3+\ldots=\tan\frac{\pi}{4}=1.$$

QUESTIONS FOR SOLUTION.

[N.B.—Question 965 by Mr. A. C. L. Wilkinson which appeared in June 1918 should be numberedas 965 A.]

969. (H. Br.):—From the vertex D of a tetrahedron ABCD straight lines DL,DM,DN, are drawn parallel and proportional to the perpendiculars from a point P onthe faces BCD, CAD, ABD, respectively. Q is the isogonal conjugate of P with respect to ABCD. Show that PD passes through the circumcentre of DLMN, and QD is perpendicular to LMN. Show further that if DL, DM, DN. are twice the lengths of the three perpendiculars, but are drawn in the opposite direction in each case, P is the circumcentre of DLMN.

970. (S. R. RANGANATHAN) :—Show that

$$x^n \prod_{r=1}^{\infty}\left(1+\frac{x^{2n}}{r^{2n}}\right) = \frac{1}{2^{n/2}\pi^n} \prod_{r=1}^{n-1} (\cosh 2\pi\alpha_r x - \cos 2\pi\beta_r x)$$

when n is even,

$$= \frac{1}{2^{\frac{1}{2}(n-r)}\pi^n} \sinh \pi x \prod_{1}^{n-2} (\cosh 2\pi\alpha_r x - \cos 2\pi\beta_r x)$$

when n is odd,

where $\alpha_r = \sin\frac{r\pi}{2n}$, and $\beta_r = \cos\frac{r\pi}{2n}$, r being an odd integer.

971. (A. A. KRISHNASWAMI AIYANGAR M.A , L.T.) :—A rigid spherical shell of radius a contains a mass M of gas in which the pressure is K times the density, and the gas is repelled from a fixed external point O (distant c from the centre) with a force per unit of mass equal to K times the distance. Prove that the resultant pressure of the gas on the shell is

$$\frac{KM}{c}\cdot\frac{5c^2-a^2}{5c^2+a^2}.$$ (Besant and Ramsay) *Hydrostatecs.*

972. (K. B. MADHAVA) ;—Solve.

$$\delta(x)+\delta(y)+\delta(z)=B.$$

$$0=[\delta(y)-\delta(x)][\delta'(y)+\delta'(z)]+[\delta(y)-\delta(z)][\delta'(y)+\delta'(x)]$$

$$0=[\delta(x)-\delta(z)][\delta'(x)+\delta'(y)]+[\delta(x)-\delta(y)][\delta'(x)+\delta'(z)]$$

where δ is the Weiertrassian Elliptic function and δ', its derivate. [This is needed in the solution of Lame's equation when $n=3$;

$$\frac{1}{w}\frac{w^2 d}{dr^2} = 12\,\delta(z)+5B.].$$

973. (A. C. L. WILKINSON) :—Prove that

$$\text{Dsn}(u_1+u_2+u_3) = \begin{vmatrix} s_1, & s_1^1, & s_1^3 \\ s_2, & s_2^1, & s_2^3 \\ s_3, & s_3^1, & s_3^3 \end{vmatrix}; \quad \text{Dcn}(u_1+u_2+u_3) = -\begin{vmatrix} c_1, & c_1^1, & c_1^3 \\ c_2, & c_2^1, & c_2^3 \\ c_3, & c_3^1, & c_3^3 \end{vmatrix};$$

$$k^4 \operatorname{Ddn}(u_1+u_2+u_3) = -\begin{vmatrix} d_1, & d_1^1, & d_1^2 \\ d_2, & d_2^1, & d_2^2 \\ d_3, & d_3^1, & d_3^2 \end{vmatrix};$$

where $\mathrm{D} = \begin{vmatrix} 1, & s_1^2, & s_1c_1d_1 \\ 1, & s_2^2, & s_2c_2d_2 \\ 1, & s_3^2, & s_3c_3d_3 \end{vmatrix}$ and $s_1 = \operatorname{sn}u_1$, $s_1' = \dfrac{d}{du_1}(\operatorname{sn}u_1) = \operatorname{cn}u_1 \operatorname{dn}u_1$ etc.

Prove that

$$\mathrm{E}(u_1)+\mathrm{E}(u_2)+\mathrm{E}(u_3)-2\mathrm{E}(x)$$
$$= \frac{k^2d_1d_2d_3d_4}{k^{12}+d_1d_2d_3d_4}\left(\frac{s_1c_1}{d_1}+\frac{s_2c_2}{d_2}+\frac{s_3c_3}{d_3}+\frac{s_4c_4}{d_4}\right)$$
$$-\frac{2k^2\operatorname{sn}x\ \operatorname{cn}x.\operatorname{dn}(x-u_3-u_4)\ \operatorname{dn}(x-u_2-u_4)\ \operatorname{dn}(x-u_2-u_3)}{k^{12}+\operatorname{dn}x\ \operatorname{dn}(x-u_3-u_4)\ \operatorname{dn}(x-u_2-u_4)\ \operatorname{dn}(x-u_2-u_3)}$$

where $x=\frac{1}{2}(u_1+u_2+u_3+u_4)$ and s_1, c_1, d_1 are written as usual for $\operatorname{sn}u_1$, $\operatorname{cn}u_1$, $\operatorname{dn}u_1$.

974. (A. C. L. Wilkinson) :—If two confocal conics are such that triangles ABC can be inscribed in the one which are circumscribed to the other and if D, E, F are the points of contact, prove that (1) the area of the triangle DEF is proportional to the product of the diameters conjugate to D, E, F: (2) the ratio of the areas of the triangles ABC. DEF is constant: (3) the loci of the points of concurrence of the normals at A, B, C and D, E, F are ellipses (4) the line joining the points of concurrence of the normals at A, B, C and D, E, F is a normal to an ellipse: (5) if from the points of concurrence the fourth normals be drawn to the two confocals the line joining their feet is also normal to an ellipse.

(Suggested by Question 947 which shows that in all triangles ABC the ratio R : r is constant.)

975. (Professor K. J. Sanjana):—A uniform right pyramidal solid on a square base is divided into two halves by a plane through the vertex and one diagonal of the base, and the two parts are hinged together at the vertex. If the solid floats in a liquid which is such that the plane of flotation is a square whose side bears to the side of the base the ratio r:1, prove that the parts will not separate if $\sec\theta \geqslant r^{\frac{1}{2}}$ when the vertex is within the fluid, and if $\operatorname{cosec}\theta \leqslant (3r^2-r^2-r)^{\frac{1}{2}}$ when the vertex is without, $-\theta$ being the dihedral angle between the planes of the base and a slant side of the pyramid.

976. (K. J. Sanjana, M. A.) :—Find the probability that the roots of (i) the biquadratic $ax^4+bx^2+c=0$, (ii) the cubic $ax^3+bx+c=0$, may be all real, where the numbers a, b, c are taken at random.

977 (M. K. Kewalramani) :—In the series

$$\frac{1}{3}-\frac{1}{7}+\frac{1}{11}-\frac{1}{15}+\frac{1}{19} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \infty$$

if the terms are deranged in such a manner that to every 6561 positive terms correspond 256 negative terms, the terms in each group having their original order, show that the sum of the new series is

$$\frac{\sqrt{2}}{8}\left[\pi-\log\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]+\log\frac{3}{2}$$

978. M. K. KEWALRAMANI):—If two circles are such that quadrilaterals can be inscribed in the one which are circumscribed to the other, prove that the ratio of the sum of the reciprocals of the diagonals of any two quadrilaterals is constant.

[Suggested by Q. 963]

979. (F. H. V. GULASEKHARAM):—If $u_1, u_2, u_3, u_4, u_5, u_6$ are variable arguments connected by the relations $u_2-u_1=u_3-u_2=u_4-u_3=u_5-u_4=u_6-s=\frac{K}{3}$

and if d_r stands for $dn\ u_r$, shew that

(1) $(d_1d_2+d_2d_3+d_3d_4+d_4d_5+d_5d_6+d_6d_1)$ is constant.

(2) $(d_1+d_4)(d_2+d_5)+(d_2+d_5)(d_3+d_6)+d_3+d_6)(d_1+d_4)$ is constant.

In each case determine the constant in terms of the modulus k and the sn, cn, dn, functions of $\frac{K}{3}$.

980. (F. H. V. GULASEKHARAM):—A triangle $\triangle$ is inscribed in a given conic $S=0$ and circumscribed to a given confocal conic $S_1=0$. A triangle $\triangle_1$ is inscribed in the conic $S_1=0$ with its sides parallel to those of $\triangle$. Shew that

(1) the circum-radius of the triangle $\triangle_1$ = the in radius of the triangle $\triangle$;

(2) the locus of the circum-centre of the triangle $\triangle_1$ for all positions of $\triangle$ is another fixed confocal conic.

981. (R. VYTHYNATHASWAMY). $A_1\ A_2 \ldots A_n$ are n unequal real numbers If $L_{pq}\ldots$ and $\ldots G_{pq}$ denote respectively the least and greatest of the numbers $A_p, A_q \ldots\ldots$, prove that

$$L_{123\ldots n}=\Sigma G_1-\Sigma G_{12}+\Sigma G_{123}-\ldots$$

$$G_{123\ldots n}=\Sigma L_1-\Sigma L_{12}+\Sigma L_{123}-$$

Use these results to express the G. C. M (L. C. M.) of n positive integers as a function of the integers and their mutual L.C M's (G.C.M's).

982. (R. VYYHYNATHASWAMY):—If I and SIS are both involutions, prove that S must be an involution unless I contains the fixed points of S

983. (R. VYTHYNATHASWAMY):—If Ω and S Ω are both periodic and of the same period, shew that thsir fixed points make with the fixed points of S ranges of equal cross ratio.

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PROGRESS REPORT.

The committee feel pleasure in announcing that *Mr. T. K. Venkataraman, M.A.* has got himself elected as a *Life-member* of our Society.

2. The following gentlemen have been elected members of our society :—

1. *Mr. Sriram Venkatasubba Setti, B.A., D.F.H., A.M.I.C.E.*—Superintendent Mechanical Engineering School and Professor, College of Engineering, Bangalore ;

2. *Mr. K. Ananda Rao, M.A.*, (*Smith's Prizeman*).—6, Singarachari Street, Triplicane, Madras ;

3. *Mr. A.V. Subba Rau Esq. M.A., L.T.*—Mathematics Assistant, Taluk High School, Tenali ;

4. *Mr. Carcherla Srinivasa Rau.*—Clerk, Collector's Office, 1/14, Prospect House, Vepery, Madras ;

3. As announced along with the last issue of our journal, a conference of our society will be held in Bombay, on or about 9th, 10th and 11th of January 1919, and consequently it is very earnestly requested that members will try to attend the Conference and to make it a success as on the last occasion. Also those of the members who are desirous to read papers at the Conference, should communicate their intention and submit synopsis of their papers for the inspection of the sub-committee appointed for the purpose by the 1st of December 1918.

POONA,
5th Oct. 1918.

D. D. KAPADIA,
Honorary Joint Secretary.

A Note in Combinatory Analysis.

By R. Vythynathaswamy.

§ 1. The following lemma will be of use in the course of the present paper.

Lemma :

The number of ways of placing λ distinct things in the first of two rows of r vacant places and μ distinct things in the second, where $\lambda \not> r$, and $\mu \not> r$, and each of the λ as well as each of the μ things should be used at least once, so as to satisfy the condition that if k_1, l_1, two members of the first row, contain the same λ-thing, k_2, l_2, the corresponding members in the second row should not contain the same μ-thing, is the coefficient of x^r in

$$\left[\Delta_1^{\lambda}\,\Delta_2^{\mu}(1+x)^{t_1 t_2}\right]_{t_1=t_2=0}$$

where the operators Δ_1 and Δ_2 refer to t_1 and t_2 respectively.

Proof. Let the λ things be $\alpha_1, \alpha_2, \ldots \alpha_\lambda$ and the μ things $\beta_1, \beta_2, \ldots \beta\mu$.

Fill the first row with k_1. α_1's, k_2. α_2's $\ldots k_\lambda$. α_λ's where $k_n (n = 1, 2, \ldots \lambda)$ is a positive non-zero integer and $\Sigma k_n = r$.

The corresponding number of ways of filling the second row will then be

$$\mu^{C_{k_1}} \cdot \mu^{C_{k_2}} \cdot \mu^{C_{k_3}} \cdots \mu^{C_{k_\lambda}}$$

Hence the total number of ways would be

$$\sum_k \mu^{C_{k_1}} \mu^{C_{k_2}} \cdots \mu^{C_{k_\lambda}} \text{ where } \Sigma k_n = r \text{ and } k_n \neq 0$$

and this is the coefficient of x^r in $[(1+x)^{\mu}-1]^{\lambda}$.

If now the condition that each of the βs should occur once at least be added, the total number of ways is

$$[(1+x)^{\mu}-1]^{\lambda} - \mu^{c}{}_1[(1+x)^{\mu-1}-1]^{\lambda} + \mu^{c}{}_2[(1+x)^{\mu-2}-1]^{\lambda} - \ldots$$

which is easily seen to be a symmetric function of λ and μ and equal to

$$\left[\Delta_1^{\lambda}\,\Delta_2^{\mu}(1+x)^{t_1 t_2}\right]_{t_1=t_2=0}.$$

This we shall write also as $\Delta_1^{\lambda}\Delta_2^{\mu}(1+x)^{0\cdot 0}$.

We may also adopt a different type of argument which is perfectly general.

If there are λ distinct things of one kind and μ distinct things of another kind, then the number of ways of forming r distinct combina-

tions of a λ and a μ thing is evidently $\lambda\mu\,{}^{C}r$, and these correspond to the number of ways of arranging the λ and μ things in two rows of r spaces each, with two dissimilar λ—things always opposite to two similar μ—things, without, however, the assurance that each μ—thing or each λ—thing occurs at least once. Therefore, if we add the condition that each λ—thing is to occur at least once, the number of ways is

$$\lambda\mu\,{}^{C}r-\lambda\,{}^{C}1(\lambda-1)\mu\,{}^{C}r+\lambda\,{}^{C}2(\lambda-2)\mu\,{}^{C}r-\ldots=\Delta^{\lambda}0\cdot\mu\,{}^{C}r$$

(cf. Whitworth: *Choice and Chance*

Adding the further condition that each μ—thing should occur also once at least, we see that the total number of ways is

$$\Delta_1{}^{\lambda}\Delta_2{}^{\mu}0\cdot0\,{}^{C}r$$

which agrees with the previous result.

Corollary. More generally, we shall also have:

The number of ways of placing $\lambda_1,\lambda_2\ldots\lambda_n$ things respecively in n rows of r spaces each, so that each thing occurs at least once, and no two columns are identical, is equal to

$$\Delta_1{}^{\lambda_1}\Delta_2{}^{\lambda_2}\ldots\Delta_n{}^{\lambda_n}0\,0\cdot0\ldots\,{}^{C}r$$

§ 2. We can now find *the number of ways in which a set of r different factors of N can be chosen so as to have the greatest common measure unity.*

We will denote this by $\phi_r(N)$ and seek to express $\phi_r(pq)$ where p and q are mutually prime in terms of $\phi_k(p)$ and $\phi_k(q)$. If p and q are mutually prime, every factor of pq is uniquely resoluble into a factor of p and a factor of q. Thus every factor is specified by a factor of p and a factor of q. To specify r distinct factors of pq, then we have to take λ factors of p and μ factors of q where, of course, neither λ nor μ is to be greater than r, and fill with them the two rows of r vacant places satisfying the conditions of the Lemma; that is

$$\phi_r(pq)=\text{coefficient of } x^r \text{ in } \Sigma\Sigma\phi_\lambda(p)\phi_\mu(q)\Delta_1{}^{\lambda}\,\Delta_2{}^{\mu}(1+x)^{0.0}$$

$$(\lambda\not>r,\ \mu\not>r.$$

$$=\Sigma\Sigma\phi_\lambda(p)\phi_\mu(q)\Delta_1{}^{\lambda}\,\Delta_2{}^{\mu}\,0.0\,{}^{C}r$$

Similarly, if $a_1, a_2,\ldots a_n$ are mutually prime, we shall have

$$\phi_r(a_1a_2\ldots a_n)=\Sigma\phi_{\lambda_1}(a_1)\phi_{\lambda_2}(a_2)\ldots\phi_{\lambda_n}(a_n)\Delta_1{}^{\lambda_1}\Delta_2{}^{\lambda_2}\ldots\Delta_n{}^{\lambda_n}000-\,{}^{C}r$$

$$[\lambda_k\not>r\ k=1,2\ldots n.]$$

We have also the result, that if p is a prime

$$\phi_r(p^{\alpha}) = \alpha^{C}r-1;$$

and therefore, that if $N = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$

$$\phi_r(N) = \Sigma \phi_{\lambda_1}(p_1^{\alpha_1}) \ldots \phi_{\lambda_n}(p_n^{\alpha_n}). \quad \Delta_1^{\lambda_1} \ldots \Delta_n^{\lambda_n} 000 \ldots C_r$$

$$= \Sigma \alpha_1^{C}{}_{\lambda_1 - 1} \cdots \alpha_n^{C}{}_{\lambda_n - 1} \cdot \Delta_1^{\lambda_1} \ldots \Delta_n^{\lambda_n} \; 000^{C_r}$$

$$= (1+\Delta_1)^{\alpha_1} \ldots (1+\Delta_n)^{\alpha_n} \Delta_1 \Delta_2 \ldots \Delta_n 000^{C_r}$$

$$= \Delta_1 \Delta_2 \ldots \Delta_n \left[\alpha_1 \alpha_2 \ldots \alpha_n {}^{C}r \right].$$

Thus if $N = 120$ and $r = 2$, $\phi_r(N) = 17$.

§ 3. If $\phi_r'(N)$ be defined as the number of sets of r factors of N (not including N itself) which can be found so as to have the greatest common measure unity, we evidently have

$$\phi_r' + \phi'_{r-1} = \phi_r \text{ (for values of } r \text{ up to 2)}$$

and we shall also suppose $\phi_1' = \phi_1 = 1$. We then have

$$\phi'_r = \phi_r - \phi_{r-1} + \phi_{r-2} - \ldots$$

$$= \Delta_1 \Delta_2 \ldots \Delta_n [\alpha_1 \alpha_2 \ldots \alpha_n C_r - \alpha_1 \alpha_2 \ldots \alpha_n C_{r-1} + \ldots]$$

$$= \Delta_1 \Delta_2 \ldots \Delta_n \left[\alpha_1 \alpha_2 \ldots \alpha_n - 1^{C_r} \right]$$

We therefore have

$$\phi_1' - \phi'_2 + \phi'_3 - \ldots = \Delta_1 \Delta_2 \ldots \Delta_n \left[\alpha_1 \alpha_2 \ldots \alpha_n - 1^{C_1} - \alpha_1 \alpha_2 \ldots \alpha_n - 1^{C_2} + \ldots \right]$$

$$= - \Delta_1 \Delta_2 \ldots \Delta_n (1-1)^{\alpha_1 \alpha_2 \ldots \alpha_n - 1}$$

from which by writing $E_r - 1$ for Δ_r and expanding, we see that the terms in which the index is not zero will vanish. If any one of the αs is different from unity, the index of none of the terms in the expanded series vanishes. Hence *when* N *contains repeated prime factors the series identically vanishes.* If on the other hand all the αs are unity, there is one term for which the index is zero; *viz* $(-)^{n+1}(1-1)^0 = (-)^{n-1}$, and therefore, *if* N *contains n prime factors none being repeated the series equals* $(-)^{n-1}$.

§ 4. We may notice that if $\psi_r(N)$ denotes the number of sets of r factors of N which can be chosen so as to have the least common multiple N, then $\psi_r(N) = \phi_r(N)$. For if we divide N by a set of factors having the least common multiple N, we get a set of factors having the greatest common factor unity, Similarly also we find $\psi_r'(N) = \phi_r'(N)$ where $\psi_r'(N)$ is the same as $\psi_r(N)$ with unity excluded.

This method of reasoning may be employed to obtain two familiar results.

Let $t_1, t_2 \ldots$ be the factors of N, N being excluded. If $\phi(N)$ be the number of factors of N we have,

$$\phi(N)-1=\Sigma\phi(t_1)-\Sigma\phi(t_{12})+\Sigma\phi(t_{123})\ldots$$

where $t_{12\ldots r}$ is the G. C. M. of $t_1, t_2, \ldots t_r$ and is thus itself one of the ts

that is, $$\phi(N)=1+\Sigma\phi(t_k)\left[1-\phi_2'\left(\frac{N}{t_k}\right)+\phi_3'\left(\frac{N}{t_k}\right)-\ldots\right]$$

$$=1+\Sigma\phi\left(\frac{N}{p_1}\right)-\Sigma\phi\left(\frac{N}{p_1p_2}\right)+\ldots$$

which is a familiar result. (Cf. Chrystal: *Algebra*, Vol. II.)

Similarly, the numbers less than N and *not* prime to it will be multiples of one or more of the factors $t_1, t_2 \ldots t_n$ and will therefore be $\sum\frac{N}{t_k}$ in all, which enumeration, however, is not simplex. The simplex enumeration is $\Sigma\frac{N}{t_1}-\Sigma\frac{N}{t_{12}}+\Sigma\frac{N}{t_{123}}-\ldots$

where $t_{12\ldots r}$ is the least common multiple of $t_1, t_2, \ldots t_r$.

This last series is by the definition of ψ'

$$=\Sigma\frac{N}{t_k}[1-\psi_2'(t_k)+\psi_3'(t_k)-\ldots]$$

and the expression, in the square brackets, is either 0 or $(-1)^{n-r}$ according as t_k contains repeated prime factors, or n non-repeated prime factors. Hence, finally, the number of numbers less than N and not prime to it is

$$\sum\frac{N}{p_1}-\sum\frac{N}{p_1p_2}+\sum\frac{N}{p_1p_2p_3}\ldots$$

which is a well known result. (Cf. Smith: *Algebra*, p. 488).

Postscript: Extension of Fermat's Theorem.

It is well known that a rational transformation of the nth degree has $n+1$ fixed points and the mth power of the transformation which is of degree n^m has n^m+1 fixed points. But if t be any factor of m, all the fixed points of S^t (where S is the rational transformation in question) are also fixed points of S^m. Hence, if $t_1 t_2 \ldots t_k$ be the factors of m, m itself being excluded, the number of *special* fixed points of S^m would be equal to

$$(n^m+1)-\Sigma(n^{t_1}+1)+\Sigma(n^{t_{12}}+1)\Sigma(n^{t_{123}}+1)$$

where $t_{12\ldots r}$ is the G.C.M of $t_1, t_2 \ldots t_r$.

This series is equal to $n^m - \Sigma n^{\frac{m}{p_1}} + \Sigma n^{\frac{m}{p_1 p_2}} - \text{etc.}$

where $p_1\ p_2 \ldots$ are the prime factors of m.

We shall shew now that if p is a special fixed point of S^m, then $S(p)$ is also a special fixed point.

It is evident of course that $S(p)$ is a fixed point of S^m. To shew that it is a special fixed point, assume it to be a fixed point of S^t where t is a factor of m. Then we have the following series

p, $S(p)$, $S(S(p)), \ldots S^{t-1}(S(p))$, $S^t(S(p))$ $\{ = S(p) \}$, etc.

It is evident from this scheme that $S^m(p) = S^{t-1}(S(p)) = S^t(p)$

Since $S^m(p) = p$, this shews that p also is a fixed point of S^t. This however is contrary to the hypothesis that p is a special fixed point. Hence $S(p)$ is a special fixed point. Hence if p_1 is one special fixed point, p_1, $S(p_1)$, $S^2(p_1)$.. $S^{m-1}(p_1)$ are all special fixed points *and are all distinct*. In fact, if $S^r(p_1) = S^{r_1}(p_1)$, then $S^r(p)$ is a fixed point of $S^{r_1 - r}$; which is impossible since $S^r(p)$ is a special fixed point. Again if p_2 be a special fixed point not included in the above series, then p_2, $S(p_2)$, $S^2(p_2) \ldots S^{m-1}(p_2)$, are all special fixed points and are all distinct both from one another and from the points of the previous series. In fact, the assumption that $S^r(p_1) = S^{r_1}(p_2)$, leads to $S^{m+r-r_1}(p_1) = p_2$, which contradicts the hypothesis that p_2 is not included in the first series.

Proceeding in this manner we finally come to the conclusion, that the special fixed point can be distributed into sets of cyclic m—groups. Hence.

If n and m be any two integers and $p_1\ p_2\ p_3 \ldots$ are the prime factors of m, then the series.

$$n^m - \Sigma n^{\frac{m}{p_1}} + \Sigma n^{\frac{m}{p_1 p_2}} - \Sigma n^{\frac{m}{p_1 p_2 p_3}} +$$

is exactly divisible by m.

If m is a prime, this reduces to Fermat's Theorem.

It will be noticed that in this extension of Fermat's theorem no restrictions are imposed on either of the two numbers n, m.

Extension of M'cay's Theorem.

By S. Narayanan, B.A., L T., Madura College.

M'Cay's extension of Feuerbach's theorem is '*If the axis major of an inscribed conic passes through the circumcentre, the auxiliary circle touches the nine-points circle.*' It is proposed, in this paper, to shew generally the relation between the auxiliary circle of *any* inconic whose axis major passes through a *given point* P and the pedal circle of P. The main theorem of this paper may be enunciated as follows:

If the axis major of an inscribed conic of a triangle passes through a given point P, the auxiliary circle of this conic and the pedal circle of P *are coaxal with the director circle of the inconic touching the axis major at P, and the common radical axis envelopes the orthopolar conic of P.*

Lemma I. If $x(y^2-z^2)=l$, $y(z^2-x^2)=m$, $z(x^2-y^2)=n$,

then $$\frac{x(y^2+z^2)}{l(m^2+n^2-l^2)}=\frac{y(z^2+x^2)}{m(n^2+l^2-m^2)}=\frac{z(x^2+y^2)}{n(l^2+m^2-n^2)}=\frac{-xyz}{lmn}.$$

This follows immediately on reduction.

Lemma II. If a, b c, denote the circles described on the sides BC, CA AB of a triangle ABC as diameters, then the director circle of any inscribed conic of ABC may be written in the form $\lambda a+\mu b+\nu c=0$.

This is also obvious by rearranging the usual equation of the director circle of an inconic in terms of the varying coefficients of the conic.

Lemma III. The tricyclic equation $al^2\alpha'+bm^2\beta'+cn^2\gamma'=0$,

represents the director circle of the inscribed conic which touches $l\alpha+m\beta+n\gamma=0$ at α', β', γ'. [Here a, b, c denote, as before, the circles on the sides as diameters, l, m, n are variables subject to the condition $l\alpha'+m\beta'+n\gamma'=0$ and α', β', γ' are the normal coordinates of any point].

This follows immediately from Lemma II.

Lemma IV. If P be any point in the plane of a triangle ABC, and if AP, BP, CP meet the opposite sides in D, E, F and if K, L, M be the orthopoles of AP, BP, CP respectively, then the locus of the orthopoles of straight lines passing through P, which is called the orthopolar conic of P, is the conic passing through K, L, M and touching the common chords of the pedal circle of P and the circles on AP, BP, CP as diameters.

Let $l\alpha + m\beta + n\gamma = 0$ be any straight line passing through α', β', γ', the normal coordinates of the orthopole $l\alpha + m\beta + n\gamma = 0$ are given by

$$\left.\begin{aligned} \alpha &= (m\cos B + n\cos C - l\cos B\cos C)\,(m\cos C + n\cos B - l) \\ \beta &= (n\cos C + l\cos A - m\cos C\cos A)\,(n\cos A + l\cos C - m) \\ \gamma &= (l\cos A + m\cos B - n\cos A\cos B)\,(l\cos B + m\cos A - n) \end{aligned}\right\} \quad (I)$$

Now, by means of $l\alpha' + m\beta' + n\gamma' = 0$, substitute for l in terms of m and n in (I), we have thus three equations in m^2, mn, n^2. And again eliminating m and n between these three equations, the result evidently assumes the form $QR = P^2$ (A)

[The details of working are omitted as being too tedious for reproduction here].

Similarly substitute for m in terms of n and l in (I) and eliminate n and l between the three equations. The result will be of the form $Q_1R_1 = P_1^2$ (B)

Again doing the same thing for n we have $Q_2R_2 = P_2^2$... (C)

On a simple inspection of the results (A), (B), (C), it will be found that the orthopolar conic in question is *circumscribed* to the triangle formed by

$$\begin{aligned} \alpha &= (\beta'\cos B - \gamma'\cos c)\,(\beta'\cos c - \gamma'\cos B), \\ \beta &= \cos c\,(\beta' + \gamma'\cos A)\,(\beta'\cos A + \gamma'), \\ \gamma &= \cos B\,(\beta' + \gamma'\cos A)\,(\beta'\cos A + \gamma'), \end{aligned}$$

and two other points whose coordinates can easily be written down from cyclic symmetry.

The same is *inscribed* in the triangle formed by

$$\begin{aligned} \alpha = -\{ &\alpha'^2(\cos^2 B + \cos^2 C) + \alpha'\beta'\cos C\,(1+\cos^2 B) \\ &+ \alpha'\gamma'\cos B\,(1+\cos^2 C) + 2\beta'\gamma'\cos B\cos C\}, \\ \beta = &\alpha'^2\cos C\,(1+\cos^2 A) + \alpha'\beta'(\cos^2 A + \cos^2 C) \\ &- \alpha'\gamma'\cos A\,(1+\cos^2 C) - 2\beta'\gamma'\cos C\cos A, \\ \gamma = &\alpha'^2\cos B\,(1+\cos^2 A) - \alpha'\beta'\cos A\,(1+\cos^2 B) \\ &+ \alpha'\gamma'(\cos^2 A + \cos^2 B) - 2\beta'\gamma'\cos A\cos B, \end{aligned}$$

and two other points whose coordinates can easily be written down from cyclic symmetry.

The three former set of points can easily be recognized to be the orthopoles K, L, M of AP, BP, CP respectively. And the joins of the three latter set of points are the chords of intersection of the pedal circle of P with the circles on AP, BP, CP as diameters.

Let us now consider the theorem proper.

Let x,y,z be the normal coordinates of the focus of an inconic whose axis major passes through $P(\alpha',\beta',\gamma')$.

The equation of the major axis is

$$\alpha x(y^2-z^2)+\beta y(z^2-x^2)+\gamma z(x^2-y^2)=0.$$

If this passes through α',β',γ', we have

$$\alpha' x(y^2-z^2)+\beta' y(z^2-x^2)+\gamma' z(x^2-y^2)=0. \quad \ldots \quad \ldots \quad (D)$$

The equation of the auxiliary circle, *viz.* the pedal circle, of (x,y,z) is

$\Sigma\beta\gamma \sin A.\ \Sigma yz \sin A.\ \Sigma x \sin A.$

$=\Sigma\alpha \sin A.\ \Sigma\alpha x \sin B \sin C\ (y+z\cos A)\ (y\cos A+z)$

Arranging this in terms of x,y,z and powers, we have

$$ax(y^2+z^2)+by(z^2+x^2)+cz(x^2+y^2)+k\,xyz=0 \ldots \quad \ldots \quad (E)$$

where a, b, c denote the circles on BC, CA, AB as diameters

i.e.

$$a\equiv\beta\gamma+\gamma\alpha\cos C+\alpha\beta\cos B-\alpha^2\cos A$$

$$b\equiv\beta\gamma\cos C+\gamma\alpha+\alpha\beta\cos A-\beta^2\cos B$$

$$c\equiv\beta\gamma\cos B+\gamma\alpha\cos A+\alpha\beta-\gamma^2\cos C$$

and

$$k\equiv 2\Sigma\beta\gamma\cos B\cos C-\Sigma\alpha^2(1+\cos^2 A).$$

Now let $x(y^2-z^2)=l$, $y(z^2-x^2)=m$, $z(x^2-y^2)=n$, the relation (D) assumes the form

$$l\alpha'+m\beta'+n\gamma'=0 \quad \ldots \quad \ldots \quad \ldots \quad (F)$$

and with the aid of Lemma I, the relation (E) becomes

$$al(m^2+n^2-l^2)+bm(n^2+l^2-m^2)+cn(l^2+m^2-n^2)=k\,l\,m\,n. \quad (G)$$

Now (G) is the auxiliary circle of the inconic whose axis major is

$$l\alpha+m\beta+n\gamma=0.$$

Transforming (G) with the aid of (F), we have

$$\Sigma\, a\, l^2\,(m\gamma'.\ \overline{\alpha'^2-\beta'^2}+n\beta'.\ \alpha'^2-\gamma'^2)$$
$$+l\,m\,n\,\{k\alpha'\beta'\gamma'+\Sigma\, a\,\alpha'\,(\beta'^2+\gamma'^2)\}=0 \quad \ldots \quad (H)$$

The co-efficient of $l\,m\,n$ in (H) can easily be identified with the pedal circle of α', β', γ'. Hence, denoting this by Θ, the auxiliary circle becomes after further reduction with the aid of $l\alpha+m\beta'+n\gamma'=0$,

$l\,m\,n\,\alpha'\,\Theta+(m\gamma'.\ \overline{\alpha'^2-\beta'^2}+n\beta'.\ \overline{\alpha'^2-\gamma'^2})\,(a\,l^2\,\alpha'+b\,m^2\beta'+cn^2\gamma')=0.$

Here let $a\,l^2\,\alpha'+bm^2\beta'+cn^2\gamma'=0$ be denoted by ϕ and $m\,\gamma'\,(\alpha'^2-\beta'^2)$ $+n\,\beta'\,(\alpha'^2-\gamma'^2)\equiv\lambda$ the auxiliary circle may be written in the form

$lmn\,\alpha'\Theta+\lambda\phi=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (J)$

Now the relation J shews that the circles ϕ and Θ are coaxal with the auxiliary circle. And Lemmas II and III tell us that ϕ is the director circle of the inconic touching $l\alpha+m\beta+n\gamma=0$ at (α',β',γ')

This proves the first part of the general theorem of this paper, *viz* that the three circles are coaxal. Now, it remains to prove the second part, *viz.* the common radical axis envelopes the orthopolar conic of $(\alpha' \beta' \gamma')$.

Theorem II.—The radical axis of θ and ϕ is found to be

$$\alpha(p_1 l^2 + p_2 m^2 + p_3 n^2) + \beta(q_1 l^2 + q_2 m^2 + q_3 n^2) + \gamma(r_1 l^2 + r_2 m^2 + r_3 n^2) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (M.)$$

where

$$p_1 = \frac{\alpha'}{\sin A}\Big\{ \alpha' \sin B \sin C\,(\beta' + \gamma' \cos A)\,(B' \cos A + \gamma') - \cos A\, \Sigma\beta'\gamma' \sin A.\ \ \Sigma\alpha' \sin A \Big\}$$

$$p_2 = \alpha'\beta' \sin C\,(\beta' + \gamma' \cos A)\,(\beta' \cos A + \gamma')$$

$$p_3 = \alpha'\gamma' \sin B\,(\beta' + \gamma' \cos A)\,(\beta' \cos A + \gamma');$$

and symmetrical values for q and r.

We have now to find the envelope of M subject to the condition $l\alpha' + m\beta' + n\gamma' = 0$. The envelope is easily seen to be of the form

$$BC\alpha' \sin A + CA\beta' \sin B + AB\gamma' \sin C = 0, \quad \ldots \quad \ldots \quad (N)$$

where

$$\frac{A\alpha'}{\sin A} \equiv \alpha p_1 + \beta q_1 + \gamma r_1$$

$$\frac{B\beta'}{\sin B} \equiv \alpha p_2 + \beta q_2 + \gamma r_2$$

$$\frac{C\gamma'}{\sin C} \equiv \alpha p_3 + \beta q_3 + \gamma r_3.$$

The relation (N) shews that the envelope which is evidently a conic is circumscribed to the triangle formed by the three straight lines A, B, C. And on finding the normal coordinates of the vertices of the triangle formed by A, B, C they are easily found to be the *former* set of points in Lemma IV.

Hence the whole theorem is proved.

Euclid's Book on Divisions of Figures. *

Euclid, it is now well known, was the author of no less than nine works. Four of these—the *Elements*, the *Data*, the *Optics*, and the *Phenomena*, have been carefully preserved and are well known, even to the extent that the first, perhaps, is in the hands of every schoolgirl. On the fifth, the *Porisms*, there is an elaborate commentary of Pappus, while in the case of the *Pseudaria*, the *Surface loci* and the *Conics* however, our fragmentary knowledge, derived wholly from Greek sources, enables us to make a conjecture of the vaguest nature, as to their possible contents. Of the ninth book, *On Division* (*of figures*), nothing however was known till about the middle of the last century ; nor, curiously enough, is there any trace of it in the extant Greek sources of mathematics, except perhaps a casual mention by Proclus. But it is however certain, that the Problems of this book, thirty-six in number, concerned mainly with the division of rectilinear figures and the circle into a certain number of 'like' and 'unlike' parts by straight lines, was very popular among Arabians, as well as in Europe. This popularity was perhaps largely due to the possible practical applications of the problems in the division of parcels of lands of various shapes, the areas of which, according to the Rhind papyrus, were discussed in empirical fashion about 1800 B. C. Indeed, in the first century before Christ, we find that Heron of Alexandria, dealt with the division of surfaces and solids in the third book of his *Metrika*, (*Surveying*) though not in the actual form in which Euclid dealt with the same. But nothing in a definite form was known until Woepcke, whose name is so familiar to every oriental researcher, discovered at the *Bibliotheque Nationale* in Paris, a treatise in Arabic on the division of plane figures, which he translated (into French) and published in the *Journal Asiatique*, September 1851, (XVIII), under the title "*Notice sur des traductions Arabes de deux ouvrages perdus d'Euclide*". In the original Arabic not only is the full account given (though unfortunately the proofs, save of four theorems, are passed over as being too easy !), but the scribe expressly attributes the authorship to Euclid, and contents himself with the unassuming office of a 'translator'. Such a treatise, significant alike for its historical interest as for its technical import, had to be restored and re-edited, if for no other reason, at least as a mark of our regard fro the author. Prof. Heiberg gives an account of the book only in a Latin publication, *Opera Omnia*, while Sir Thomas Heath, the foremost living authority in England on Greek Mathematics, dismisses the work in the brief space of two pages in his *Thirteen Books of Euclid's Elements* ; there is no reference to it even in the article on Euclid in the last

* By R. C. Archibald, Ph. D. Cambridge University Press, 1915.

edition of the *Encyclopœlia Brittanica.* Under these circumstances, Dr. R. H. Archibald of Providence, Rhode Island, has done a lasting service to the English reader in restoring in the English language this long buried treasure; and the scholarly way in which he has done his task is proved by the readiness with which the publication of the book was undertaken by the Cambridge University Press.

To follow in historical sequence how this treatise came to be unearthed or to dismiss the myth of Dee's making a copy of any Cottonian MSS earlier than 1563, is not our purpose in this article; the interested reader must be referred to the learned discussion in Archibald's book itself. But it must be noticed to what extent Woepcke's translation referred to above, is in agreement with and is supplemented by the writings of *Leonardo Pisano* (*Fibonaci*), who occupies such an important place in the history of Mathematics of the 13th century. It is well known that in 1220 he wrote his *Practica Geometriae*, a book largely used by the Mathematicians of the middle ages, published nearly 650 years after its compilation by Prince Boncompagni, and still preserved in MS in the Vatican library. Section IV of this book is wholly devoted to the enunciation and proof and numerical illustration of propositions concerning the division of figures. No less than twenty-two of Woepcke's propositions are practically identical in statement with propositions in Leonardo; the solutions of eight more of Woepcke are either given or are clearly indicated by Leonardo, and all the six of the remaining Woepcke's are assumed as known in the book of Leonardo. Indeed the two works have a remarkable similarity. Not only are practically all of Woepcke's theorems in Leonardo, but proofs called for by the order of the theorems and by the auxiliary propositions are invariably the kind of proofs which Euclid might have given—no other propositions than those which had gone before or which were to be found in the *Elements* being regarded in the successive constructions.

In this treatise of Euclid, the figures divided are the triangle, the parallelogram, the trapezium, the quadrilateral, the circle and a figure formed by the arc of a circle and two lines; and the division is by transversals through assigned points or parallel to given lines into two or several parts in a given ratio, equality included. It is difficult to choose from this interesting collection but the three following are specially instructive, and suggest generalisations. Thus Steiner solved in 1827 how *through a given point on a sphere to draw arc of a great circle cutting two given great circles such that the intercepted area is equal to*

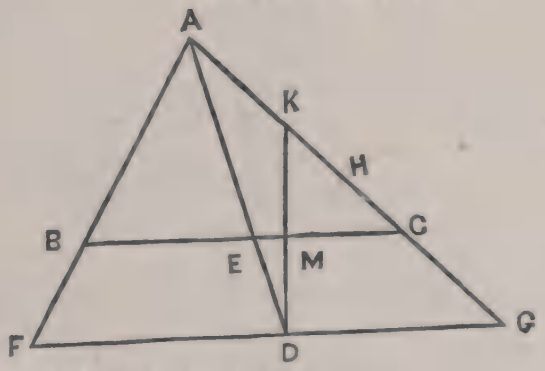
given area; and Bobillier solved: *to draw by means of planes and spheres only, a given line in a plane which shall cut off from a given cone of revolution a required volume.* The three theorems referred to above are given as propositions 27, 28 and 29 by Woepcke and are extracted below:

Proposition 27.

To cut off a certain fraction of a triangle by a straight line drawn from a given point situated outside of the triangle:

Let ABC be the given triangle and D, the given point outside. It is required to cut off from the triangle a certain fraction, say a third, by a line through D. Join AD cutting BC in E. If either BE or EC be a third of BC, then the line AD cuts off the required fraction from the given triangle. If it is not, produce AB and AC to meet in F and G respectively a line drawn through D parallel to BC. Make area $DG \cdot CH = \frac{1}{3}$ area $AC \cdot CB$ and apply to the line CH, a rectangle equal to the rectangle $GC \cdot CH$, but exceeded by a square. Then

$$GC \cdot CH = HK \cdot KC$$



and draw the line KD cutting BC in M.

Then KMC shall be a third of the triangle ABC.

Proof. For since area $GC.CH$ = area $CK \cdot KH$,

and therefore $GC : CK = KH : HC$,

it follows that $GK : CK = CK : CH$,

But $GK : KC = DG : CM$,

Hence $DG \cdot CM = CK : CH$

i.e. $CK \cdot CM = CH \cdot DG$.

But area $DG \cdot CH = \frac{1}{3}$ area $AC \cdot CB$,

since $\triangle KCM : \triangle ACB = CK \cdot CM : CA \cdot CB$, it follows that $\triangle KCM = \frac{1}{3} \triangle ACB$.

In a similar manner any part of a triangle may be cut off by a straight line drawn from a given point on a side of the triangle produced, &c.

Q.E.F.

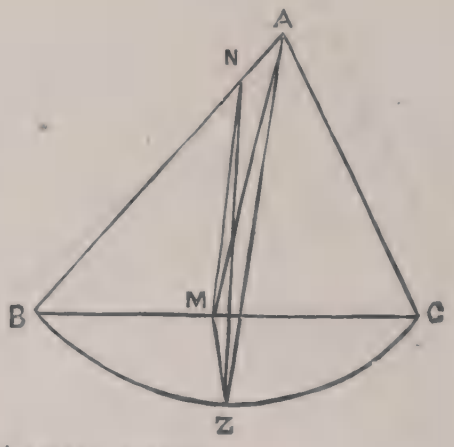
Proposition 28.

To divide into two equal parts a given figure bounded by an arc of a circle and by two straight lines which form a given angle.

Let ABC be the given figure bounded by the arc BC and two lines AB and AC which include the angle BAC. It is required to divide by means of a straight line the figure ABC into two equal parts.

Construction. Bisect the line BC at M, and erect MZ perpendicular to BC join MA. If AMZ are in a line, obviously the figure is divided into two equal parts by AZ. If not; join AZ, and through M draw a line parallel to AZ cutting AB at N; and join NZ.

Then NZ shall be the required line, and the figure BNZ shall be equal in area to the figure NACZ.



Proof. Obviously, $\Delta ABM = \Delta ACM$.

and figure BMZ = figure CMZ.

Hence figure ABZM = figure ACZM.

But $\Delta MAZ = \Delta NAZ$.

Therefore, figure ACZN = figure ACZ + ΔNAZ.

= figure ACZ + ΔMAZ.

= figure ACZM,

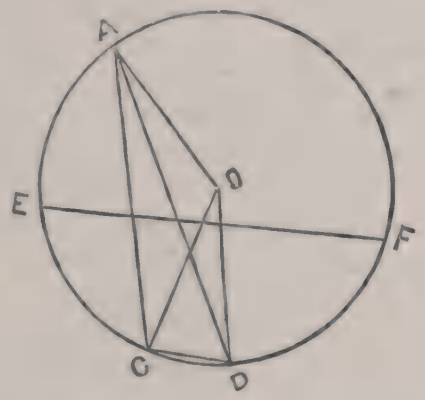
= $\frac{1}{2}$ the figure ABZC.

Q E,F.

Proposition 29.

"*To draw in a given circle two parallel lines cutting off a certain fraction, say, a third, from the circle.*

Take the circle with centre O, and inscribe in it the equilateral triangle ABC. Join AO and CO., and draw OD parallel to AC. Join CD. Bisect the arc AC at E and draw EF parallel to CD. Then shall the figure EC DF be a third of the given circle. Join AD.



Proof. Since AC and OD are parallel,

$$\triangle ACO = \triangle ACD ;$$

and, adding to each side the figure AEC, the figure AECD = sector ACO = $\frac{1}{3}$ of the circle.

Again, since EF and CD are parallel,

arc DF = arc EC

= the equal arc AE.

Add arc ED to these.

$\therefore$ arc AD = arc EF,

and chord AD = chord EF ;

and segment AED = segment ECDF

Take away the common segment CD,

and the figure ECDF = figure AECD = $\frac{1}{3}$ of the circle,

as was required to be shown.

Q. E. F.

In the Arabic Manuscript it seems to have been stated that the method is also available, when the fraction is *any* fraction. But Dr. Archibald hastens to add in a footnote that this problem is clearly not susceptible of solution with ruler and compasses, in such a case as when a

'certain fraction" $1/n$, is one-seventh. In fact the only cases in which the problem is possible, for a fraction of this kind, is when n is of the form $2^p(2^{2^{s_1}}+1)(2^{2^{s_2}}+1)\ldots(2^{2^{s_m}}+1)$ where p, and s's (all different), are positive integers or zero, and $2^{2^{s_m}}+1 (m=1, 2,\ldots m)$ is a prime number.

It is no wonder that such interesting theorems were considered independently or by suggestion by mathematicians, great and small. Thus the problems "*To bisect, by a straight line, an equilateral, but not equiangular heptagon*", "*Through a point without a triangle to draw a line which will cut off a third*" went as a challenge from L. F. (L. Ferari) to N. T (N. Tartaglia). Kepler, in his investigations of the path of a planet was led in 1609 to the problem "*To divide the area of a semicircle in a certain ratio by a straight line drawn through a given point on the diameter or on the diameter produced*" and this was attacked among others by Wallis, Cassini, Lagrange and Laplace. Newton and Bernoulli, D'Alembert and Euler, Huyghens and L'Hospital attacked several of the problems that have been given in this treatise of Euclid. These theorems, or their immediate generalisations have also fascinated the readers of popular magazines: thus in 1814, in the *Gentlemen's Diary* was solved "*within a given triangle to find thus, that if lines be drawn from it to cut each side at right angles, the three parts into which the triangle thus becomes divided, shall obtain a given ratio*"; while a century later in 1914 we find in the two most popular mathematical journals, the problem how "*to divide a circle into two equal parts by means of an arc with its centre on the circumference of the given circle*" (Q. 4327, *L'Intermediare des Mathematiciens*) and "*to divide a square into five right angled triangles the areas of which shall be in Arithmetic progression*" (Q. 17197 *Educational Times*). Among Indians who interested themselves in such problems, Dr. Archibald mentions the name of the late Mr. S. Radhakrishna Aiyar of Pudukotah.

SHORT NOTES.

On Legendre's relation in Elliptic Functions.

§ 1. The relation $EK'+E'K-KK'=\frac{\pi}{2}$,

connecting the two kinds of complete Elliptic integrals was first given by Legendre in his *Exercises de Calcul*, and several proofs of it are well known. [Cf. Dixon: *Elliptic Functions* § 65; Greenhill: *Elliptic Functions* § 121; Whittaker and Watson: *Modern Analysis*, § 22·735, Appellet Lacour: *Fonctions Elliptiques*, § 80; or Jacobi: *Fundamenta Nova*, p. 161]. Following however, the proof (cf. Forsyth: *Theory of Functions*, § 129 or Henry: *Fonctions Elliptiques*, p. 53) of the analogous relation in the Weierstrassian theory, viz, $\eta_1\omega_2-\eta_2\omega_1=\frac{\pi i}{2}$, we may proceed directly as follows.

We assume of course that

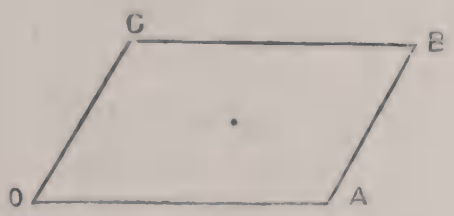
$$E(u)=\int_0^u \mathrm{dn}^2\, u\, du$$

and that the function dn u is a doubly periodic function of u with periods $2K$ and $4iK'$, that it is analytic except at the points congruent to iK' and to $3iK'$ which are simple poles, the residues thereat being $-i$ and $+i$ respectively.

It follows that $\mathrm{dn}\, u=-i/u+O(u)$ at $u=iK'$

$$\mathrm{dn}^2\, u=-1/u^2+O(1)$$

whence we notice that $E(u)$ has got a simple pole with residue 1 at $u=iK'$ which is the only pole in the parallelogram $[-K, K, K+2iK', -K+2iK']$



Integrating now $E(u)$ along the parallelogram OABC, we have

$$\int_{ABCO} E(u)du=2\pi i\times \text{the sum of the residues}=2\pi i,$$

$$i.e.\int_{-K}^{K} E(u)du+i\int_{0}^{2K} E(K+iv)dv+\int_{K}^{-K} E(u+2iK')du+i\int_{2K'}^{0} E(-K+iv)dv \Bigg\} =2\pi i,$$

$$i.e., \quad \int_{-K}^{-K}[E(u+2iK')-E(u)]du - i\int_{0}^{2K'}[E(-K+iv)-E(K+iv)]dv = 2\pi i$$

$$i.e., \quad \int_{-K}^{-K} 2i(K'-E')du+i\int_{0}^{2K'} 2E\,du = 2\pi i$$

by the periodicity formulæ, cf. Dixon *Elliptic Functions* § 64.

$$i.e., \qquad 4i(EK'+E'K-KK')=2\pi i$$

which establishes Legendre's relation connecting the two complete Elliptic Integrals.

§ 2. In passing, it may also be noticed that the above relation is only a special case of a more general result in the theory of Abelian functions (Cf. Baker: *Abelian Functions* p.p. 197—200): that between the modulii of periodicity A', B',...of a function P (z) of the first kind (Normal function on a Riemann Surface) and A, B,...of a function Q (z) of the second kind, all the infinities of which are simple,

$$\sum_{r=1}^{p}(A_rB_r'-B_rA_r')+2\pi i\Sigma H_s\left(\frac{dP}{dz}\right)_{a_s}$$

where a_s is a simple pole of $Q(z)$ and H_s its residue thereat. Legendre's relation is of course included in this: for, we know $P(z)\equiv F(z)$ in the notation of the Elliptic Functions

$$=\int_0^z[(1-z^2)(1-k^2z^2)]^{-\frac{1}{2}}dz$$

$$\therefore \frac{dP}{dz}=\frac{1}{K} \text{ in the lower sheet, and} -\frac{1}{K} \text{ in the upper for } Z=\infty$$

and $$Q(z)\equiv E(z)=\int_0^z\left[\frac{1-K^2Z^2}{1-Z^2}\right]^{\frac{1}{2}}dz.$$

This is infinite at $z=\infty$ in each sheet, and the residue is $+K$ in the upper and $-K$ in the lower sheet, so that $\Sigma H_s\left(\frac{dP}{dz}\right)_{a_s}=-2$.

Moreover A_1' and B_1' are $4K$ and $2iK'$

A_1 and B_1 are $4E$ and $2i(K'-E')$, respectively.

Hence substituting in the above we have immediately

$$4E.2iK'-4K.2i(K'-E')-4\pi i=0,$$

which leads to $$EK'+KE'-KK'=\pi/2.$$

K. B. Madhava.

Astronomical Notes.

Eclipse of the Sun.

The third and last eclipse of the year will occur on December 3. This will be an annular eclipse of the sun. The central line runs from a point just inside the west coast of Africa to a point in the Pacific Ocean, crossing the Argentine Republic. The eclipse is partially visible over nearly the whole of South America and a considerable portion of Africa. The southern limit does not quite reach the South Pole.

Nova Aquilae.

The new star in Aquila was discovered by about a dozen different observers in England within a few hours of one another, and also by numerous other observers in different parts of the world.

Both as regards magnitude and changes in its spectrum, it seems to have followed pretty close the changes observed in other novae. Perhaps the most interesting point about it is that it appears to be identical with a star which has been photographed several times during the last quarter of a century, the magnitude of which appears to have varied from the eighth to the tenth magnitude. With the exception of the nova known as T Coronæ, (which was also exceptional in that it did not appear in the Milky Way), there is no other case in which a star is known to have existed previously in the place where a nova has appeared.

New Comet.

The first discovery of a comet this year was made by Mr Reid at the Cape of Good Hope in June, when it appeared as a faint round nebulosity. As yet only very meagre details are to hand, so that it is impossible to give any further information at present.

New Trojan Planet.

A new planet having the same mean distance as Jupiter and thus illustrating Lagranges particular Solution of problem of the three bodies has been discovered. This is the fifth such planet known, three of the planets librate in the neighbourhood of the triangular point on one side of the line joining the Sun and Jupiter, and the other two, of which the new discovery is one, librate about the apex of the equilateral triangle on the opposite side of the Sun—Jupiter line.

R. J. Pocock.

SOLUTIONS.

Question 685.

(S. MALHARI RAO) :—If n and p are any two positive integers, show that the fraction $\frac{1}{n}$ may be expressed as the sum of a series of p different fractions, the numerator of each of which is 1 and the denominator a positive integer. If the least of the fractions be $(n-1)!/(n+p-1)!$ find the rth fraction.

Solution by Prof. N. B. Mitra and N. Sankara Aiyar M. A.

The question is vague ; for, $\frac{1}{n}$ can be resolved into the sum of p unit-numerator fractions in several ways. Thus, one way would be as follows :—

We easily see that $$\frac{1}{n}=\frac{1}{n(n+1)}+\frac{1}{n+1}. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

By applying the same formula to $\frac{1}{n+1}$, the second fraction on the right, we have,

$$\frac{1}{n}=\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}+\frac{1}{n+2}.$$

By repeating the same process we get,

$$\frac{1}{n}=\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}+\ldots\ldots+\frac{1}{(n+p-2)(n+p-1)}+\frac{1}{n+p-1}.$$

Another way is to substitute in the first fraction on the right of (1), the value of $1/n$ given by (1). If this is repeated $p-1$ times we get ultimately

$$\frac{1}{n}=\frac{1}{n(n+1)^{p-1}}+\frac{1}{(n+1)^{p-1}}+\frac{1}{(n+1)^{p-2}}+\ldots\ldots+\frac{1}{n+1}.$$

A third way is as follows :—In (1) apply the same formula to $\frac{1}{n+1}$ in the first fraction on the right. We get

$$\frac{1}{n}=\frac{1}{n(n+1)(n+2)}+\frac{1}{n(n+2)}+\frac{1}{n+1}.$$

By applying the process altogether $p-1$ times, we get

$$\frac{1}{n} = \frac{1}{n(n+1)\ldots(n+p-1)} + \frac{1}{n(n+1)\ldots(n+p-3)(n+p-1)}$$
$$+ \frac{1}{n(n+1)\ldots(n+p-4)\,n+p-2)} + \ldots + \frac{1}{n(n+2)} + \frac{1}{n+1}$$
$$= \lfloor n-1 \left[\frac{1}{\lfloor n+p-1} + \frac{1}{\lfloor n+p-3\,(n+p-1)} \right.$$
$$\left. + \frac{1}{\lfloor n+p-4\,(n+p-2)} + \ldots + \frac{1}{\lfloor n-1\,(n+1)} \right]$$

The r^{th} fraction on the right is evidently $\frac{\lfloor n-1}{\lfloor n+p-r-1\,(n+p-r+1)}$

the least being the first.

This is perhaps what the proposer seeks by the second part of his question.

Question 954.

(K. B. Madhava):—When asked his age De Morgan once humorously observed: "I am one of those whose age shall be in a certain year belonging to the century of their birth the square root of that year. "For being born in 1806 A. D., he was 43 years old in 1849 and 1849=43. Show that the same observation will be true in the case of those born in 9 particular years in all subsequent centuries.

Solution by N. B. Mitra, K. J. Sanjana and R. D. Karve.

It would appear that the figure 9 in the last line of the question is wrong. The problem may be solved thus:—Let x be the year of birth, y the age at which the year is square of the age. Then we have to solve the indeterminate equation $y^2=x+y$ in non-zero positive integers subject to the conditions (1) that $x>1900$ and (2) that $y+$ Least positive residue of x, modulus $100>100$ this residue itself being >0. This gives 15 sets of values of x and y which are exhibited below:—

$x=2450$; 3306 ; 3422 ; 3540 ; 3906 ; 4032 ; 4422 ; 4830 ; 5112 ; 5402 ; 6006 ; 6320 ; 6806 ; 7310 ; 8010.

$y=50$; 58 ; 59 ; 60 ; 63 ; 64 ; 67 ; 70 ; 72 ; 74 ; 78 ; 80 ; 83 . 86 ; 90.

It may be interesting to know the years in the centuries previous to the 20th in which De Morgan's observation was true. They are given below together with the corresponding ages at which the year of the Christian era was square of the age:—

$x=2$; 6 ; 12 ; 20 ; 30 ; 42 ; 56 ; 72 ; 90 ; 110 ; 132 ; 156 ; 182 ; 210 ; 240 ; 272 ; 306 ; 342 ; 380 ; 420 ; 462 ; 506 ; 552 ;

$y=2$; 3 ; 4 ; 5 ; 6 ; 7 ; 8 ; 9 ; 10 ; 11 ; 12 ; 13 ; 14 ; 15 ; 16 ; 17 ; 18 ; 19 ; 20 ; 21 ; 22 ; 23 ; 24 ;

$x=600$; 702 ; 756 ; 812 ; 870 ; 930 ; 1056 ; 1122 ; 12600 ; 1332 ; 1406 ; 1560 ; 1640 ; 1722 ; 1806 (De Morgan's case.)

$y=26$; 27 ; 28 ; 29 ; 30 ; 31 ; 33 ; 34 ; 36 ; 37 ; 38 ; 49 ; 41 ; 42 ; 43 ;

Question 831.

(Enquirer) Find, by elementary methods, the invariant relation expressing the condition that hexagons may be inscribed in the conic $S=0$ which are also circumscribed to the conic $S'=0$.

If a hexagon ABCDEF circumscribes a conic S and is inscribed in S', then the Brianchon point of the Pascal line of the hexagon are pole and polar.

Solution by A. C. L. Wilkinson.

Projecting S' into a circle with the Brianchon point as centre we have hexagons inscribed in $S' \equiv x^2+y^2-c^2=0$ and circumscribed to

$$S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0.$$

To find the relation between a, b, c consider that hexagon ABCDEF of which A and D lie on the minor axis of the ellipse and BC, EF touch the ellipse at the extremities of the major axis.

Let AB touch the ellipse at $(a\cos\alpha, b\sin\beta)$ then A is $\left(0, \frac{b}{\sin\alpha}\right)$ and B is $\left(a, b\tan\frac{\alpha}{2}\right)$.

Hence $$c^2=a^2+b^2\tan\frac{2a}{2}=b^2\operatorname{cosec}^2\alpha$$

giving $$(c^2-a^2+b^2)^2=4c^2(c^2-a^2)$$

or $$3c^4-2c^2(a^2+b^2)-(a^2+b^2)^2+4a^2b^2=0 \quad \ldots \quad \ldots \quad (1)$$

Also $\Delta=-\frac{1}{a^2b^2}$, $\Theta=-\frac{1}{a^2b^2}(a^2+b^2+c^2)$, $\Theta'=-\frac{a^2b^2+a^2c^2+b^2c^2}{a^2b^2}$, $\Delta'=-c^2$

write $p=\frac{\Theta}{\Delta}=a^2+b^2+c^2$, $q=\frac{\Theta}{\Delta}a^2b^2+a^2c^2+b^2c^2$, $r=\frac{\Delta'}{\Delta}=a^2b^2c^2$.

Then a^2, b^2, c^2 are the roots of

$$x^3-px^2+qx-r=0$$

and from (1) $$3x^2-2x(p-x)-(p-x)^2+\frac{4r}{x}=0$$

or $$4x^3-p^2x+4r=0.$$

Write $2p$ for p, so that $x^3-2px^2+qx-r=0$, $x^3-p^2x+r=0$; whence

$$2x^2-2px+q-p^2=0 \text{ and } 2px^2-(p^2+q)x+2r=0;$$

which gives

$$\{(p^4-q^2)+4pr\}(p^2-q)+2\{p^3-pq+2r\}^2=0,$$

or $$(3p^2+q)(p^2-q)^2+12pr(p^2-q)+8r^2=0,$$

Hence the invariant relation is
$(3\Theta^2+4\Delta\Theta')(\Theta^2-4\Delta\Theta')^2+96\Theta\Delta'^2\Delta'(\Theta^2-4\Delta\Theta')+512\Delta'^3\Delta'^4=0$

This will be found to agree with $\begin{vmatrix} D, & E \\ E, & F \end{vmatrix}=0$ of Salmon after removing the factor $\Theta^2-4\Delta\Theta'$, and is easily identified with Halphen's form which is
$4(\Theta^3-4\Delta\Theta\Theta'+8\Delta^2\Delta')^2+4(\Theta^3-4\Delta\Theta\Theta'+8\Delta^2\Delta').8\Delta^2\Delta'-(\Theta^2-4\Delta\Theta')^3=0.$

Question 844.

(A. A. KRISHNASWAMIENGAR, M. A.) :—Shew that the locus of the centre of the conic of minimum eccentricity having contact of the third order with the conic $Ax^3+By^3=1$, has for its equation $(x^3+y^3)\sqrt{(Ax^2+By^2)}=(Ax^3-By^3)\left(\frac{1}{A}-\frac{1}{B}\right)$

Solution (1) *by N. B. Mitra, Hemraj, and S. V. Venkatachala Aiyar;*
(2) *by F. H. V. Gulasekharam.*

(1) The equation giving the eccentricity of the general conic in rectangular coordinates is $\frac{e^4}{1-e^2}=\frac{(a+b)^2}{ab-n^2}-4$

If $e<1$, e will be least when $\frac{(a+b)^2}{ab-n^2}$ is least since $\frac{e^4}{1-e^2}$ in this case diminishes continually with e.

Now the conic having four pointic contact with $Ax^2+By^2=1$...(1) at the point $\left(\frac{\cos\theta}{\sqrt{A}}, \frac{\sin\theta}{\sqrt{B}}\right)$ has for its equation
$Ax^2+By^2-1-\lambda(\sqrt{A}x\cos\theta+\sqrt{B}y\sin\theta-1)^2=0$ (2)

The eccentricity of this is least when
$$\frac{\{A(1-\lambda\cos^2\theta)+B(1-\lambda\sin^2\theta)\}^3}{AB(1-\lambda\cos^2\theta)(1-\lambda\sin^2\theta)-AB\sin^2\theta\cos^2\theta}$$
or $\frac{\{A+B-\lambda(A\cos^2\theta+B\sin^2\theta)\}^3}{AB(1-\lambda)}$ or $\frac{(A\cos^2\theta+B\sin^2\theta)^3}{AB(1-\lambda)\left\{\frac{A\sin^2\theta+B\cos^2\theta}{A\cos^2\theta+B\sin^2\theta}+1-\lambda\right\}}$
is least or putting $p=A\sin^2\theta+B\cos^2\theta$ & $q=A\cos^2\theta+B\sin^2\theta$, when
$\frac{q^3}{AB}\left\{\frac{\frac{p^3}{q^3}}{(1-\lambda)}+2\frac{p}{q}+(1-\lambda)\right\}$ is least *i.e.* when $1-\lambda=\frac{p}{q}$
or $\lambda=\frac{(A-B)\cos 2\theta}{q}$... (3).

The centre of (2) is given by $A(1-\lambda\cos^2\theta)x-\lambda\sqrt{AB}\cos\theta\sin\theta y+\lambda\sqrt{A}\cos\theta=0.$

$$\&-\lambda\sqrt{AB}\cos\theta\sin\theta x+B(1-\lambda\sin^2\theta)y+\lambda\sqrt{B}\sin\theta=0.$$

i.e. by $\lambda\cos\theta(\sqrt{A}x\cos\theta+\sqrt{B}y\sin\theta-1)=\sqrt{A}.x\ldots(4).$

and $\lambda\sin\theta(\sqrt{A}x\cos A+\sqrt{B}y\sin\theta-1)=\sqrt{B}.y\ldots(5).$

Multiplying (4) by $\cos\theta$ & (5) by $\sin\theta$ and adding

$$\lambda(\sqrt{A}x\cos\theta+\sqrt{B}y\sin\theta-1)=\sqrt{A}x\cos\theta+\sqrt{B}y\sin\theta$$

or $\sqrt{A}x\cos\theta+\sqrt{B}y\sin\theta=\dfrac{\lambda}{\lambda-1}=\dfrac{(B-A)\cos^2\theta}{p}$ from (3) ... (6)

also from (4) and (5) $\dfrac{\sqrt{A}x}{\cos\theta}=\dfrac{\sqrt{B}y}{\sin\theta}=\sqrt{\dfrac{Ax^2+By^2}{1}}=K$ suppose.

$$\therefore\ \cos\theta=\frac{\sqrt{Ax}}{k},\ \sin\theta=\frac{\sqrt{By}}{k}\therefore\cos^2\theta=\frac{Ax^2-By^2}{k^2}$$

$$\text{and } p=\frac{AB(x^2+y^2)}{k^2}$$

$$\therefore\ \text{from (6)},\frac{Ax^2+By^2}{k}=\frac{(B-A)(Ax^2-By^2)}{AB\ (x^2+y^2)}\ \text{or}\ (x^2+y^2)\sqrt{(Ax^2+By^2)}=\left(\frac{1}{A}-\frac{1}{B}\right)(Ax^2-By^2)$$

[See: Wolstenholme, Ex. 1209, p. 202.]

(2) The equation to the conic having contact of the third order with $Ax^2+By^2=1$ at any point P (x_1, y_1) on it is

$$Ax^2+By^2-1-k(Axx_1+Byy_1-1)^2=0 \quad \ldots \quad \ldots \quad (1)$$

The coordinates (x, y) of the centre Q of this conic are given by the equations,

$$\frac{x}{x_1}=\frac{y}{y_1}=k(Axx_1+Byy_1-1)$$

$$=\lambda \text{ say} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Then since (x_1, y_1) is a point on the given conic,

$$\left.\begin{aligned}\lambda^2&=Ax^2+By^2\\ \text{and}\qquad k&=\frac{\lambda}{\lambda-1}\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Also from (2), it is clear that the centre of the conic (1) lies on the diameter through P of the given conic.

Again from Ex. 6 Chap XI, C. Smith's Conic Sections (Revised Edition), "Of all conics which have a third order contact with a given conic at a given point on it, the one of least eccentricity will be that in which one of the equi-conjugate diametres passes through that point."

Hence if r_1, r_2 be the semi—axes of the conic (1),

$$2PQ^2 = r_1^2 + r_2^2$$

$$\therefore 2[(x-x_1)^2+(y-y_1)^2] = \frac{A+B-k(A^2x_1^2+B^2y_1^2)}{\frac{1}{2}AB(1-k)}$$

(Use the equation to determine the length of the axes of a central conic given by the general equation of the second degree.)

Now substituting for x_1, y_1 and k from (2) and (3), the above equation becomes.

$$2\left[1-\frac{1}{\lambda}\right]^2(x^2+y^2) = \frac{A+B-\frac{A^2x^2+B^2y^2}{\lambda(\lambda-1)}}{\frac{AB}{(\lambda-1)^2}},$$

which, when simplified and Ax^2+By^2 is written for λ^2, reduces to

$$(x^2+y^2)\sqrt{Ax^2+By^2} = \left(\frac{1}{A}-\frac{1}{B}\right)(Ax^2-By^2)$$

giving the equation of the locus required.

Question 845.

(M. K. KEWALRAMANI) :—ABC is a triangle such that cos 2 B, cos 2 A, cos 2 C are in A. P. A rectangular hyperbola circumscribes this triangle and passes through its centroid. AD is the common chord of this hyperbola and the osculating circle at A meeting the opposite side in D. show that AD divides BC in the ratio $\sin \overline{A-B}\,[b \sin \overline{B-C} - a \sin \overline{C-A}] : \sin \overline{C-A}\,[c \sin \overline{B-C} - a \sin \overline{A-B}]$.

Solution by Hemraj, S. V. Venkatachala Aiyar.

If $\cos 2B + \cos 2C = 2\cos 2A$, it is easy to show that

$$b^2+c^2 = 2a^2, \frac{\sin(B-C)}{\sin(C-A)} = -2\frac{b}{a}, \frac{\sin(B-C)}{\sin(A-B)} = -2\frac{c}{a} \quad \ldots \quad \ldots \quad (1)$$

From the last two relations of (1), it follows at once that the ratio $\sin \overline{A-B}\,[b \sin \overline{B-C} - a \sin \overline{C-A}]$:

$$\sin \overline{C-A}\,[C \sin \overline{B-C} - a \sin \overline{A-B}] = 2b^2+a^2 : 2c^2+a^2.$$

So we have to show that AD divides CB in the ratio

$$2b^2+a^2 : 2c^2+a^2.$$

Now take the triangle ABC as the triangle of reference, and the equation of the rectangular hyperbola is

$$uyz+vzx+wxy=0$$

with the conditions $bc \cos A\, u + ca \cos B\, v + ab \cos C\, w = 0$
and $u + v + w = 0$,

from which we have $\dfrac{u}{b^2-c^2} = \dfrac{v}{c^2-a^2} = \dfrac{w}{a^2-b^2}$.

Hence the equation of the rectangular hyperbola takes the form

$$(b^2-c^2)\,yz + (c^2-a^2)\,zx + (a^2-b^2)\,xy = 0.$$

A is (1, 0, 0) and the tangent there at is $(a^2-b^2)\,y + (c^2-a^2)\,z = 0$; *i.e.* $y+z=0$ from the first relation of (1).

And any line through A is $\mu y + \nu z = 0$.

Hence the equation of a conic having a contact of the 2nd order at A with the hyperbola is

$$2(b^2-c^2)\,yz + 2(c^2-a^2)\,zx + 2(a^2-b^2)\,xy + (y+z)(\mu y + \nu z) = 0$$

It represents a circle if

$$\frac{\mu+\nu-2\left(b^2-c^2+\frac{1}{2}\mu+\frac{1}{2}\nu\right)}{a^2} = \frac{\nu-2(c^2-a^2)}{b^2} = \frac{\mu-2(a^2-b^2)}{c^2},$$

which gives

$$\mu = 2(a^2-b^2) - 2\frac{c^2}{a^2}(b^2-c^2) = \frac{2}{a^2}[(c^2-a^2)\,a^2 - c^2(b^2-c^2)]$$

$$= \frac{2}{a^2}(c^2-a^2)(2c^2+a^2)$$

and $\nu = 2(c^2-a^2) - 2\dfrac{b^2}{a^2}(b^2-c^2) = \dfrac{2}{a^2}[(a^2-b^2)\,a^2 - b^2(b^2-c^2)]$

$$= \frac{2}{a^2}(a^2-b^2)(2b^2+a^2) \text{ from (1).}$$

$$\therefore \quad \nu/\mu = (2b^2+a^2)/(2c^2+a^2).$$

Now the equation of the chord of curvature is evidently $\mu y + \nu z = 0$ and it divides CB in the ratio $\dfrac{-y}{z}$ *i.e.* $\dfrac{\nu}{\mu}$ *i.e.* $(2b^2+a^2)/(2c^2+a^2)$. Hence the result.

Question 862.

(Communicated by Mr. Hem Raj) :—Angles A, B, C of a triangle are trisected by AQ, AR; BR, BP; CP, CQ. Shew that the triangle PQR is equilateral.

Solution (1) *by F. H. V. Gulaskaram, H. R. Kapadia, Sadanand, K. R. Rama Iyer,* (2) *by Hemraj and S. V. Venkatachala Aiyar.*

(1) Let P, Q, R be such that $\hat{CBP} = \frac{B}{3}$, $\hat{PCB} = \frac{C}{3}$, $QCA = \frac{C}{3}$, $\hat{CAQ} = \frac{A}{3}$, $\hat{RAB} = \frac{A}{3}$, $RBA = \frac{B}{3}$,

Let BR, CQ meet at D ; CP, AR meet at E, and AQ, BP meet at F.
Hence P is the incentre of the $\triangle$ DBC

$$\therefore \quad BP = a \sin\frac{C}{3} \operatorname{cosec}\left(\frac{B+C}{3}\right)$$

$$= a \sin\frac{C}{3} \operatorname{cosec}\left(60-\frac{A}{3}\right)$$

Also $$a = 2R \sin A = \frac{R}{2} \sin A/3 \sin\left(60+\frac{A}{3}\right) \sin\left(60-\frac{A}{3}\right)$$

$$\therefore \quad BP = \frac{R}{2} \sin\frac{A}{3} \sin\frac{C}{3} \sin(60+A/3).$$

Similarly $$BR = \frac{R}{2} \sin\frac{A}{3} \sin\frac{C}{3} \sin(60+C/3)$$

$$\therefore \quad PR^2 = \frac{R^2}{4}\left\{\sin^2\left(\frac{A}{3}\right)\sin^2\left(\frac{C}{3}\right)\right\}\left[\sin^2\left(60+\frac{A}{3}\right)\right.$$

$$\left. + \sin^2\left(60+\frac{C}{3}\right) - 2\cos\frac{B}{3}\sin\left(60+\frac{A}{3}\right)\sin\left(60+\frac{C}{3}\right)\right]$$

$$= \frac{R^2}{4} \sin^2(A/3) \sin^2(B/3) \sin^2(C/3).$$

$$\therefore \quad PR = \frac{1}{2} R \sin\frac{A}{3} \sin\frac{B}{3} \sin\frac{C}{3}.$$

Similarly $RQ = QP = \frac{1}{2} R \sin A/3 \sin B/3 \sin C/3$.

Hence the triangle PQR is equilateral.

We can similarly shew that the $\triangle$ DEF is also equilateral.

(2) Put $A = 3\lambda, B = 3\mu, C = 3\nu$. In $\triangle AQR$,

$$\frac{\sin\alpha}{\sin\beta} = \frac{AR}{AQ} = \frac{c \sin\mu}{\sin(\lambda+\mu)} \times \frac{\sin(\lambda+\nu)}{b \sin\nu}$$

$$= \frac{\sin 3\nu \sin\mu \sin(\lambda+\nu)}{\sin 3\mu \sin\nu \sin(\lambda+\mu)}$$

$$= \frac{(3-4\sin^2\nu)\sin(60^\circ-\mu)}{(3-4\sin^2\mu)\sin(60^\circ-\nu)} = \frac{\sin(60^\circ+\nu)}{\sin(60^\circ+\mu)}$$

There arise three cases.

(*a*) $\alpha = 60^\circ+\nu$, $\beta = 60^\circ+\mu$.

(*b*) $\alpha = 60^\circ+\nu$, $\beta = 120^\circ-\mu$; or $\alpha = 120^\circ-\nu$, $\beta = 60+\mu$; (inadmissible).

(*c*) $\sin\alpha = k_1 \sin(60^\circ+\nu)$, $\sin\beta = k_1 \sin(60^\circ+p_1)$.

It is easy to show that (*c*) is also inadmissible, as

$$\frac{\sin\alpha}{\sin(60^\circ+\nu)} = \frac{\sin\beta}{\sin(60^\circ+\mu)}$$

and $$\frac{AR}{\sin\alpha} = \frac{AQ}{\sin\beta} = \frac{QR}{\sin\lambda}, \quad \therefore \quad \frac{AR}{k_1\sin(60^\circ+\nu)} = \frac{AQ}{k_1\sin(60^\circ+p_1)} = \frac{QR}{\sin\lambda}$$

i.e. $$\frac{AR}{\sin\alpha'} = \frac{AQ}{\sin\beta'} = \frac{QR}{\sin\lambda}$$

Now $\alpha+\beta+\lambda=180^\circ$ and $60^\circ+\nu+60+\mu+\lambda=180^\circ$, it follows that $\alpha'+\beta'+\lambda \neq 180^\circ$ unless $k_1=1$

Hence (a) is the only admissible solution.

$\therefore \quad \alpha=60^\circ+\nu,\ \beta=60^\circ+\mu,\ r=60^\circ+\lambda$

$\angle RPQ=360^\circ-\angle BPC-\alpha-\beta=60^\circ=\angle PRQ=\angle PQR$

$\therefore$ PQR is equilateral.

Question 913.

(Prof. K. J. Sanjana) :—If the expansion of $(\sec x+\tan x)$ is written in the form

$$1+S_1\frac{x}{1!}+S_2\frac{x^2}{2!}+S_3\frac{x^3}{3!}+\ldots\ldots$$

prove that

$$S_n=2^{n+2}\,I/\pi^{n+1}$$

where $I\equiv\int_0^a \frac{(\log\sec x)^n}{\sin x}\,dx$ or $\log(\sqrt{2}+1)\int_0^b(\log \operatorname{cosech} x)^n/\cosh x\,dx$,

according as n is odd or even and $a=\frac{1}{2}\pi$, $b=\log(\sqrt{2}+1)$.

Solution by Hemraj, R. Srinivasan, S. R. Ranganathan, C. Krishnamachary, H. R. Kapadia, A. K. Krishnaswami Iyer, and M. K. Kewalramani.

From Chap. XVII. Hobson's *Trignometry*, or Arts. 573, 574: *Diff. Cal.* by Edwards, we have

$$\sec x+\tan x=1+S_1\frac{x}{1!}+S_2\frac{x^2}{2!}+S_3\frac{x^3}{3!}+\ldots$$

where

$$S_n=\frac{2^{n+2}n!}{\pi^{n+1}}\left\{1+\left(-\frac{1}{3}\right)^{n+1}+\left(\frac{1}{5}\right)^{n+1}+\left(\frac{1}{7}\right)^{n+1}+\left(\frac{1}{9}\right)^{n+1}+\ldots\right\}$$

Hence we have to show that

$$(i)\int_0^a \frac{(\log\sec x)^{2n-1}}{\sin x}\,dx=(2n-1)!\left\{1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\ldots\right\}$$

$$(ii)\int_0^b \frac{(\log \operatorname{cosech} x)^m}{\cosh x}\,dx=2n!\left\{1-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\frac{1}{7^{2n+1}}+\ldots\right\}.$$

Question 919

(C. KRISHNAMACHARY) :—Prove that

$$1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{9}-\frac{1}{11}-\frac{1}{13}+\ldots=\frac{1}{\sqrt{2}}\log(1+\sqrt{2})$$

Solution (1) *by K. J. Sanjana and H. R. Kavadia*; (2) *by S. R. Ranganathan, R. Srinivasan and G. A. Kamtekar*, (3) *Hemraj, Sadanand and N. B. Mitra.*

Let $$S=x-\frac{x^3}{3}-\frac{x^5}{5}+\frac{x^7}{7}+\frac{x^9}{9}-\ldots;$$

then $\frac{dS}{dx}$

is the sum of two G.P.'s. Finding their sums and integrating we shall be able to evaluate S. The following is a trigonometrical method.

In Gregory's series change x to $x\sqrt{\iota}$ and multiply by $\sqrt{\iota}$; we get

$$\sqrt{\iota}\tan^{-1}(x\sqrt{\iota})=\iota\left(x-\frac{x^5}{5}+\frac{x^9}{9}-\ldots\right)+\left(\frac{x^3}{3}-\frac{x^7}{7}+\frac{x^{11}}{11}=\ldots\right)$$
$$=\iota S_1+S_2, \text{ suppose.}$$

Then $S=S_1-S_2$, and we have thus to separate $\sqrt{\iota}\tan^{-1}x\sqrt{\iota}$ into real and imaginary parts. Taking $+\sqrt{\iota}\equiv(1+\iota)/\sqrt{2}$, we have

$$\frac{1+i}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}+\frac{x}{\sqrt{2}}2\right.$$
$$=\frac{1+i}{\sqrt{2}}\left\{\frac{1}{2}\tan^{-1}\frac{x\sqrt{2}}{1-\frac{1}{2}x^2-\frac{1}{2}x^2}+\frac{1}{2}i\tanh^{-1}\frac{x\sqrt{2}}{1+\frac{1}{2}x^2+\frac{1}{2}x}\right\}$$
$$=\frac{1+\iota}{2\sqrt{2}}\left\{\tan^{-1}\frac{x\sqrt{2}}{1-x^2}+i\tanh^{-1}\frac{x\sqrt{2}}{1+x^2}\right\},$$

by the usual procedure. This gives at once

$$S_1=\frac{1}{2\sqrt{2}}\left(\tan^{-1}\frac{x\sqrt{2}}{1-x^2}+\tanh^{-1}\frac{x\sqrt{2}}{1+x^2}\right),\ S^2=\frac{1}{2\sqrt{2}}$$
$$\left(\tan^{-1}\frac{x\sqrt{2}}{1+x^2}-\tanh^{-1}\frac{x\sqrt{2}}{1+x^2}\right);$$

whence

$$S=\frac{1}{2\sqrt{2}}\left\{2\tanh^{-1}\frac{x\sqrt{2}}{1+x^2}\right\}=\frac{1}{2\sqrt{2}}\log\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}.$$

The result in the question follows by making $x=1$, which is allowable as the signs are $-$ and $+$ in pairs.

We also get

$$S_1+S_2=\frac{1}{\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2},$$

so that

$$1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\ldots=\frac{\pi}{2\sqrt{2}}.$$

(2)

We have for $-\frac{\pi}{2}<x<\frac{\pi}{2}$

(*Bromwich*: Art. 90. Ex. 2).

$$e^{xi}-\frac{e^{3xi}}{3}+\frac{e^{5xi}}{5}-\ldots=\frac{\pi}{4}+\tfrac{1}{2}i\log(\sec x+\tan x)$$

Equating the imaginary parts, we have,

$$\sin x-\frac{\sin 3x}{3}+\frac{\sin 5x}{5}-\ldots=\tfrac{1}{2}\log(\sec x+\tan x)$$

Putting $x=\frac{\pi}{4}$,

we get,

$$1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{9}-=-\sqrt{2}\log(\sqrt{2}+1)$$

$$=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$$

(3) It follows from $\tan^{-1}(\cos\theta+i\sin\theta)$ *or* $\frac{1}{2}\log\left(\frac{1+Z}{1-Z}\right)$

that if $0<x<\frac{\pi}{2}$ $\sin x-\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x-\frac{1}{7}\sin 7x+\ \ldots\ \ldots$

$=\frac{1}{2}\log(\sec x+\tan x)$

Putting $x=\frac{\pi}{4}$ we have the result.

Questions 919, & 968.

919. (C. KRISHNAMACHARY):—Prove that

$$1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{9}-\frac{1}{11}-\frac{1}{13}+=\frac{1}{\sqrt{2}}\log(1+\sqrt{2})$$

968. (V. ANANTARAMAN):—Sum the series

$$\frac{1}{3}-\frac{1}{7}+\frac{1}{11}-\frac{1}{15}+\frac{1}{19}\ldots$$

Solution (i) *by T. H. V. Gulasekharam, N. G. Leather and S. Muthukrishnan*; (ii) *L. S. Vaidyanathan and M. K. Kewalramani.*

(i) We have on page 303, Hobson's *Plane Trigonometry*, the formula

$\log(2\cos\frac{1}{2}\theta)=\cos\theta-\frac{1}{2}\cos 2\theta+\frac{1}{3}\cos 3\theta-\frac{1}{4}\cos 4\theta+\ldots$;

write $\theta=\frac{\pi}{4}$.

Then remembering that

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots$$

we deduce

$$1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} + \ldots = \frac{1}{\sqrt{2}} \log (1+\sqrt{2}). \ldots \quad \ldots (1)$$

Again, we have the formula

$$\tfrac{1}{2}\theta = \sin\theta - \tfrac{1}{3}\sin 2\theta + \tfrac{1}{5}\sin 3\theta - \tfrac{1}{4}\sin 4\theta + \ldots\ldots;$$

write $\theta = \frac{\pi}{4}$.

Then remembering

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$$

we deduce that

$$1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \ldots = \frac{\pi}{2\sqrt{2}} \quad \ldots \quad \ldots \quad (2)$$

Hence subtracting (1) from (2) and dividing the result by 2,

$$\frac{1}{3} - \frac{1}{7} + \frac{1}{11} - \frac{1}{15} + \frac{1}{19} - \ldots = \frac{\pi}{4\sqrt{2}} - \frac{1}{2\sqrt{2}} \log (1+\sqrt{2}).$$

(ii) Consider the equality.

$$\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \ldots = \int_0^1 \frac{x^{a-1}}{1+x^b}\, dx \quad (a, b, >0)$$

Here by giving particular values to a and b we can get the sum of not only the above series, but also of many others of its type.

The given series corresponds to the values $a=3$ and $b=4$.

$$\text{Hence } S = \int_0^1 \frac{x^2 dx}{1+x^4}$$

$$i.e.\ S = \frac{1}{4\sqrt{2}} \left[\log \frac{x^2 - x\sqrt{2}+1}{x^2 + x\sqrt{2}+1} \right]_0^1 + \frac{1}{2\sqrt{2}} \left[\tan^{-1}(x\sqrt{2}+1) + \tan^{-1}(x\sqrt{2}-1) \right]$$

$$= \frac{1}{4\sqrt{2}} \log \frac{2-\sqrt{2}}{2+\sqrt{2}} + \frac{1}{2\sqrt{2}} \left[\tan^{-1}(\sqrt{2}+1) + \tan^{-1}(\sqrt{2}-1) \right]$$

$$= \frac{1}{4\sqrt{2}} \log \frac{\sqrt{2}-1}{\sqrt{2}+1} + \frac{\pi}{4\sqrt{2}}$$

$$= \frac{\pi}{4\sqrt{2}} - \frac{1}{2\sqrt{2}} \log (\sqrt{2}+1)$$

Question 921.

(H. R. KAPADIA) :—Prove that the infinite product

$$(1+x^{2n})\left(1+\frac{x^{2n}}{3^{2n}}\right)\left(1+\frac{x^{2n}}{5^{2n}}\right) \quad \ldots \quad \ldots$$

$$=\frac{1}{2^{\frac{n}{2}}}\prod_{1}^{n-1}(\cosh \pi x\alpha_r+\cos\pi x\beta_r)$$

$$or=\frac{1}{2^{\frac{1}{2}(n-1)}}\cosh\frac{\pi x}{2}\prod_{1}^{n-2}(\cosh \pi x\alpha_r+\cos\pi x\beta_r).$$

according as n is even or odd, if α_r, β_r denote $\sin\frac{r\pi}{2n}$ and $\cos\frac{r\pi}{2n}$ respectively, where r is an odd number.

Solution by Hemraj and M. K. Kewalramani.

Consider the equation $\frac{x^{2n}}{k^{2n}}+1=0$; clearly $\frac{x}{k}=\cos\frac{2\rho+1}{2n}\pi+i\sin\frac{2\rho+1}{2n}\pi$ where $\rho=0, 1, 2 \quad \ldots \quad \ldots \quad (2n-1)$

i.e. if r is odd $=\cos.\frac{r\pi}{2n}+i\sin\frac{r\pi}{2n}=\beta_r+i\alpha_r$

If one root is w, the other must be $-w$, the third w^{-1}, and the fourth $(-w^{-1})$,—provided n is even. But if n is odd, there are two roots $\pm i$ corresponding to $\rho=\frac{n-1}{2}$ and $\rho=n+\frac{n-1}{2}$; and the other $(2n-2)$ roots can be divided into fours. Also it is evident that if any other factor be equated to zero, the roots will depend on k and w's must be the same.

Hence $$\frac{x^{2n}}{k^{2n}}+1=\prod_{\rho=0}^{n-1}\left(\frac{x^2}{k^2}-\omega^2{}_\rho\right)=(-1)^n\prod_{0}^{n-1}(\omega\rho)^2$$

$$\times\prod_{0}^{n-1}\left(1-\frac{x^2}{k^2\rho^2}\right)=\prod_{0}^{n-1}\left(1-\frac{x^2}{k^2\omega^2{}_\rho}\right)$$

$\therefore \prod(\omega_\rho{}^2)=\pm 1$ according as n is even or rodd.

Hence $\prod_{=1}^{\infty}\left(1+\frac{x^{2n}}{k^{2n}}\right)=\prod_{k=1}^{\infty}\left[\prod_{\rho=0}^{n-1}\left(-\frac{x^2}{k^2\omega_\rho^2}\right)\right]$

$$=\prod_{0}^{n-1}\left[\prod_{0}^{\infty}\left(1-\frac{x^2}{k^2\omega_\rho{}^2}\right)\right]$$

$$=\prod_{0}^{n-1}\left(\cos\frac{x}{2\omega_\rho}\right) \qquad \therefore k \text{ is odd} \quad \ldots \quad \ldots \quad (A)$$

$$=\prod_{0}^{2n-1}\left\{\cos\frac{\pi x}{2}(\beta_r-i\alpha_r)\right\} \qquad \therefore r=2\rho+1$$

$$=\prod_{1}^{2n-1}(\cos\tfrac{1}{2}\pi x\,\beta_r|\cosh\tfrac{1}{2}\pi x\,\alpha_r+i\sin\tfrac{1}{2}\pi x\,\beta_r\sinh\tfrac{1}{2}\pi x\,\alpha_r)$$

$$=\prod_{1}^{n-1}(\cos^2\tfrac{1}{2}\pi x\,\beta_r\cosh^2\tfrac{1}{2}\pi x\,\alpha_r+\sin^2\tfrac{1}{2}\pi x\,\beta_r\sinh^2\tfrac{1}{2}\pi x\,\alpha_r)$$

(n even)

or $=\cosh\tfrac{1}{2}\pi x\prod_{1}^{n-2}(\cos^2\tfrac{1}{2}\pi x\,\beta_r\cosh^2\tfrac{1}{2}\pi x\,\alpha_r+\sin^2\pi x\,\beta_r\sinh^2\tfrac{1}{2}\pi x\alpha_r)$

(n odd)

[Notice that $\cosh\frac{1}{2}\pi x$ corresponds to $\omega_\rho=i$ and that if one factor in (A) is $\cos\frac{\pi x}{2\omega_\rho}$, the other must be $\cos\frac{\pi x}{2}\omega_\rho$]

$$=\frac{1}{2^{\frac{1}{2}n}}\prod_{1}^{n-1}(\cosh\pi x|\alpha_r+\cos\pi x\,\beta_r) \text{ or } =$$

$$\frac{\cosh\frac{1}{2}\pi x}{2^{\frac{1}{2}(n-1)}}\prod_{1}^{n-1}(\cosh\pi x\,\alpha_r+\cos\pi x\,\beta_r)$$

Note.—This question was originally constructed by Glaisher. See Hobson's Trigonometry. Q. 32 Ch. XVII.

Question 924.

(S. Mallhari Rao.):—Complete the following pandiagonqual Bears.

24				
	30			
		1		
			7	
				13

Solution by R. J. Pocock and Sadanand

The numbers given may be written

$$20+4,\ 25+5,\ 0+1,\ 5+2,\ 10+3.$$

Consider the following squares A, B and C. If each number in A is multiplied by 5 and added to the corresponding number in as we get the required pandiagonal magic square C.

A

4	0	3	5	1
2	5	1	4	0
1	4	0	2	5
0	2	5	1	4
5	1	4	0	2

B

4	3	2	1	5
1	5	4	3	2
3	2	1	5	4
5	4	3	2	1
2	1	5	4	3

C

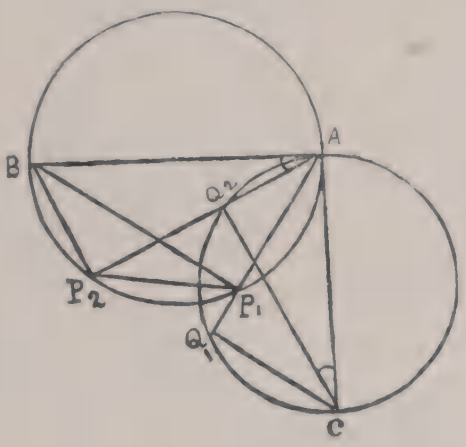
24	3	12	26	10
11	30	9	23	2
8	22	1	15	29
5	14	28	7	21
27	6	25	4	13

Question 926.

(S. R. RANGANATHAN):—If AB, AC be two straight lines intersecting at right-angles P_1, P_2 any two points on the circle on A B as diameter and Q_1, Q_2 the points of intersection of P_1 A and P_2 A respectively with the circle on AC as diameter, show that P_1P_2 and Q_1Q_2 are at right angles to each other.

Solution by R.J. Pocock, P.M.H. Baker, C. Krishnamachary, R. Srinivasan K.B. Madhava, P.R. Venkatakrishnaiya, Hemraj and many others.

In the circle AP_1P_2B $\angle P_2AB = \angle P_2P_1B$.



Also $P_2AB = AQ_1Q_2$ since BA touches the circle ACQ_1Q_2.

$\therefore$ $P_2P_1Q_1 + P_1Q_1Q_2 = BP_1Q_1 = 90°$.

Hence P_1P_2 and Q_1Q_2 are at right angles.

[An obvious extension of the Question is the following :—

"Any two circles intersect at A and A P_1Q_1, A P_2Q_2 are drawn to cut them again in (P_1,Q_1), (P_2,Q_2), Shew that the angle between P_1P_2 and Q_1Q_2 is equal to the angle between the circles."]

Questions 942 and 943.

942. (HEMRAJ) :—Show that if p be prime

$$\left[\sum_{1}^{p-1}\left(\frac{1}{n}\right) \div \frac{p^2}{p!}\right] \text{ is an integer}$$

943. (HEMRAJ) :—If x be prime and

$$(x+1)\,(x+2)\ldots\ldots(x+n-1)$$
$$\equiv x^{n-1}+A_1\,x^{n-2}+A_2\,x^{n-3}+\ldots A_{n-1}$$

shew that all the odd A's except A_1 are divisible n^2.

Solution and Remarks by H. Br.

Writing $f(x)$ for the given expression, we have

$$\begin{aligned}
f(-x) &= (-x+1)\,(-x+2)\ldots\ldots(-x+n-1)\\
&= (x-1)\,(x-2)\ldots\ldots(x-n+1)\\
&= (x-n+n-1)\,(x-n+n-2)\ldots(x-n+1)\\
&= f(x-n)\\
&= f(x)-nf'(x)+\frac{n^2}{2}f''(x)-\ldots\ldots\\
&= f(x)-n\{(n-1)\,x^{n-2}+(n-2)A_1\,x^{n-3}+\ldots\ldots\}+\ldots\ldots\\
&= f(x)-n\,(n-1)\,x^{n-2}+n^2\,F(x),
\end{aligned}$$

where $F(x)$ is a polynomial in x with integral co-efficients. [This follows from the fact that the A's, upto A_{n-2} are multiples of n.]

Hence

$$f(x)-f(-x)\equiv n\,(n-1)\,x^{n-2}+0\ (\text{mod},\ n^3).$$

This being an identity, we can equate co-efficients of powers of x. We get

$$A_1=\tfrac{1}{2}\,n\,(n-1)\,;\ A_3\equiv A_5\equiv\ldots\ldots\equiv A_{n-2}\equiv 0\ (\text{mod},\ n^2).$$

Note : By considering terms involving n^2 and n^3, we get

$$2\,A_k\equiv n\,(n-k)\,A_{k-1}$$

where k is odd and >3. This is easily proved.

Remarks on Q. 943 by Mr. N. G. Leathar St. Stephen's College, Delhi.

This problem was set by me in the Punjab University M. A. Exam. of this year in the following form :—

By considering the equation $(x-1)(x-2)\ldots(x-p+1)$

$$\equiv x^{p-1}-C_1x^{p-2}+C_2x^{p-3}+\ldots(-)^{p-1}(p-1)\,!=0$$

shew from the results of Qu. IV that, if p be a prime all the coefficients $C_1, C_2,\ldots$ are divisible by p, and that the odd coefficients $C_3, C_5,\ldots$ are divisible by p^2.

This was set as a rider to Newton's relations between the sums of the powers of the roots and the coefficients of an algebraical equation.

The reference to Qn. IV is to the theorem that :—

If S_r denote the sum of the r^{th} powers of the first $(p-1)$ natural numbers, $S_{2r+1}=S_1^2$. $F(S_1)$ and $S_{2r}=S_2\ G\ (S_1)$, where F and G are polynomials of degree $(r-1)$.

Newton's relations give

$$r\,C_r-S_1C_{r-1}+S_2C_{r-2}-S_3C_{r-3}+\ldots+(-)^rS_r=0 \quad \ldots \quad \ldots \quad (A)$$
$$(r=1, 2, \ldots, p-1)$$

I. $S_1, S_2, \ldots S_r$ are all divisible by p (assumed); therefore since r is prime to p, C_r is divisible by p, for all values of r from 1 to $(p-1)$.

II. If r is odd and $\not< 3$, S_r is divisible by p^2,

$S_1, S_2 \ldots S_{r-1}$ are all divisible by p,

and $C_{r-1}, C_{r-2} \ldots C_1$ are also all divisible by p.

Thus every term of (A), except the first, is divisible by p^2. Also r is prime to p.

Hence C_r is divisible by p^2, if r is odd and $\not< 3$

Question 944.

(Sadanand) :—Prove the identity

$$\phi(x)+S_1\phi'(x).\frac{x}{1!}-S_3\phi'''(x).\frac{x^3}{3!}+\ldots\ldots$$

$$\equiv\phi(2x)-S_2\phi''(2x)\frac{x^2}{2!}+S_4\phi''(2x).\frac{x^4}{4!}\ldots\ldots$$

where S_{2n} is the n^{th} Eulerian number and S_{2n-1} is the n^{th} "prepared Bernoullian" number.

Deduce that $\log 2=\frac{S_1}{1}-\frac{S_2}{2^2.1}-\frac{S_3}{3}+\frac{S_4}{2^5.2}+\frac{S_5}{5}-\ldots\ldots$

Solution by A. C. L. Wilkinson.

In the identity $\frac{2e^x}{e^x+e^{-x}}=1+S_1x-S_3\frac{x^3}{3!}+S_5\frac{x^5}{5!}-\ldots$

where $S_1, S_3, S_5, \ldots$ are the "prepared Bernoullians," put $\lambda\frac{d}{dx}$ for x and operate with both sides on $\phi(x)$, we get

$$\phi(x)+\lambda S_1\phi'(x)-\frac{\lambda^3}{3!}S_3\phi'''(x)+\ldots=\frac{2}{e^{\lambda\frac{d}{dx}}+e^{-\lambda\frac{d}{dx}}}\,\phi(x)$$

$$\frac{2}{e^{\lambda\frac{d}{dx}}+e^{-\lambda\frac{d}{dx}}}\,\phi(x+\lambda)$$

$$= \phi(x+\lambda) - \frac{\lambda^2}{2!}S_2\phi''(x+\lambda) + \frac{\lambda^4}{4!}S_4\phi''''(x+\lambda) - \ldots$$

Put $\lambda = x$ and we have the result given.

If we put $\phi(x) = \log x$ and then put $\lambda = 1$, we get $\log 2$ in the form given which is however a divergent series.

Question 949.

(C. KRISHNAMACHARY) :—With the usual notation for the numbers of Bernoulli and Euler, show that

$$2n = B_1 2^2 (2^2-1)\binom{2n}{1} - B_3 2^4 (2^4-1)\binom{2n}{4} + B_3 2^6 (2^6-1)\binom{2n}{6} - \ldots + (-1)^{n-1} B_n 2^{2n} (2^{2n}-1)$$

$$(2n+1) E_n = B_n 2^{2n} (2^{2n}-1)\binom{2n+1}{1} - B_{n-1} 2^{2n-2} (2^{2n-2}-1)\binom{2n+1}{3} + \ldots + (-1)^{n-1} B_1 2^2 (2^2-1)\binom{2n+1}{2} + (-1)^n (2n+1).$$

Solution by A. C. L. Wilkinson.

The first result is proved in Edwards: *Differential Calculus*, second edition, *p.*502. It can also be proved as follows :

$$\frac{2x}{e^{2x}-1} = 1 - n + \Sigma(-)^{n-1}\frac{2^{2n} B_n x^{2n}}{(2n)}$$

$$\frac{4x}{e^{4x}-1} = 1 - 2x + \Sigma(-)^{n-1}\frac{2^{4n} B_n x^{2n}}{(2n)};$$

whence subtracting

$$\frac{2x}{e^{2x}+1} = x + \Sigma(-)^n \frac{2^{2n}(2^{2n}-1) - B_n x^{2n}}{(2n)!}$$

or
$$e^{-x} = \left\{1 + \Sigma(-)^n 2^{2n}\frac{(2^{2n}-1) B_n x^{2n-1}}{2n!}\right\} \times \left\{1 + \Sigma\frac{x^{2n}}{2n!}\right\}.$$

Equating the coefficients of x^{2n-1}, we obtain the first result.

Again, in

$$\frac{2e^x}{e^{2x}+1} = \operatorname{sech} x = 1 + \Sigma(-)^n E_n \frac{x^{2n}}{(2n)!}$$

$$= \left\{1 + \Sigma(-)^n B_n \frac{2^{2n}(2n-1)}{(2n)!} x^{2n-1}\right\} \times \left\{1 + \Sigma\frac{x^n}{n!}\right\},$$

equating the coefficient of x^{2n} on both sides, we obtain the second result.

QUESTIONS FOR SOLUTION.

984. (S. R. Ranganatuan) :—A prism has for its cross section a quadrilateral ABCD with $B=75^\circ$, $C=105^\circ$ and $D=90^\circ$ and $\sqrt{3}\,AB=\sqrt{2}\,BC$. Show that a ray of light incident on the face AB at the angle $\sin^{-1}\frac{M}{2}$ and emerging out after once being totally reflected would undergo a deviation of 90°, M being the coefficient of refraction of the substance of the prism.

985. (M. K. Kewalramani) :—If A′, B′, C′ are points in the sides of the triangle ABC, such that

$$BA' : A'C = CB' : B'A = AC' : C'B = M : 1,$$

find an expression for the radius of the circumcircle of the triangle XYZ formed by the intersections of AA′, BB′, CC′.

986. (M. K. Kewalramani) :—At any point P of a curve equal arcs PQ, PR are measured in the same direction along the curve and along the parabola of closest contact. Prove that ultimately

$$QR=\left\{3+\tfrac{1}{3}\left(\frac{d\rho}{ds}\right)^2-\rho\frac{d^2\rho}{ds^2}\cdot\right\}\frac{(ds)^4}{24\rho^3}$$

987. (V. Tiruvenkatachari) :—Show that

$$\int_0^{\frac{1}{2}\pi}\sin\theta(\log\cos\theta)d\theta=-\tfrac{1}{3}\left\{1+\frac{2}{5}+\frac{2.4}{5.7}+\frac{2.4.6}{5.7.9}+\dots\right\}=-1.$$

988. (S. Malhari Rao) :—Complete the following magic squares of 16 cells, by placing 8 other prime numbers in the vacant cells :

7	43	71	113
73	97	23	41

989. (S. Malhari Rao) :—Give three solutions in positive integers of the equation

$$(x+y+z)^2=xyz.$$

990. (S. Malhari Rao) :—Give a solution in positive integers of

$$(x_1+x_2+\dots x_n)^{\frac{1}{2}(n-1)}=x_1.x_2.x_3\dots x_n$$

when n is odd ; and derive a method of solution of

$$(x_1+x_2+\dots x_n)^{\frac{1}{2}n}=x_1x_2\dots x_n$$

when n is even and greater than 2.

Hence deduce an expression for any fraction $\left(\frac{1}{n}\right)$ as the sum of a series of r fractions

$$\frac{1}{x_1}+\frac{1}{x_2}+\dots\frac{1}{x_r}.$$

991. (S. NARAYANAN) :—ABC is a triangle whose incentre is I and orthocentre H. If the incentre to the ex-medial triangle of AB I be K and KH bisected at L, prove that the nine-points centre of ABC is the middle point of LI.

992. (S. NARAYANAN):—The internal bisectors of the angles A, B,C of a triangle meet the sides of the medial triangle DEF at P, Q, R. Prove that the conic which touches these sides at P, Q, R touches also the principal Feuerbach tangent of DEF.

993 (COMMUNICATED BY PROF. HEMRAJ) :—From point P perpendiculars PL, PM, PN are drawn to the sides of a triangle ABC. If AL, BM, CN meet at a point Q, find the locus of P and prove that PQ passes through a fixed point.

994. (K. B. MADHAVA) :—Conduct the following game of Bridge with Hearts as trumps in which A has to lead and A and B together have to make 7 tricks.

	Hearts : Q. Clubs : 7 ; 6.	Diamonds : A ; 3 ; 2. Spades : K ; 8.
Hearts : 7 ; 6. Diamonds : J ; 9 ; 8. Clubs : Q ; J : 9. Spades : None.	B Y Z A	Hearts : 9, 8. Diamonds : 10 ; 5. Clubs : None. Spades : Q ; 7 ; 6 ; 5.
	Hearts : A ; K ; 5. Clubs : A ; 10 ; 8 ; 4.	Diamonds : K. Spades : None.

995. (T. P. TRIVEDI, M.A., LL.B.):—Prove that

$$\int_0^1 x^x x\,dx = \sum_{n=0}^{n=\infty} (-1)^n \left\{ \frac{1}{1^{n+1}} - n.\frac{n+1}{2^{n+2}} + \frac{n^2}{2!}\frac{(n+1)(n+2)}{3^{n+1}} - \frac{n^3}{3!}\frac{(n+1)(n+2)(n+3)}{4^{n+4}} + \cdots \right\}$$

996. (T. P. TRIVEDI, M.A, LL. B.) :—Find the conditions that the conics

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

and

$$a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0$$

have one asymptote in common.

997. (C. KRISHNAMACHARY) :—A uniform chain in partly coiled on a table with one end just carried over a smooth pulley at a height $2h$ vertically above the coil, and to this end is attached a mass M. Discuss the motion, shewing that the system will come to rest when M reaches the table provided its weight is nearly equal to that of a length h of the chain.

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PROGRESS REPORT.

2. The following gentlemen have been elected members of the Society :—

1. *Mr. Vaman M. Gaitonde, B.A.*, (Hon.)—Lecturer in Mathematics, Wilson College, Chowpatty, Bombay.
2. *Mr. Tenneti Surya Narayana*, B.A., (Cantab)—Lecturer in Mathematics, Ceded Districts College, Anantapur ;
3. *Mr. B. T. Krishna Aiyangar*, B.A.—Mathematics Assistant, Central College, 21 Kottige Lane, Bangalore (City) ;
4. *Mr. Bhimasena Rao*,—Research Student, Mysore University, Central College, Bangalore, (at concessional rate) ;
5. *Mr. M. V. Sundaresan*, B.A.—Teacher, Pachaiyappa's College School, Madras. (Concessional rate) ;
6. *Mr. S Muthukrishnan*, M.A.—Asst. Lecturer, Government College, Kumbakonam.

2. The recent Influenza Epidemic worked its cruel ravages in our society and we to-day mourn the sad and untimely death of the following members. All the five were young, promising and industrious workers in their own lines of action.

(1) *Mr. R. J. Pocock*, B.A., B.Sc.—Director, Nizamiah Observatory, Begumpet ;

(2) *Mr. J. R. Gunjikar*, B.A., B.Sc.—Assistant Accountant General, Calcutta ;

(3) *Mr. S. V. Setti*, B.A., A.M.I.C.E.—Professor, College of Engineering, Bangalore.

(4) *Mr. D. G. Dandekar* B. Sc., Professor of Physics, Maharajah College, Jaipur.

(5) *The Rev. T. Noronha*, St. Aloysius College, Mangalore.

3. The attention of our members is invited to the leaflet circulated along with the Journal, in connection with the next Conference of our Society.

D. D. KAPADIA,

Honorary Joint Secretary.

POONA,
1*st Dec.* 1918.

Infinite Series and Arithmetical Functions.

By Prof. F. Hallberg.

(*Concluded from p.* 264, *Vol. X.*)

IV. Laguerre's Series—(*continued.*)

17. Before proceeding with our analysis of Laguerre's series, it will be well to examine on what foundations our theory rests, in order to distinguish between established and conjectured results.

The following two theorems given by Bromwich (*Infinite Series*, p. 171, ex. 15), are of importance for our purpose.

I. If $v_n(x)$ decreases as n increases and

$$\operatorname*{Lt}_{x\to 1} v_n(x)=1,$$

and if Σa_n is convergent (or divergent to $+\infty$), then

$$\operatorname*{Lt}_{x\to 1} \Sigma a_n v_n(x)=\Sigma a_n \text{ (or } +\infty).$$

II. If $A_n=\sum_{h=1}^{n} a_h$, $B_n=\sum_{h=1}^{n} b_h$,

and if

$$\operatorname*{Lt}_{n\to\infty} B_n/A_n=l,$$

then

$$\operatorname*{Lt}_{x\to 1}\left\{\sum b_n v_n(x)\right\}\Big/\left\{\sum a_n v_n(x)\right\}=l,$$

provided that A_n is always positive, and that A_n tends to ∞.

18. Put now

$$\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n}=\frac{x}{1-x}\sum_{n=1}^{\infty} a_n v_n(x) \quad \ldots \quad \ldots \quad (18.1)$$

Thus

$$a_n=\frac{f(n)}{n}, \quad \ldots \quad \ldots \quad \ldots \quad (18.11)$$

$$v_n(x)=\frac{nx^{n-1}}{1-x^n}(1-x)$$

$$=\frac{x^{n-1}+x^{n-1}+\ldots+x^{n-1}}{1+x+\ldots+x^{n-1}} \quad \ldots \quad \ldots \quad (18.12)$$

$$\therefore \quad \operatorname*{Lt}_{x\to 1} v_n(x)=1 \quad \ldots \quad \ldots \quad (18.2)$$

Also

$$v_n(x)-v_{n+1}(x)=$$

$$=\frac{(x^{n-1}-x^n)+(x^{n-1}-x^{n+1})+\ldots+(x^{n-1}-x^{2n-1})}{(1+x+\ldots+x^{n-1})(1+x+\ldots+x^{n-1}+x^n)}>0, \quad \ldots \quad (18.21)$$

if $0<x<1$.

$$\therefore \qquad v_n(x)>v_{n+1}(x) \quad \ldots \quad \ldots \quad \ldots \quad (18.22)$$

Thus the necessary conditions for $v_n(x)$ are satisfied, and we may apply Bromwich's theorems to Laguerre's series.

We have thus justified to some extent the use to which the series has already been put in the calculation of asymptotic mean values of arithmetical functions. By far the most difficult question, however—whether the asymptotic mean value really *exists*—cannot be solved in this way. But we have a powerful instrument for its calculation, provided its existence is *established* by other means, or *assumed*.

In general, our result, obtained by the aid of Cesaro's theorem, may be written:

$$\operatorname*{Lt}_{n\to\infty}\frac{1}{n}\sum_{h=1}^{n}\psi(h)=\sum_{n=1}^{\infty}\frac{f(n)}{(n)} \quad \ldots \quad \ldots \quad (18.3)$$

or

$$\frac{1}{n}\sum_{h=1}^{n}\psi(h)\sim\sum_{n=1}^{\infty}\frac{f(n)}{n}, \quad \ldots \quad \ldots \quad (18.4)$$

provided the limit exists (*or tends to* ∞) *and the series converges* (*or diverges to* ∞).

We will now consider some additional examples.

19. Using the notation of (11·21), we have

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^{s-1}}\cdot\frac{x^n}{1-x^n}=\sum_{n=1}^{\infty}x^n\cdot\sum_{d:n}\frac{f(d)}{d^{s-1}} \quad \ldots \quad \ldots \quad (19\cdot1)$$

Thus (18·4) gives

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}\frac{f(d)}{d^{s-1}}\sim\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=F(s), \quad \ldots \quad \ldots \quad (19\cdot11)$$

provided the *asymptotic mean value* (which will in the following be denoted A. M. V.) on the left exists and the Dirichlet's series converges.

Similarly

$$\sum_{n=1}^{\infty} \mu(n) \frac{f(n)}{n^{s-1}} \quad \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} x^n . \sum_{d\,:\,n} \mu(d) \frac{f(d)}{d^{s-1}}, \quad \ldots \quad (19{\cdot}2)$$

from which follows by (11·22), $f(n)$, satisfying the condition (11·2)

$$\frac{1}{n} \sum_{h=1}^{n} \sum_{d\,;\,h} \mu(d) \frac{f(d)}{d^{s-1}} \backsim \sum_{n=1}^{\infty} \mu(n) \frac{f(n)}{n^s} = \frac{1}{\mathrm{F}(s)}, \quad \ldots \quad (19{\cdot}21)$$

provided the A. M. V. exists and the Dirichlet's series converges.

The formulae (19·11), (19·21) may be regarded as generalisations of (13·24).—An interesting problem would be to examine, whether the A. M. V. and the series have the same convergence abscissa. This may be considered as very probable.

Using certain well-known formulae from the theory of Dirichlet's series*, we have thus in all probability :

For $f(n) = t(n)$:

$$\frac{1}{n} \sum_{h=1}^{n} \sum_{d\,:\,h} \frac{t(d)}{d^{s-1}} \backsim \zeta^2(s). \quad \ldots \quad \ldots \quad \ldots \quad (19{\cdot}12)$$

For $f(n) = \int(n)$:

$$\frac{1}{n} \sum_{h=1}^{n} \sum_{d\,:\,h} \frac{\int(d)}{d^{s-1}} \backsim \zeta(s).\, \zeta(s-1). \quad \ldots \quad \ldots \quad (19{\cdot}13)$$

For $f(n) = \phi(n)$:

$$\frac{1}{n} \sum_{h=1}^{n} \sum_{d\,:\,h} \frac{\phi(d)}{d^{s-1}} \backsim \frac{\zeta(s-1)}{\zeta(s)}, \quad \ldots \quad \ldots \quad (19{\cdot}14)$$

For $f(n) = (-1)^{n-1}$:

$$\frac{1}{n} \sum_{h=1}^{n} \sum_{d\,:\,h} \frac{(-1)^{d-1}}{d^{s-1}} \backsim (1-2^{1-s}).\, \zeta(s). \quad \ldots \quad (19{\cdot}15)$$

For $f(n) = 1$:

$$\frac{1}{n} \sum_{h=1}^{n} \sum_{d\,:\,h} \frac{1}{d^{s-1}} \backsim \zeta(s), \quad \ldots \quad \ldots \quad \ldots \quad (19{\cdot}16)$$

* See " Encyclopedie " T. I, Vol. 3, p. 268.

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}\frac{\mu(d)}{d^{s-1}} \sim \frac{1}{\zeta(s)}. \qquad \ldots \qquad \ldots \quad (19\cdot22)$$

Of these, (19·12), (19·16), (19·22) are probably valid for R $(s)>1$, (19·13), (19·14) for R $(s)>2$, (19·15) for R $(s)>0$.——Formula (19·16) is nothing else than (13·24).

20. Let $f(z)$ be any analytic function, and thus

$$f(z)-f(0)=\frac{z}{1}\cdot f'(0)+\frac{z^2}{2!}f''(0)+ \qquad \ldots \qquad \ldots \quad (20\cdot1)$$

Then

$$\sum_{n=1}^{\infty}\frac{z^n}{(n-1)!}f^{(n)}(0).\frac{x^n}{1-x}=\sum_{n=1}^{\infty}x^n.\sum_{d:n}\frac{z^d}{(d-1)!}f^{(d)}(0), \quad \ldots \quad (20\cdot11)$$

from which follows

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}\frac{z^d}{(d-1)!}f^{(d)}(0) \sim f(z)-f(0), \qquad \ldots \quad (20\cdot12)$$

for a sufficiently small $|z|$, provided the A. M. V. exists.

In this way we obtain, for instance

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}z^d \sim -\log(1-z), \qquad \ldots \qquad \ldots \quad (20\cdot13)$$

when $|z|<1$.

For $z=-1$, the last formula becomes

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}(-1)^{d-1} \sim \log 2 \qquad \ldots \qquad \ldots \quad (20\cdot131)$$

Many more examples could of course be given, but would give us nothing fundamentally new.

It is further evident, that any other expansion of a function in the form of an infinite series will furnish similar results. To go into details would carry us to far. The following examples must suffice.

Formula (7.31), for instance, gives

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}\frac{2dx^2}{x^2-d^2\pi^2} \sim x\cot x-1, \qquad \ldots \qquad \ldots \quad (20\cdot2)$$

and the corresponding (7·33)

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:\,h}\mu(d)\left[x\cot\frac{d}{x}-d\right]\infty\frac{2x^2}{x^2-\pi^2}, \qquad \ldots \quad (20{\cdot}21)$$

provided the limits on the left exist.

Similarly formula (11·31) gives

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:\,h}d\left(\frac{s}{d}-\frac{s}{s+d}\right)\infty s(\psi(s)+C)+1, \qquad \ldots \quad (20{\cdot}3)$$

when s is not a negative integer, provided the A. M. V. exists ; and (11.33) gives a corresponding formula.

21. Let p be a given prime number.

Define $f_p(n)=1$, if $n=p^r$, $r=$positive integer.
$f_p(n)=0$, if n contains other prime factors than p.
$f_p(1)=0$.
$\psi_p(n)=r$, if $n=p^r n'$, $n'\not\equiv 0 \pmod p$.
$\psi_p(n)=0$, if $n\not\equiv 0 \pmod p$.

Thus

$$\sum_{r=1}^{\infty}\frac{x^{p^r}}{1-x^{p^r}}=\sum_{n=1}^{\infty}\psi_p(n)\,x^n. \qquad \ldots \qquad \ldots \quad (21.1)$$

From this we obtain in the usual manner

$$\frac{1}{n}\sum_{h=1}^{n}\psi_p(h)\infty\sum_{r=1}^{\infty}p^{-r}=\frac{1}{p-1}. \qquad \ldots \qquad \ldots \quad (21.11)$$

This result is capable of the following generalisation.

Let $(P)=(p_{n_1}, p_{n_2},\ldots, p_{n_k})$ be a given finite set of primes, k in number.

Define $f_p(n)=1$, if n is made up entirely of primes, belonging to the given set.

$f_p(n)=0$, if n contains other prime factors than those belonging to the given set.

$f_p(1)=0$.

and $\psi_p(n)=\prod_{i=1}^{k}(1+r_i)-1$, if $n=n'.\prod_{i=1}^{k}p_{n_i}^{r_i}$

$n'\not\equiv 0 (\bmod p_{n_i};\ i=1, 2,\ldots, k)$.

Thus

$$\sum_{n=1}^{\infty} f_p(n).\frac{x^n}{1-x^n}=\sum_{n=1}^{\infty}\psi_p(n).x^n, \qquad \ldots \qquad \ldots \qquad (21.2)$$

from which follows

$$\frac{1}{n}\sum_{h=1}^{n}\psi_p(h) \sim \sum\frac{1}{\pi_p-1} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (21.21)$$

where the sum to the right is extended over all numbers π_p, made up entirely of primes belonging to (P), and not forming an r^{th} power of any factor of $\prod_{i=1}^{k} p_{n_i}$, r being ≥ 2.

22. Comparing (12·1), (12.11), (12·12) with the corresponding

$$\sum_{n=1}^{\infty} f_1(n)\ \frac{x^n}{1-x^n}=\sum_{n=1}^{\infty}x^n\,\psi_1(n), \qquad \ldots \qquad \ldots \qquad (22.1)$$

where

$$\psi_1(n)=\sum_{d\,:\,n} f_1(d), \qquad \ldots \qquad \ldots \qquad \ldots \qquad (22.11)$$

$$f_1(n)=\sum_{d\,:\,n}\mu(d)\psi_1\left(\frac{n}{d}\right), \qquad \ldots \qquad \ldots \qquad (22\,12)$$

and where $f_1(n)$, and hence by (22.11) also $\psi_1(n)$, are *positive*, we obtain easily from Bromwich's theorem II and Cesaro's theorem of comparison for two divergent power series *, the following interesting result, provided

$\sum_{n=1}^{\infty}\frac{f_1(n)}{n}$ *diverges*, and *both limits exist*:

$$\operatorname*{Lt}_{n\to\infty}\frac{\sum_{h=1}^{n}\frac{f(h)}{h}}{\sum_{h=1}^{n}\frac{f_1(h)}{h}}=\operatorname*{Lt}_{n\to\infty}\frac{\sum_{h=1}^{n}\psi(h)}{\sum_{h=1}^{n}\psi_1(h)} \qquad \ldots \qquad \ldots \qquad (22.2)$$

(For the proof, use the value for $v_n(x)$ given in (18.12)).

* See Bromwich, *Infinite Series*, p. 131.

Ex 1. Let

$$\left.\begin{matrix} f(n)=\mu^2(n) \\ f_1(n)=1 \end{matrix}\right\} \quad \therefore \left.\begin{matrix} \psi(n)=p(n) \\ \psi_1(n)=t(n) \end{matrix}\right\}$$

Thus

$$\frac{\sum_{h=1}^{n} \frac{\mu^2(h)}{h}}{\sum_{h=1}^{n} \frac{1}{h}} \sim \frac{\sum_{h=1}^{n} p(h)}{\sum_{h=1}^{n} t(h)} \sim \frac{6}{\pi^2}$$

by (13.41), (13.12).

Hence

$$\sum_{h=1}^{n} \frac{\mu^2(h)}{h} \sim \frac{6}{\pi^2} \cdot \log n. \qquad \ldots \qquad \ldots \quad (22.21)$$

Ex. 2. Let

$$\left.\begin{matrix} f(n)=\sum_{d:n} \mu(d) \cdot \phi\left(\frac{n}{d}\right) \\ f_1(n)=n \end{matrix}\right\} \left\{\begin{matrix} \psi(n)=\phi(n) \\ \therefore \psi_1(n)=\int(n) \end{matrix}\right\}$$

Thus

$$\frac{\sum_{h=1}^{n} \frac{1}{h} \cdot \sum_{d:h} \mu(d) \cdot \phi\left(\frac{h}{d}\right)}{\sum_{h=1}^{n} \frac{1}{h} \cdot h} \sim \frac{\sum_{h=1}^{n} \phi(h)}{\sum_{h=1}^{n} \int(h)} \sim \frac{36}{\pi^4}$$

by (13.32), (13.22)

Hence

$$\sum_{h=1}^{n} \frac{1}{h} \cdot \sum_{d:h} \mu(d) \cdot \phi\left(\frac{h}{d}\right) \sim \frac{36}{\pi^4} \cdot n. \qquad \ldots \quad (22.22.)$$

Ex. 3. Let

$$\left.\begin{matrix} f(n)=\phi(n) \\ f_1(n)=n \end{matrix}\right\} \quad \therefore \left.\begin{matrix} \psi(n)=n \\ \psi_1(n)=f(n) \end{matrix}\right\}$$

Thus

$$\frac{\sum_{h=1}^{n} \frac{\phi(h)}{h}}{\sum_{h=1}^{n} \frac{h}{h}} \sim \frac{\sum_{h=1}^{n} h}{\sum_{h=1}^{n} f(h)} \sim \frac{6}{\pi^2},$$

by (13.22).

Hence

$$\sum_{h=1}^{n} \frac{\phi(h)}{h} \sim \frac{6}{\pi^2} \cdot n. \quad \ldots \quad \ldots \quad \ldots (22\,23)$$

Putting $\psi(n) = \frac{\phi(n)}{n}$ in formula (12.1), we obtain from (18.3) by the aid of (22.23)

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{d\,:\,n} \mu\left(\frac{n}{d}\right) \frac{\phi(d)}{d} =$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{d\,:\,n} \mu(d) \cdot d \cdot \phi\left(\frac{n}{d}\right) = \frac{6}{\pi^2}. \quad \ldots (22.231)$$

We know, however, from (11.12), that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2},$$

from which we conclude

$$\sum_{d\,:\,n} \mu(d) \cdot d \cdot \phi\left(\frac{n}{d}\right) = \mu(n), \quad \ldots \quad \ldots (22.3)$$

a formula, which is easily verified.

Formula (22.3) may be written

$$\frac{\mu(n)}{n} = \sum_{d\,:\,n} \mu(d) \frac{\phi\left(\frac{n}{d}\right)}{\frac{n}{d}},$$

from which follows, by (12.12)

$$\frac{\phi(n)}{n} = \sum_{d\,:\,n} \frac{\mu(d)}{d}. \quad \ldots \quad \ldots \quad \ldots \quad (22.31)$$

Various other general results similar to (22.2) may be obtained by using Frobenius's theorem with Bromwich's extension (See Bromwich, *Infinite Series*, pp. 132, 313).

23. *μ-inversion of Laguerre's series.*

Let in (12.1), $f(n)$ satisfy the condition (11.2), and μ-invert.

The result is

$$\sum_{r=1}^{\infty} \mu(r).\, f(r).\, \sum_{n=1}^{\infty} x^{nr}.\, \psi(n) = \frac{x}{1-x}. \quad \ldots \quad \ldots \quad (23.1)$$

Hence:

$$\sum_{d\,:\,n} \mu(d).\, f(d).\, \psi\left(\frac{n}{d}\right) = 1, \quad \ldots \quad \ldots \quad (23.11)$$

provided always

$$f(m\,m') = f(m)f(m'). \quad \ldots \quad \ldots \quad \ldots (23.111)$$

Let now $\psi(n)$ satisfy (11.2), and $\mu-$invert.

We obtain

$$\sum_{r=1}^{\infty} \mu(r).\, \psi(r).\, \sum_{n=1}^{\infty} \frac{x^{nr}}{1-x^{nr}}.f(n) = x$$

$$= \sum_{n=1}^{\infty} \frac{\mu(n)x^n}{1-x^n}, \text{ by (14.1).} \quad \ldots \quad \ldots \quad (23.2)$$

Hence:

$$\sum_{d\,:\,n} \mu(d).\, \psi(d).\, f\left(\frac{n}{d}\right) = \mu(n) \quad \ldots \quad \ldots \quad (23.21)$$

for $n>1$, provided always

$$\psi(mm') = \psi(m)\psi(m') \quad \ldots \quad \ldots \quad (23{\cdot}211)$$

We will consider a few simple examples.

Ex. 1. Put

$$f(n) = 1. \quad \therefore\ \psi(n) = t(n).$$

Hence by (23·11)

$$\sum_{d\,:\,n}\mu(d).\ t\left(\frac{n}{d}\right)=1=f(n),$$

which is evidently correct because of (12·12)

Ex. 2. Put

$$f(n)=n. \quad \therefore\ \psi(n)=\int(n)$$

Hence by (23.11).

$$\sum_{d\,:\,n}\mu(d).\ d.\ \int\left(\frac{n}{d}\right)=1\ .$$

The last formula may be written

$$\sum_{d\,:\,n}\mu(d).\ \frac{\int\left(\frac{n}{d}\right)}{\left(\frac{n}{d}\right)}\ =\ \frac{1}{n}\ ,$$

from which follows, by (12·12), (12· 11)

$$\frac{f(n)}{n}=\sum_{d\,:\,n}\frac{1}{d}=\sum_{d\,:\,n}\frac{d}{n}=\frac{1}{n}.\sum_{d\,:\,n}d,$$

an evident identity.

Ex. 3. Put

$$f(n)=\phi(n). \quad \therefore\ \psi(n)=n.$$

In this case, (23·21) becomes (22·3).

For examples of a more general nature, use for $f(n)$ or $\psi(n)$ the arithmetical functions defined in § 3. (towards the end of p. 175, Vol. IX).

V. Differentiation of Laguerre's Series.

24. The series (12.1) evidently admits of differentiation. The result may be written:

$$\sum_{n=1}^{\infty}\frac{f(n)}{n}.\frac{n^2.\ x^{n-1}}{(1+x+x^2+\ldots+x^{n-1})^2}=\frac{(1-x)^2}{x}.\sum_{n=1}^{\infty}n.\ \psi(n).\ x^n. \quad (24.1)$$

If we here put

$$v'_n(x)=\frac{n^2.\ x^{n-1}(1-x)^2.}{(1-x^n)^2} \quad \ldots \qquad \ldots \qquad \ldots(24.11)$$

we find that the condition

$$\operatorname*{Lt}_{x\to 1} v'_n(x)=1, \qquad \ldots \qquad \ldots \qquad \ldots \quad (24.12)$$

is satisfied.

Let $0<y<1$.

$$\therefore \quad 1+(1+y)+(1+y+y^2)+\ldots+(1+y+y^2+\ldots+y^{n-1})>$$
$$>y^{2n}.\ 1+y^{2n-1}.\ (1+y)+y^{2n-2}.\ (1+y+y^2)+\ldots$$
$$+y^{n+1}(1+y+y^2+\ldots+y^{n-1}).$$

Multiply by $(1-y)>0$.

$$\therefore \quad (1-y)+(1-y^2)+(1-y^3)+\ldots+(1-y^n)>$$
$$>y^{n+1}(1-y^n)+y^{n+2}(1-y^{n-1})+\ldots+y^{2n}(1-y).$$

$$\therefore \quad n+ny^{2n+1}>y(1+y+y^2+\ldots+y^{2n-1})=y.\ \frac{1-y^{2n}}{1-y}.$$

$$\therefore \quad n-ny+ny^{2n+1}-ny^{2n+2}>y-y^{2n+1}.$$

$$\therefore \quad \frac{n}{1-y^{2n}}>\frac{(n+1)y}{1-y^{2(n+1)}}.$$

Square, put $y=\sqrt{x}$, and multiply by $x^{n-1}(1-x)^2$, when we obtain

$$\frac{n^2x^{n-1}(1-x)^2}{(1-x^n)^2}>\frac{(n+1)^2x^n(1-x)^2}{(1-x^{n+1})^2},$$

or

$$v'_n(x)>v'_{n+1}(x) \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (24.13)$$

for $0<x<1$.

We may therefore apply Bromwich's theorem I, on the left side of (24.1), using Cesaro's theorem on the right, when we obtain

$$\frac{2}{n^2}\sum_{h=1}^{n}{}' h\,\psi(h)\sim\sum_{n=1}^{\infty}{}'\frac{f(n)}{n}. \qquad \ldots \qquad \ldots \qquad \ldots \quad (24.2)$$

provided the limit exists (or tends to ∞), and the series converges (or diverges to ∞).

We may put (24.2) into a slightly different form.

For this purpose, assume $n\,\psi(n)=\Theta(n)$;

$$\therefore \quad f(n)=\sum_{d:n}{}'\mu(d)\,\psi\left(\frac{n}{d}\right)=\sum_{d:n}{}'\mu\left(\frac{n}{d}\right)\psi(d)=$$
$$=\frac{1}{n}\sum_{d:n}{}'\mu(d).\,d.\,\Theta\left(\frac{n}{d}\right)=\sum_{d:n}{}'\mu\left(\frac{n}{d}\right)\frac{\Theta(d)}{d},$$

Hence :

$$\frac{2}{n^2}\sum_{h=1}^{n}\Theta(h) \backsim \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{d:n}\mu(d).\, d.\,\Theta\left(\frac{n}{d}\right) =$$

$$= \sum_{n=1}^{\infty}\frac{1}{n}\sum_{d:n}\mu\left(\frac{n}{d}\right)\frac{\Theta(d)}{d}, \quad \ldots \quad \ldots \quad (24{\cdot}21)$$

valid under similar conditions.

We will now consider some examples.

25. As in § 19., we have, putting $\frac{f(n)}{n^{s-1}}$ instead of $f(n)$ in (24·2)

$$\frac{2}{n^2}\sum_{h=1}^{n} h \sum_{d:h}\frac{f(d)}{d^{s-1}} \backsim \mathrm{F}(s), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25{\cdot}1)$$

and similarly, *if the condition* (11· 2) *is satisfied* :

$$\frac{2}{n^2}\sum_{h=1}^{n} h \sum_{d:h}\mu(d)\frac{f(d)}{d^{s-1}} \backsim \frac{1}{\mathrm{F}(s)}, \quad \ldots \quad \ldots \quad \ldots \quad (25{\cdot}2)$$

always provided that the limits on the left exists and that the Dirichlet's series on the right converges.

Ex. : For $f(n)=1$, formula (25.1) gives

$$\frac{2}{n^2}\sum_{h=1}^{n} h^{2-s} \sum_{d\,\cdot\,h} d^{s-1} \backsim \zeta(s), \quad \ldots \quad \ldots \quad \ldots \quad (25.11)$$

probably valid for $\mathrm{R}(s)>1$.

The case $s=2$ is particularly important. We obtain

$$\sum_{h=1}^{\infty}\int(h) \backsim \frac{\pi^2}{12}n^2, \quad \ldots \quad \ldots \quad \ldots \quad (25.111)$$

in agreement with Dirichlet's result, given in formula (13.22). Hence the alternative result (see p. 257 of the present paper) is incorrect.

The case $s=1$ shows that $\sum_{h=1}^{n} h.\, t(h)$ is of an order greater than $n.^2$

For $f(n)=1$, formula (25.2) gives

$$\frac{2}{n^2}\sum_{h=1}^{n} h^{2-s}\sum_{d\,:\,h}\mu\left(\frac{h}{d}\right)d^{s-1}\sim\frac{1}{\zeta(s)}, \quad \ldots \quad \ldots \quad (25.1)$$

which for $s=2$ becomes

$$\sum_{h=1}^{\infty}\phi(h)\sim\frac{3}{\pi^2}n^2 \quad \ldots \quad \ldots \quad \ldots \quad (25.21)$$

in agreement with (13.32).

Other special results may easily be obtained in a way similar to that of § 19

The matter of § § 20, 21, may also be easily extended on the present lines.

26. It is further evident that we may, by the aid of Bromwich's theorem II, obtain a result corresponding to (22·2). Under the same conditions as in § 22, we have :

$$\operatorname*{Lt}_{n\to\infty}\frac{\sum_{h=1}^{n}\frac{f(h)}{h}}{\sum_{h=1}^{n}\frac{f_1(h)}{h}}=\operatorname*{Lt}_{n\to\infty}\frac{\sum_{h=1}^{n}h\,\psi(h)}{\sum_{h=1}^{n}h\,\psi_1(h)} \quad \ldots \quad \ldots \quad (26\cdot1)$$

From this result, new asymptotic formula may be obtained

We note (see § 22., Ex. 3.) $$\frac{\sum_{h=1}^{n}h\int(h)}{\sum_{h=1}^{n}h^2}\sim\frac{\pi^2}{6},$$

But

$$\sum_{h=1}^{n}h^2=\tfrac{1}{6}n(n+1)(2n+1)\sim\frac{n^3}{3}$$

Hence

$$\sum_{h=1}^{n} h\Big[(h) \sim \frac{\pi^2}{18} n^3. \qquad \ldots \qquad \ldots \qquad \ldots \quad (26{\cdot}2)$$

27. We may apply μ inversion to formula (24.1) written in the form

$$\sum_{n=1}^{\infty} nf(n)\frac{x^n}{(1-x^n)^2} = \sum_{n=1}^{\infty} n\psi(n)x^n, \qquad \ldots \qquad \ldots \quad (27.1)$$

but the results are the same as those of § 23.

We will consider specially the formula (14.22) (where the brackets in the denominator of the fraction under the sum should be removed), comparing it with the corresponding

$$\sum_{n=1}^{\infty} \frac{n\phi(n)x^n}{(1-x^n)^2} = \frac{x(1+x)}{(1-x)^3}, \qquad \ldots \qquad \ldots \quad (27.2)$$

We obtain

$$\frac{\sum_{n=1}^{\infty} \frac{\phi(n)}{n} v_n(x)}{\sum_{n=1}^{\infty} \frac{\phi(n)}{n} v_n'(x)} = \frac{1}{1+x}, \qquad \ldots \qquad \ldots \quad (27.21)$$

and hence

$$\operatorname*{Lt}_{x\to 1} \frac{\sum_{n=1}^{\infty} \frac{\phi(n)}{n} v_n(x)}{\sum_{n=1}^{\infty} \frac{\phi(n)}{n} v_n'(x)} = \frac{1}{2}, \qquad \ldots \qquad \ldots \quad (27.3)$$

This is an interesting relation between two divergent series, corresponding terms of which tend to equality. The anomaly is of course due to the fact, that $v_n(x)$ and $v_n'(x)$ approach unity in a different manner.

VI. Integration of Laguerre's Series.

28. Dividing both sides of (12.1) by x and integrating, we obtain

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \cdot \log(1-x^n) = -\sum_{n=1}^{\infty} \frac{\psi(n)}{n} \cdot x^n, \quad \ldots \quad \ldots \quad (28.1)$$

or

$$\prod_{n=1}^{\infty} (1-x^n)^{\frac{f(n)}{n}} = e^{-\sum_{n=1}^{\infty} \frac{\psi(n)}{n} \cdot x^n} \quad \ldots \quad \ldots \quad (28.11)$$

no constant of integration being required.

From this result we may deduce many interesting identities.

We note:

$$\log \prod_{n=1}^{\infty} (1-x^n) = -\sum_{n=1}^{\infty} \frac{f(n)}{n} x^n. \quad \ldots \quad \ldots \quad (28.2)$$

$$\log \prod_{n=1}^{\infty} (1-x^n)^{\frac{1}{n}} = -\sum_{n=1}^{\infty} \frac{t(n)}{n} x^n. \quad \ldots \quad \ldots \quad (28.3)$$

$$\log \prod_{n=1}^{\infty} (1-x^n)^{\frac{\mu(n)}{n}} = -x. \quad \ldots \quad \ldots \quad \ldots \quad (28.4)$$

$$\log \prod_{n=1}^{\infty} (1-x^n)^{\frac{\phi(n)}{n}} = -\frac{x}{1-x}. \quad \ldots \quad \ldots \quad \ldots \quad (28.5)$$

29. Formula (28.1) may be written

$$\frac{\sum_{n=1}^{\infty} \frac{\psi(n)}{n} \cdot x^n}{\sum_{n=1}^{\infty} \frac{1}{n} \cdot x^n}$$

$$= \sum_{n=1}^{\infty} \frac{f(n)}{n} + \frac{\sum_{n=1}^{\infty} \frac{f(n) \log}{n} \cdot \frac{\log(1+x+\ldots+x^{n-1})}{\log n}}{\log(1-x)} \quad \ldots \quad (29.1)$$

Put

$$v''_n(x) = \frac{\log(1+x+\ldots+x^{n-1})}{\log n} \quad \ldots \quad \ldots \quad (29.11)$$

$$\therefore \quad \operatorname*{Lt}_{x\to 1} v''_n(x) = 1. \quad \ldots \quad \ldots \quad (29.111)$$

Also

$$v''_n(x) > v''_{n+1}(x), \quad \ldots \quad \ldots \quad (29.112)$$

since the fraction $v''_n(x)$ is <1 for $x<1$, and the numerator is increased by a smaller quantity than the denominator, when $v''_n(x)$ is changed into $v''_{n+1}(x)$.

Hence Bromwich's theorem I, may be applied to the series in the numerator of the last term in (29.1), so that, in particular this series approaches

$$\sum_{n=1}^{\infty} \frac{f(n)\log n}{n},$$

if the latter series converges.

Hence we derive the following theorem, by the aid of Cesaro's theorem:

$$\frac{1}{\log n} \sum_{h=1}^{n} \frac{\Psi(n)}{n} \sim \sum_{n=1}^{\infty} \frac{f(n)}{n}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (29.2)$$

provided the limit exists, and the series

$$\sum_{n=1}^{\infty} \frac{f(n)\log n}{n}$$

is convergent.

(This is of course only a particular case of a more general theorem, where the condition for the validity of (29.2) is that the last term in (29.1) should approach zero, as x tends to unity).

30. As an example of the last theorem, substitute in (29.2) $\frac{f(n)}{n^{s-1}}$ for $f(n)$

We deduce, as in §§ 19., 25.,

$$\frac{1}{\log n} \sum_{h=1}^{n} \frac{1}{h} \sum_{d:h} \frac{f(d)}{d^{s-1}} \sim F(s), \quad \ldots \quad \ldots \quad (30{\cdot}1)$$

provided the limit on the left exists and the Dirichlet's series converges.

The last condition is obtained from the theorem * that if $F(s)$ converges for $s=s_o$,

$$F'(s) = -\sum_{n=1}^{\infty} \frac{f(n)\log n}{n^s} \quad \ldots \quad \ldots \quad (30{\cdot}11)$$

converges for $s>s_o$. Hence, when $R(s)$ is equal to the convergence-abscissa of $F(s)$, a special examination is necessary.

Ex. For $f(n)=1$, formula (30·1) gives

$$\frac{1}{\log n} \sum_{h=1}^{n} \frac{1}{h^s} \sum_{d\,:\,h}^{n} d^{s-1} \sim \zeta(s), \quad \ldots \quad \ldots \quad (30\,12)$$

which for $s=2$ becomes

$$\sum_{h=1}^{n} \frac{f(h)}{h} \sim \frac{\pi^2}{6} \log n \quad \ldots \quad \ldots \quad \ldots \quad (30{\cdot}12)$$

Under conditions similar to the above we obtain in addition

$$\frac{1}{\log n} \sum_{h=1}^{n} \frac{1}{h} \sum_{d\,:\,h}^{n} \mu(d) \frac{f(d)}{d^{s-1}} \sim \frac{1}{F(s)}, \quad \ldots \quad \ldots \quad (30{\cdot}2)$$

provided the condition (11.2) *is satisfied.*

Ex. For $f(n)=1$, formula (30·2) becomes

$$\frac{1}{\log n} \sum_{h=1}^{n} \frac{1}{h^s} \sum_{d\,:\,h}^{n} \mu\left(\frac{h}{d}\right) d^{s-1} \sim \frac{1}{\zeta(s)} \quad \ldots \quad \ldots \quad (30{\cdot}21)$$

which for $s=2$ gives

$$\sum_{h=1}^{n} \frac{\phi(h)}{h^2} \sim \frac{6}{\pi^2} \log n \quad \ldots \quad \ldots \quad \ldots \quad (30{\cdot}212)$$

31. From (28·1) we deduce, putting $f(n)=\frac{1}{n}$,

$$-\frac{\pi^2}{6} \log(1-x) - \sum_{n=1}^{\infty} \frac{1}{n} x^n \sum_{d\,:\,n} \frac{1}{d} =$$

* *See* Landau : *Handbuch der Lehre von der Verteilung der Primzahlen*, pp. 108, 109.

$$= \sum_{n=1}^{\infty} \frac{x^n}{n}\left(\frac{\pi^2}{6} - \frac{f(n)}{n}\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \log\,(1+x+\ldots+x^{n-1}). \qquad \ldots \quad (31.1)$$

Hence :

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\pi^2}{6} - \frac{f(n)}{n}\right) = \sum_{n=1}^{\infty} \frac{\log n}{n^2}, \quad \ldots \qquad \ldots \quad (31.11)$$

provided the series on the left converges.—This is a formula very similar to (13.52).

Similarly, for $f(n)=\mu(n)$,

$$-\log\,(1-x) \sum_{n=1}^{\infty} \frac{\mu(n)}{n} - x = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1+x\ldots+x^{n-1}). \quad \ldots \quad (31.2)$$

But the first term on the left vanishes by (11.13).

Hence

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log\,(1+x+\ldots+x^{n-1}) = -x, \qquad \ldots \qquad \ldots \qquad (31.21)$$

from which again follows the result (15.3) for $x \to 1$, the series on the left supposed convergent for $x=1$.

By differentiation of (31.21), or directly from (14.1), we deduce

$$\sum_{n=2}^{\infty} \frac{\mu(n)}{n} \cdot \frac{1+2x+3x^2+\ldots+(n-1).\,x^{n-2}}{1+x+x^2+\ldots+x^{n-2}+x^{n-1}} = -1. \qquad \ldots \quad (31.22)$$

This series oscillates for $x=1$.

We obtain, however, symbolically

$$\sum_{n=2}^{\infty} \frac{\mu(n)}{n} \cdot \frac{\frac{(n-1).\,n}{2}}{n} \backsim -1,$$

and hence

$$\sum_{n=2}^{\infty} \mu(n)\left(1-\frac{1}{n}\right) \backsim \sum_{n=1}^{\infty} \mu(n)\left(1-\frac{1}{n}\right) \backsim$$

$$\backsim \sum_{n=1}^{\infty} \mu(n) - \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \backsim -2,$$

from which follows

$$\sum_{n=1}^{\infty} \mu(n) \backsim -2. \quad \ldots \quad \ldots \quad \ldots \quad (31.3)$$

We may therefore regard it as highly probable, that

$$\operatorname*{Lt}_{x \to 1} \sum_{n=1}^{\infty} \mu(n).\, x^n = -2. \quad \ldots \quad \ldots \quad (31.31)$$

This agrees very well with the theorem of Frobenius. In fact, we find, putting

$$\sigma(n) = \sum_{r=1}^{n} \mu(r), \quad \ldots \quad \ldots \quad \ldots \quad (16.12)$$

for N = 83, 91, 126, 127, 128, 131, 134, 144, etc.

that
$$\frac{1}{N} \sum_{n=1}^{N} \sigma(n) = -2$$

exactly.

VII. Conclusion.

In the present paper I have tried to show, how asymptotic formulæ in the theory of numbers may easily be obtained in a manner, somewhat different from the usual, which rests chiefly on the theory of Dirichlet's series. Incidentally I have derived many other interesting results, several of which appear to be novel. I have dealt mainly with the properties of a class of series, bearing the name of Laguerre, and closely associated with that of Dirichlet. The literature, dealing with the latter class, however, is rapidly attaining enormous proportions while that of the former is very inconsiderable, except in the special case, known as Lambert's series. Laguerre's class of series thus deserves a closer attention than hitherto has been paid to it by workers in the theory of numbers. There is reason to believe that the theory of these series might be brought to at least the same degree of perfection as that of Dirichlet's series at present. Probably new light would thus be thrown on some of the most difficult parts of the subject in particular on the still unproved, highly important, proposition of Riemann about the imaginary roots of the ζ—function. The chief usefulness of the series in question lies in the fact, that they form a connecting link between Dirichlet's series on one hand and ordinary power-series on the other, they being associated in a simple manner with both.

SHORT NOTES.

Note on Triangles inscribed in an Ellipse.*

§ 1. Consider a triangle ABC inscribed in the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

and let the eccentric angles of its corners be α, β, γ respectively.

If (x, y) be the centre of either of the in and ex-circles of the triangle ABC, and p_1, p_2, p_3 the perpendiculars from its vertices on the line $x=X$, we shall obviously have

$$BC.p_1 \pm CA.p_2 \pm AB.p_3 = 0\ .$$

But the equation of a circle having double contact with the ellipse along $x=X$ is,

$$x^2+y^2-2e^2xX+e^2X^2-b^2=0,$$

and therefore, if t_1, t_2, t_3 be the tangents from A,B,C, we have

$$t_1=ep_1\ ;\ t_2=ep_2 \text{ and } t_3=ep_3;$$

and consequently,

$$BC.\ t_1 \pm CA.\ t_2 \pm AB.\ t_3=0\ ;$$

which shows that the circumcircle of ABC touches the circle having double contact with the ellipse along the line $x=X$ parallel to its major axis. Similarly for the circle having double contact along the line $y=Y$.

Hence, if the circle ABC be $(x-h)^2+(y-k)^2=R^2$, the abscissae x_1, x_2, x_3, x_4 of the four incentres of ABC, are given as the roots of

$$[2h(h-e^2X)+2k^2-h^2-k^2+R^2+e^2X^2-b^2]^2=R^2\{4(h-e^2X)^2+4k^2\}\ .$$

Hence, if p_1' p_2' p_3' and p_4' be the perpendiculars from the four incentres of ABC, upon a chord $x=d$,

$$\begin{aligned} e^4p_1'p_2'p_3'p_4' &= e^4(d-x_1)(d-x_2)(d-x_3)(d-x_4) \\ &= (h^2+k^2-2e^2dh+e^2d^2+R^2-b^2)-4R^2\{(h-e^2d)^2+k^2\} \\ &= (\delta^2+R^2-r^2)-4R^2\delta^2 \\ &= [\delta^2-(R-r)^2][\delta^2-(R+r)^2]=t^2t'^2, \end{aligned}$$

where δ is the distance between the circumcentre of ABC and the centre of the circle having double contact along $x=d$; and t, t' the lengths of the common tangents of these circles. From this result, it follows that if the circle round ABC were to touch the circle of double contact, since either t or t' is zero, one at least of the quantities p'_1, p'_2, p'_3, p'_4 is zero, so that the chord of contact with the ellipse must pass through the in-or excentre of the triangle.

* Extracted from a paper read by the author before the Conference of the Indian Mathematical Society, December, 1916.

Again, if $(x-X)^2+(y-Y)^2-r_1^2=0$ be one of these circles the lines through the origin parallel to the lines of intersection of this circle with the original ellipse, are.

$$\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-(x^2+y^2)=0,$$

so that $\lambda=\rho^2$, where ρ is the radiant of the ellipse parallel to the chord of intersection. But the invariant relation between the ellipse and the circle gives

$$\lambda^3\Delta-\lambda^2\Theta+\lambda\Theta'-\Delta'=0$$

so that $\Sigma\rho_1^2=\Theta/\Delta$; $\Sigma\rho_1^2\rho_2^2=\Theta'/\Delta$ and $\rho_1^2\rho_2^2\rho_3^2=\Delta'/\Delta$
and the relation $\Theta^2=4\Delta\Theta'$ becomes $(\Sigma\rho_1)^2-4\Sigma\rho_1^2\rho_2^2=0$
or
$$\rho_1\pm\rho_2\pm\rho_3=0$$
i.e. the algebraic sum of the three radiants is zero.

§ 2. The following general results, which follow as corollaries from the preceding, have been kindly suggested by Prof. A. C. L. Wilkinson.

If ABCD is a quadrilateral inscribed in a circle, the sixteen inscribed and escribed centres of the four triangles ABC, ABD, BCD, ACD lie by fours on two of perpendicular lines:

Also, $$r=\frac{1}{2R}[(x-h)^2+(y-k)^2-R^2],$$ for any incircle

Hence

$$\begin{aligned} r_1+r_2&=\frac{1}{2R}[(x_1-h)^2+(y_1-k)^2-R^2+(x_2-h)^2+(y_2-k)^2-R^2]\\ &=\frac{1}{2R}[(x_1-h)^2+(y_2-k)^2-R^2+(x_2-h)^2+(y_1-k)^2-R^2]\\ &=r_3+r_4; \end{aligned}$$

i.e. the sum of the radii of the circles inscribed in ABC and ACD is equal to the sum of the corresponding radii of ABD and BCD. We have also similar theorems for the sets of escribed circles. Again when we make

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda(x-X)^2=0,$$ a circle

we also make $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda(x-X)^2+\mu[(x-h)^2+(y-k)^2-R^2]=0$$

a circle, and the radical axis with

$$(x-h)^2+(y-h)^2-R^2=0$$

is the same for both, so that if it touches the first it also touches the second. Hence the point on the circle ABCD where the circle having double contact with the ellipse touches the given circle is the same for all conics through ABCD. Therefore in the special case of the line-pair (as conic) we obtain the following theorem :

If $S_1, S_2 \ldots S_8$ are the points of contact of the 8 circles touching two given straight lines AC, BD and a given circle, then the same 8 points occur also as the points of contact of circles touching AB, CD and the given circles, as well as of the circles touching BC, AD and the given circle.

A. N. RAGHAVACHAR.

On poristic Polygons.

The following result (due to Salmon) furnishes a simple method of determining the condition for the porism of a polygon in terms of the invariants of its in-and circum-conics :—

If a triangle is inscribed in a conic $S=0$ *whose sides touch respectively the conics*

$$S'+lS=0,\ S'+mS=0 \text{ and } S'+nS=0,$$

then

$$\{\odot'-\triangle(mn+nl+lm)\}^2=4[\triangle'+lmn\triangle][\odot+\triangle(l+m+n)]$$

where $\odot$, $\odot'$, $\triangle$, $\triangle'$ *are the invariants of the conics* S and S'.

(*Conic Sections*, p. 343.)

For, consider a polygon $A_1A_2 \ldots A_n$ inscribed in a conic $S=0$, and such that all its sides, except A_1A_n, touch a conic S'. Then it is seen that A_1A_n touches a conic of the form $S'+nS=0$, by a repeated application of Salmon's theorem thus: put $l=m=0$, and $n=l_1$; then the side A_1A_3 of the triangle $A_1A_2A_3$ touches a conic $S'+l_1S=0$; again put $l=l_1$, $m=0$, $n=l_2$, then the side A_1A_4 of the $\triangle A_1A_3A_4$ touches $S'+l_2S=0$; and so on. Now since any side A_pA_q may be looked upon as the open side of a polygon, it follow that A_pA_q touches a conic whose equation is of the form $S'+l_{p,q}S=0$, so that if we apply Salmon's result to the two triangles $A_1A_nA_{n-1}$ and $A_1A_{n-1}A_{n-2}$, we obtain

$$[\odot'-\triangle l_{1,n-2}\, l_{1,n}]^2=4\triangle'[\odot+\triangle(l_{1,n-1}+l_{1,n})]$$

and

$$[\odot'-\triangle l_{1,n-2}\, l_{1,n-1}]^2=4\triangle'[\odot+\triangle(l_{1,n-2}+l_{1,n-1})].$$

On simplification these give the result

$$l_{1,n}=\frac{2\odot}{\triangle}\frac{1}{l_{1,n-1}}+\frac{4\triangle'}{\triangle}\frac{1}{l^2_{1,n-2}}-l_{1,n-2}$$

which may be looked upon as a recurring relation from which any $l_{1,n}$ may be determined if the two preceding ones are known.

Obviously the condition that is necessary and sufficient for the polygon to be poristic is that $l_{1,\ n}=0$.

Putting successively $n=3,\ 4$ in the above recurring formula, we get

$$4\triangle\triangle'\ l_{1,3}=\odot'^2-4\triangle'\odot,$$

and $$(\odot'^2-4\triangle'\odot)^2\ l_{1,4}=8\triangle'\ (\odot'^3-4\triangle\odot'\odot+8\triangle'^2\triangle)$$

which give the well-known results for the case of the triangle and the quadrilateral.

This general method is obviously laborious for higher values of n, but there may exist shorter methods for special cases.

In the case of a pentagon, it is obvious that A_1A_3, A_1A_4 may be taken to touch the same conic $S'+l_{1,3}\ S=0$, while of course, A_3A_4 touches $S=0$. Directly applying salmon's theorem, we obtain

$$[\odot'-\triangle l_{1,}{}^2{}_3]^2=4\triangle'\ [\odot+2\triangle l_{1,3}]$$

which gives on reduction

$$(\odot'^2-4\triangle'\odot)^3=32\triangle'^2\triangle\ [\theta'^3-4\triangle'\odot'\odot+8\triangle'^2\triangle].$$

Generally when n is of the form $2m+1$, it will be seen that $l_{1,\ m}$ $l_{1,\ m+1}$.

In the case of a hexagon, similarly it is obvious that the sides of the triangle $A_1A_3A_5$ all touch the same conic $S'+l_{1,3}\ S=0$. Hence Salmon's theorem directly gives

$$[\odot'-3\triangle l_{1,3}]^2=4\ [\triangle'+\triangle l_{13}']\ [\odot+3\triangle l_{1,3}]$$

On substituting the value of $l_{1,3}$ this will be seen to be equivalent to

$$3\ [\odot'^2-4\triangle'\odot]^3+16\triangle'\odot[\odot'^2-4\triangle'\odot]^2+96\triangle'^2\triangle\odot'(\odot'^2-4\triangle'\odot)$$
$$+512\triangle'^4\triangle^2=0.$$

In the solution to Q. 831 (Vol. V. J.I.M.S.), Prof. Wilkinson has obtained the condition for the porism of a hexagon in this form. Generally, when n is of the form $3p$, Salmon's theorem directly gives the condition for porism in the form

$$[\odot'-3\triangle l_{1,}{}^2{}_{p+1}]^2=4[\triangle'+\triangle l_{1,}{}^3{}_{p+1}][\odot+3\triangle l_{1,\ p+1}].$$

F. H. V. GULASEKHARAM.

A Geometrical Problem.

Given the circumcircle and the orthocentre of a triangle to find the locus of the in- and ex-centres.

1. Let S, H denote the circumcentre and the orthocentre, and I the *in* or *ex*-centre. Then

$$SI^2=2R.NI, \qquad \ldots \qquad \ldots \quad (1)$$

where N is the mid-point of SH or *nine points centre*, and R the *given* circumradius. Thus the locus of I is defined by equation (1), and is readily seen to be a *Cartesian*.

For, the above equation may be written

$$(x^2+y^2)^2=4R^2\{(x-c)^2+y^2\} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

S being the origin of cartesian coordinates, and N the point $(c, 0)$.

Equation (2) may again be put into the form

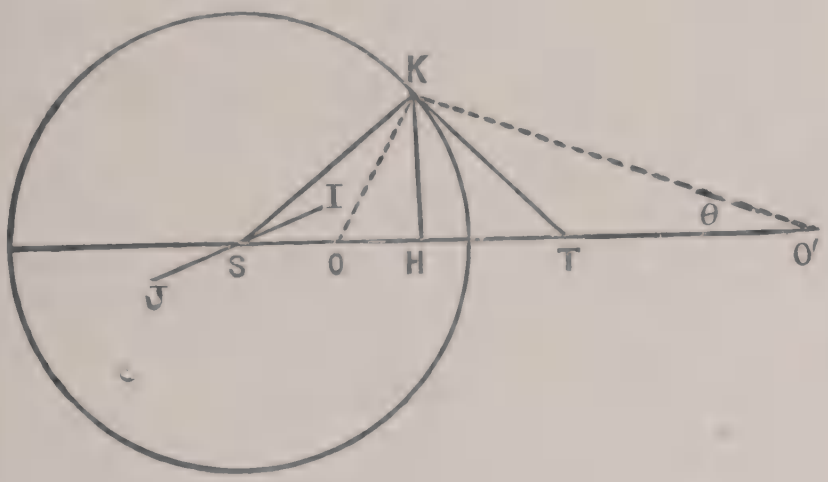
$$\{(x^2+y^2-2R^2\}^2+4R^2(2cx-R^2-c^2)=0,$$

or

$$S^2+I^2u=0, \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where s is a circle, u a straight line and I the line at infinity. The form (3) shows that the locus is a cartesian, whose double tangent is $u=0$; the points of contact being the real points of intersection of the curve and the circle s.

2. The focal properties of the curve are found as follows :—



From T the inverse of H with respect to the circumcircle draw TK tangent to the circle. Measure TO, TO′ equal to TK. Then O, O′ are two foci of the Cartesian.

For, the circle OKO′ touches SK at K, and therefore SO/SK $=$OK/O′K $=$cot KO O′ $=m$ (say)

$$\therefore \quad SO=mR.$$

and

$$SO'=R/m.$$

Also

$$SH.ST=SK^2,$$

$$\therefore \quad cR\left(m+\frac{1}{m}\right)=R^2.$$

$$\therefore \quad c=Rm/(1+m^2).$$

Denoting IO, IO′ by ρ, ρ', we have

$$\rho^2=(x-Rm)^2+y^2;$$
$$\rho'^2=(x-R/m)^2+y^2.$$

$\therefore$ $$\rho^2-m^2\rho'^2=(x^2+y^2-R^2)(1-m^2),$$
$$\rho^2+m^2\rho'^2=(x^2+y^2+R^2)(1+m^2)-4xmR.$$

$\therefore$ $$\{\rho^2-m^2\rho'^2\}^2-2k^2\{\rho^2+m^2\rho'^2\}$$
$$=(x^2+y^2-R^2)^2(1-m^2)^2-2k^2\{(x^2+y^2+R^2)(1+m^2)-4xmR\}$$
$$=(1-m^2)^2(x^2+y^2)^2-2(x^2+y^2)\{R^2(1-m^2)^2+k^2(1+m^2)\}$$
$$+8xk^2mR+R^4(1-m^2)^2-2k^2R^2(1+m^2)$$
$$=k^4, \text{ after reduction by using (2),}$$

where $k^2=R^2(1-m^2)^2/(1+m^2)=(SO'-SO)^2/(1+1/m^2)=OK^2$.

Hence, the relation between ρ and ρ' is

$$\{\rho^2-m^2\rho'^2\}^2-2k^2(\rho^2+m^2\rho'^2)+k^4=0,$$

which can be reduced to the form

$$\rho\pm m\rho'\pm k=0,$$

shewing that the foci are O, O′.

3. Now $\rho+m\rho'>m(\rho+\rho')$, since $m<1$ in the figure; and $(\rho+\rho')>OO'>KO'$ or k/m.

Hence $\rho+m\rho'>k$, for real values of ρ, ρ'.

The *conjugate ovals* are therefore given by $\rho-m\rho'\pm k=0$, the upper and lower signs corresponding to the inner and outer ovals.

4. The equations of the ovals in question may be written in the form

$$\rho-\rho'.\ \tan\theta\pm 2a\ \sin\theta=0;$$

or $$\rho\cos\theta-\rho'\sin\theta\pm a\sin 2\theta=0;$$

if $2a$ is written for OO′.

The fundamental property of this oval may therefore be stated as follows:—

If I is a point on the inner oval, produce IS to J such that JS=SI, and with J as centre and a radius equal to 2R describe a circle cutting the outer oval in I_1, I_2, I_3. Then the system $I\,I_1\,I_2\,I_3$ will be orthocentric, and the harmonic triangle ABC which is inscribed in the circle S will have the fixed point H for its orthocentre.

M. T. Naraniengar.

Astronomical Notes.

In Memorium.

During the last two years the Notes under this head were being written by Mr. R.J. Pocock, B.A., B.Sc, F.R.A.S., late Director of the Nizamiah Observatory, Hyderabad (Deccan), who fell a victim early last month to that fell disease, the Spanish Influenza. The deceased was much attached to his work at the Observatory, and in 1917 published a "*Star Chart*", which consists of a single circular disc which can be made to rotate so as to show at a glance the stars visible in India for any hour in the year. Very recently in the current year, H. E. H. the Nizam's Government published his "*Astrographic Catalogue*." He wrote frequently in the *Monthly Notices of the Royal Astronomical Society* ; and in India he appealed to the popular reader through his charming weekly notes on astronomical topics in the *Madras Times*. The Society regrets the loss of such an enthusiastic member ; and this column, in particular, will be poorer for the lack of his contributions.

In his memory this column in the current issue has been left vacant.

SOLUTIONS.

Question 930.

(S. Malhari Rao) :—Fill up the vacant cells in the following figure with the remaining numbers of the series 1, 2, 3 ..63, 64, so that the whole figure and each of the four corner subsquares of 16 cells may all be pandiagonal magic squares. (The figure of the question is the square printed below of which only the left hand bottom corner subsquare of 16 cells was filled up. *Ed*).

Solution by R. J. Pocock, B. M. Vasavada B. Sc. and Sadanand.

[*N. B.*—There is a misprint in the question where the number 24 n the cell (6, 1) should be 34].

We may write the given quarter square in the form

$4\times8+7$	$3\times8\times4$	$4\times8\times6$	$3\times8+1$
$4\times8+2$	$3\times8+5$	$4\times8+3$	$3\times8+8$
$3\times8+3$	$4\times8+8$	$3\times8+2$	$4\times8+5$
$3\times8+6$	$4\times8+1$	$3\times8+7$	$4\times8+4$

We can construct 3 other cells in which the cofactors of 8, viz. 4 and 3 are replaced by 5, 2 ; 6, 1 ; 7, 0 respectively, we have then merely to the numbers 1, 2, 3...8 in the subsquares so that each and the whole will be a magic square. We begin by writing down the numbers 1, 2, 3, 4 arrange respectively in four superposable solutions of the eight queens problem which are consistent with the numbers already given. The remaining squares are then easily filled in, adding we obtain the solution as follows :—

$5\times8+5$	$2\times8+2$	$5\times8+8$	$2\times8+3$	$7\times8+1$	6	$7\times8+4$	7
$5\times8+4$	$2\times8+7$	$5\times8+1$	$2\times8+6$	$7\times8+8$	3	$7\times8+5$	2
$2\times8+1$	$5\times8+6$	$2\times8+4$	$5\times8+7$	5	$7\times8+2$	8	$7\times8+3$
$a\times8+8$	$5\times8+3$	$2\times8+5$	$5\times8+2$	4	$7\times8+7$	1	$7\times8+6$
$4\times8+7$	$3\times8+4$	$4\times8+6$	$3\times8+1$	$6\times8+3$	$8+8$	$6\times8+2$	$8+5$
$4\times8+2$	$3\times8+5$	$4\times8+3$	$3\times8+8$	$6\times8+6$	$8+1$	$6\times8+7$	$8+4$
$3\times8+3$	$4\times8+8$	$3\times8+2$	$4\times8+5$	$8+7$	$6\times8+4$	$8+6$	$6\times8+1$
$3\times8+6$	$4\times8+1$	$3\times8+7$	$4\times8+4$	$8+2$	$6\times8+5$	$8+3$	$6\times8+8$

Question 945.

(SADANAND) :—Fill up the vacant cells of the following magic square :—(The Square printed below with only the two diagonals filled)

Solution (1) *by B. M. Vasavada. B. Sc.,* (2) *by N. P. Pandya,* (3) *by E. Rama Iyer, and* (4) *by Mehta Kanhyalal.*

(1)

46	47	31	65	16
80	54	8	39	24
5	52	27	37	84
1	50	81	35	38
73	2	58	19	43

(2)

46	24	66	53	16
3	54	64	3	45
20	41	27	48	69
63	50	25	35	32
73	36	23	30	43

(3)

46	57	33	13	16
40	54	30	29	42
36	21	27	41	80
10	50	86	35	24
73	23	29	37	43

(4)

46	44	51	48	16
47	54	37	39	28
31	32	27	49	66
8	50	60	35	52
7	25	30	34	43

Question 947.

(M. K. KEWALRAMANI) :—A triangle ABC inscribed in any ellipse touches a confocal ellipse at the points D,E,F respectively. Show that the ratio of the triangle DEF to ABC is r/2R, where r and R refer to the triangle ABC.

Solution (1) *by K. B. Madhava and Hemraj,* (2) *by A. C. L. Wilkinson.*

(1) It is known that D, E and F are the points of contact of the escribed circles of ABC with its sides : cf. Smith : *Geometrical conics* P. 179 Ex 68. (Hence the actual areals of D, E, F are O, $(s-b)/a$,$(s-c)/a$; etc. We have therefore

$$\frac{\triangle DEF}{\triangle ABC} = \frac{1}{abc}\begin{vmatrix} o & s-b & s-c \\ s-a & o & s-c \\ s-a & s-b & o \end{vmatrix}$$

$$= 2(s-a)(s-b)(s-c)/(abc)$$

$$= 2.\frac{\triangle^2}{s.}\cdot\frac{1}{4R\triangle} = \frac{r}{2R}.$$

(2) The ellipse in trilinears being $lyz+mzx+nxy=0$, we have for the tangential equation of any confocal—

$$p^2l^2+q^2m^2+r^2n^2-2plm-2qrmn-2rpnl$$

$$+\lambda(p^2+q^2+r^2-2pq\ \cos C-2qr\ \cos A-2pr\ \cos B)=0$$

This touches the sides of the triangle of reference if

$$l^2=m^2=n^2=-\lambda$$

Thus we take $yz+zx+xy=0$ for the ellipse circumscribing ABC and for the confocal

$$pq\ \sin\frac{^2C}{2}+qr\ \sin\frac{^2A}{2}+rp\ \sin\frac{^2B}{2}=0.$$

The tangential equation of the point of contact with (1, 0, 0) is

$$q\ \sin\frac{^2C}{2}+r\ \sin\frac{^2B}{2}=0.$$

Whence D is (0, $s-b$, $s-c$).

It follows that $AF=CD=s-b$; $BF=CE=s-a$; $AE=BD=s-c$ and the ratio of the areas of the triangles ABC, DDF is at once seen to be $\frac{r}{2R}$.

Question 948.

(M. K. KEWALRAMANI) :—A perfectly elastic particle, acted on by on forces is projected from the centre of a rectangle whose sides are $2a$ and $2b$ $(a > b)$ to strike the bigger side first and then go on rebounding from its sides. If it ever pass through an angular point show that the direction of projection makes with the smaller side an angle whose tangent is $\frac{a}{b}.\frac{2n+1}{2m+1}$ where m and n are integers; and if it ever strike any point of the smaller side other than the extremities, the tangent of the angle must be of the form $\frac{a}{b}.\frac{2n+1}{2m}$

Solution by S. R. Ranganathan, M.A., L.T.

Let the particle first impinge on the longer side at a distance of x from its midpoint. Then the angle of projectionis $\tan^{-1}\frac{x}{b}$, so that it has to be shown that if the particle ever passes through a corner, x must be of the form $\frac{a(2n+1)}{2m+1}$.

Measuring distances parallel to the longer side from its midpoint and neglecting the *sense* of the distances, it is easily seen, because the particle is perfectly elastic, that between two consecutive impacts on a longer side, the particle is carried through a distance $2x$, irrespective of its impinging on a shorter side in the interval.

Hence, after m impacts! at the longer sides, the particle will be at a distance of $(2m+1)x$. If it ever paases through a coner, this distance must be a, or $(2n+1)a$, n being any integer. Hence, we get

$$x = \frac{a(2n+1)}{2m+1}.$$

The second part of the question does not seem to be correct. For taking the simple case of $a = \frac{3}{2}b$ and the angle of projection being $\frac{\pi}{4}$, it is easily sesn, that the second impact of the particle will be with a shorter side at a point other than an extremity. But, in such a case, according to the question $\tan\frac{\pi}{4}$ *i.e.* 1 must be of the form

$$\frac{a}{b}.\frac{2n+1}{2m} \text{ i.e. } \frac{3}{2}.\frac{2n+1}{2m}$$

which is untrue, since $3(2n+1)$ is odd and $4m$ is even.

But the condition that the angle should be of the given form is necessary that the particle may impinge at the midpoint of a shorter side.

Question 949.

(C. KRISHNAMACHARY) :—With the usual notation for the numbers of Bernoulli and Euler, show that

$$(1)\quad 2n = B_1 2^2(2^2-1)\binom{2n}{2} - B_2 2^4(2^4-1)\binom{2n}{4} + B_3 2^6(2^6-1)\binom{2n}{6} - \ldots + (-1)^{n-1} B_n 2^{2n}(2^{2n}-1)$$

$$(2)\quad (-1)^n E_n(2n+1) = (-1)^n B_n 2^{2n}(2^{2n}-1\binom{2n+1}{1} + (-1)^{n-1} B_{n-1} 2^{2n-2}(2^{2n-2}-1)\binom{2n+1}{3} \ldots + \ldots - B_1 2^2(2^2-1)\binom{2n+1}{2} + (2n+1).$$

Solution by Hemraj.

(1) This is given in the reverse after putting

$$S_{2n-1} = \frac{2^{2n}(2^{2n}-1)}{2n\ B_{2n-1}} B.$$

in Art. 575 (7) of Edwards Diff. Cal.

(2) Following the notation of Arts: 571, 572, 573, we have

$$\sin x = \cos x \left[S_1 \frac{x}{1!} + S_3 \frac{x^3}{!} + S^5 \frac{x^5}{5!} + \ldots\ldots \right]$$

Expanding sin x and cos x, and equating the coefficients of x^{2n+1}, we get

$$S_{2n+1} = {}^{2n+1}C_2 S_{2n-1} - {}^{2n+1}C_4 S_{2n-3} + {}^{2n+1}C_6 S_{2n-5} - \ldots \ldots + (-1)^n \ldots \quad \ldots \quad \ldots \quad (A)$$

Again $\sec x + \tan x = 1 + S_1 \frac{x}{1!} + S_2 \frac{x^2}{2!} + S_3 \frac{x^3}{3!} + \ldots = \sigma$ (say)

Differenting

$$\sec x\ (\sec x + \tan x) = \frac{d\sigma}{dx} \quad i.e.\ \sec x + \tan x = \cos x \frac{d\sigma}{dx}$$

Equating the coefficients of x^{2n}, we have

$$\begin{aligned} S_{2n} &= S_{2n+1} - 2nC_2 S_{2n-1} + \ldots (-1)^n S_1 \\ &= ({}^{2n+1}C_2 - {}^{2n}C_2) S_{2n-1} - ({}^{2n+1}C_4 - {}^{2n}C_4) S_{2n-3} + ({}^{2n+1}C_6 - {}^{2n}C_6) S_{2n-5} - \ldots \\ &\quad + (-1)^{n-1}({}^{2n+1}C_{2n} - {}^{2n}C_{2n}) S_1 + (-1)^n \qquad \text{from (A)} \\ &= {}^{2n}C_1 S_{2n-1} - {}^{2n}C_3 S_{2n-3} + {}^{2n}C_5 S_{2n-5} - + (-1)^n \\ &= 2^{2n}(2^{2n-1}) B_{2n-1} - \frac{2n(2n-1)}{3!} B_{2n-3} 2^{2n-2}(2^{2n-2}-1) + \ldots\ldots \\ &\quad + (-1)^{n-1} {}_{2n}B_1 \frac{2^2(2^2-1)}{2} + (-1)^n. \end{aligned}$$

$$\therefore\ (-1)^n(2n+1)\ E_n = (-1)_n B^n 2^{2n}(2^{2n}-1)\binom{2n+1}{1} + (-1)^{n-1} B_{n-1} 2^{2n-2}(2^{2n-2}-1)\binom{2n+1}{3}$$

$$+\ldots-B_1 2^2(2^2-1)\frac{(2n+1)}{2}+(2n+1).$$

or $$\tan x = S_1 x + S_3 \frac{x^3}{3!}+\ldots\ldots$$

Differentiating $$\sec x = \cos x\left[S_1+S_3\frac{x^2}{3!}+\ldots\right]$$

i.e. $$1+S_2\frac{x^2}{2!}+\ldots\ldots = \left[1-\frac{x^2}{2!}+\ldots\right]\left[S_1+S_3\,\delta\frac{x^2}{2!}+\ldots\right]$$

Equating coefficients of x^{2n}, we get the result stated.

Question 950.

(C. KRISHNAMACHARI);—Collect the coefficients of x^n in the series

$$\frac{1}{1-x}-\frac{x}{(1-x^2)}+\frac{1.3}{2!}\frac{x^2}{(1-x)^3}-\frac{1.3.5}{3!}\frac{x^3}{(1-x)^4}+\ldots$$

and show that their sum is zero when n is odd, and is

$$(-1)^m\cdot\frac{1.3.5\ldots(2m-1)}{2.4.6\quad\ (2m)}$$

when n is $2m$.

Solution by Hemraj, Sadanand, Alpha, B. M. Vasavada and S. R. Ranganathan.

The given series is equal to

$$\frac{1}{1-x}\left\{1-\tfrac{1}{2}\left(\frac{2x}{1-x}\right)+\frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\left(\frac{2x}{1-x}\right)^2-\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}\left(\frac{2x}{1-x}\right)^3+\ldots\right\}$$

$$=\frac{1}{1-x}\left\{1+\frac{2x}{1-x}\right\}^{-\frac{1}{2}}$$

$$=\frac{1}{1-x}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}$$

$$=(1-x^2)^{-\frac{1}{2}}$$

$$=1+\frac{1}{2}x^2+\frac{1.3}{2,4}x^4+\frac{1.3.5}{2.4.6}x^6+\ldots+\frac{1.3.5.-\ldots(2m-1)}{2.4.6.\qquad(2m)}x^{2m}+\ldots$$

The absence of an odd power of x clearly shows that the sum of the coefficient of x^n is zero when n is odd.

The coefficient of x^{2m} is $\dfrac{1.3.\ldots.(2m-1)}{2.4.\ldots.\ (2m)}$, which shows that the sum is $\dfrac{1.3.5.\ldots.(2m-1)}{2.4.6.\ldots.\ (2m)}$ when n is equal to $2m$. The factor $(-1)^m$ with which it is multiplied in the question as printed is incorrect.

Question 951

(C. KRISHNAMACHARY):—Prove that

$$(1)\quad \int_0^{\frac{\pi}{4}} (\log \tan x)^{2n}\, dx = \frac{E_n}{2^{2n+2}}\pi^{2n+1},$$

$$(2)\quad \int_0^{\frac{\pi}{4}} \frac{(\log \tan x)^{2n-1}}{\cos x\,(\cos x+\sin x)}\, dx = -\frac{(2^{2n-1}-1)}{2n} B_n \pi^{2n},$$

$$(3)\quad \int_0^{\frac{\pi}{4}} \frac{(\log \tan x)^{2n-1}}{\cos^2 x}\, dx = -\frac{(2^{2n}-1)}{4n} B_n \pi^{2n},$$

$$(4)\quad \int_0^{\frac{\pi}{4}} \frac{(\log \tan x)^{2n-1}}{\cos x\,(\cos x-\sin x)}\, dx = -\frac{2^{2n-1}}{2n} B_n \pi^{2n}.$$

Solution (1) *by A. C. L. Wilkinson* ; (2) *by Hemraj and Sadanand.*

For (4), we have

$$\int_0^1 \frac{(\log x)^{2n-1}}{1-x}\, dx = \sum_{r=0}^{\infty} \int_0^1 (\log x)^{2n-1} x^r dx$$

$$= -\sum_0^{\infty} \frac{(2n-1)}{(r+1)^{2n}} = -\frac{2^{2n-1} B_n}{2n} \pi^{2n}.$$

(See : Carr : *Synopsis*, It 2395—2397, whence the formula

$$B_n = 4^n \int_0^{\infty} \frac{y^{2n-1}}{e^{2\pi y}-1} dy$$

is derived ; also, Ramanujan : I. M. J., Vol III, *p.* 226.)

For (2), write x^2 in place of x and use the formula

$$\frac{2x}{1-x^2} = \frac{1}{1-x} - \frac{1}{1+x}.$$

For (3), add (2) and (4).

For (1), we repalce it by

$$\int_0^1 \frac{(\log x)^{2n}}{1+x^2}\, dx = \sum_0^{\infty} (-) \frac{(2n)!}{(2r+1)^{n+1}}.$$

In this write $x = e^{-\pi x}$ and deduce the integrals

$$\int_0^\infty \frac{\cosh 2ax}{\cosh \pi x}\,dx=\frac{1}{2}\sec a,\ \int_0^\infty \frac{\cos 2ax}{\cosh \pi x}\,dx=\frac{1}{2}\operatorname{sech} a,\left(-\frac{\pi}{2}<a<\frac{\pi}{2}\right)$$

(See Boole : *Finite Differences* p. 110).

[Compare also : Bromwich *Infinite Series*—*pp.* 452 and 473.]

(2) In each case the result follows by putting log tan $x=z$ and applying Edwards, *Diff. Calc.* § 574.

Question 952.

(HEMRAJ):—The polar reciprocal of an equiangular spiral with respect to a rect. hyperbola having its centre at the pole and touching the spiral is the curve itself.

Solution (1) *by Sadanand*, (2) *by Alpha and L. S. Vaidyanathan, Sadanand.*

(1) Let the hyperbola and the spiral be $pr=a^2$, (2) and $r\sin\alpha=p$. Then if (R,β) be any point, its polar with respect to the hyperbola is $r\cos(\theta+\beta)=\frac{a^2}{R}$.

If it touches the spiral, then $\frac{a^2}{R}=p=r\sin\alpha$, giving $Rr=\frac{a^2}{\sin\alpha}$. The symmetry of this result shows that the polar will touch the spiral at (R,β).

Also it is clear that the curves touch where $R=r$ and hence the result.

(2) Let the rectangular hyperbola be $r^2\cos 2\theta=c^2$, and the spiral be

$$r=a\,e^{\theta\cot\alpha}.$$

These two touch at the point whose vectorial angle is $\frac{1}{2}(\pi-\alpha)$, if

$$c^2=a^2.\sin\alpha.e^{\left(\frac{\pi}{2}-\alpha\right)\cot\alpha}, \quad \ldots \quad \ldots \quad (1)$$

Now, the tangent to the spiral at (r_1, θ_1) is

$$r\cos(\theta-\theta_1-90+\alpha)=r_1\sin\alpha=a.\sin\alpha.\,e^{\theta_1\cot\alpha}.$$

i.e. $$r\sin(\alpha+\theta_1-\theta)=a\sin\alpha\,e^{\theta_1\cot\alpha} \quad \ldots \quad \ldots \quad (2)$$

If the pole of this with respect to the rect. hyp. be (ρ,ϕ), its equation must be identical with

$$r.\rho.\cos(\theta+\phi)=c^2. \quad \ldots \quad \ldots \quad (3)$$

Hence equating coeffts of $r\cos\theta$ and $r\sin\theta$ in (2) and (3) we get

$$\frac{\sin(\alpha+\theta_1)}{\rho.\cos\phi}=\frac{\text{ccs. }(\alpha+\theta_1)}{\rho.\sin\phi}=\frac{a.e^{\theta_1\cdot\cot\alpha}.\sin\alpha}{c^2};$$

whence $\tan\phi=.\cot(\alpha+\theta_1)$, or $\phi=\frac{\pi}{2}-(\alpha+\theta_1)$;

and
$$\frac{1}{\rho}=\frac{a.e^{\theta_1\cot\alpha}\sin\alpha}{c^2}$$

$$=\frac{a\sin\alpha\, e^{\left(\frac{\pi}{2}-\alpha-\phi\right)\cot\alpha}}{a^2.\sin\alpha\, e^{\left(\frac{\pi}{2}-\alpha\right)\cot\alpha}}=\frac{e^{-\phi\cot\alpha}}{a}$$

or
$$\rho=a\,e^{\phi\cot\alpha}.$$

Hence the polar reciprocal is the locus of (ρ, ϕ) and is the original spiral itself'

Question 953.

(Communicated by Hemraj):—Two of the common tangents to a circle S and a conic T meet in P and the other two in Q. Show that P and Q lie on the same confocal to T.

Solution by A. C. L. Wilkinson.

Consider the following theorem.

S. S′ are two conics, C is a conic having double contact with S, then a conic can be described through the intersections of S′ and C which will touch a pair of common chords of S and S′ at the points where the chord of contact of the conics S, C meets them.

Let AB, CD be a pair of common chords of S and S′ and take ABC as triangle of reference; so that

S is $yz+zx+xy=0$,

S′ is $ayz+bzx+cxy=0$.

Then CD is $(a-c)\,y+(b-c)\,x=0$.

Then C is $xy+yz+zx+(\lambda x+\mu y+\nu z)^2=0$,

and consider the conic

$$xy+yz+zx+(\lambda x+\mu y+\nu z)^2-\frac{1}{c}(ayz+bzx+cxy)=0,$$

This passes through the intersections of S′ and C and since it can be written

$$z\{(c-a)y+(c-b)x\}+c(\lambda x+\mu y+\nu z)^2=0,$$

it is a conic touching $z=0$, $(c-a)y+(c-b)x=0$ at the points where these lines are met by $\lambda x+\mu y+\nu z=0$.

The reciprocal theorem is

S. S′ are two conics and C a conic having double contact with S, then a conic can be described touching the four common tangents of C and S′ and passing through two of the opposite vertices of the quadrilateral formed by the four common tangents to S and S′ and the tangents at these points are the lines joining them to the pole of the chord of contact of the conics S and C with respect to S or C.

Let C degenerate to two points on S and then project these points into the circular points at infinity and the theorem as stated follows with the additional theorem that the tangents at P and Q to the confocal pass through the centre of thegiven circle. This is of course immediately obvious geometrically.

It is curious to notice that we can project this theorem into its own converse, for if we project P, Q into the circular points at infinity, the theorem to be proved becomes :

If from two points Ω, Ω′ lying on a conic C tangents be drawn to a confocal S, these tangents will touch a circle.

To prove the theorem in this form, reciprocate with respect to a focus, and we get the following theorem.

If two tangents to a circle meet another circle in four points, these four points lie on a conic having one focus at a limiting point of the two circles

This may be proved as follows :—

Take the pair of tangents

$$S\Sigma-\Delta(lx+my+n)^2=0$$

to the circle $S=0$ and consider conics through their intersection with $S'=0$, then if $S+\lambda S'=0$ be a point circle,

$$(S+\lambda S')\Sigma-\Delta(lx+my+n)^2=0$$

is a conic fulfilling the required conditions.

QUESTIONS FOR SOLUTION.

998. (F. H. V. GULASEKHARAM) :—If $u_1+u_2+u_3+u_4=2x$;

$$s_r=\mathrm{sn}\,(u_r, k);\ c_r=\mathrm{cn}(u_r, k)_3 d_r=\mathrm{dn}\,(u_r, k)\ ;$$

$$s'_r = \frac{\partial s_r}{\partial u_r};\ c'_r=\frac{\partial c_r}{\partial u_r};\ d'_r=\frac{\partial d_r}{\partial u_r},$$

prove that $\mathrm{E}(u_1)+\mathrm{E}(u_2)+\mathrm{E}(u_3)+\mathrm{E}(u_4)-2\mathrm{E}(x)$

$$=\frac{k^2c_1c_2c_3c_4}{k'^2-k^2c_1c_2c_3c_4}\sum_{r=1}^{4}\frac{c_r'}{c_r}$$

$$+\frac{2k^2\,\mathrm{sn}\,x\,\mathrm{dn}\,x\,\mathrm{cn}(x-u_3-u_2)\,\mathrm{cn}(x-u_2-u_4)\,\mathrm{cn}\,(x-u_4-u_3)}{k'^2-k^2\,\mathrm{cn}\,x\,\mathrm{cn}\,(x-u_3-u_2)\,\mathrm{cn}\,(x-u_2-u_4)\,\mathrm{cn}\,(x-u_4-u_3)}$$

[*Suggested by Q.* 973.]

Prove also that the above expression is equal to the symmetrical result

$$\frac{k^2k'^2\,s_1s_2s_3s_4\sum_{r=1}^{4}\frac{s'_r}{s_r}+k^2c_1c_2c_3c_4\sum_{r=1}^{4}\frac{c'r}{cr}-d_1d_2d_3d_4\sum_{r=1}^{4}\frac{d'r}{dr}}{k'^2k^2k'^2s_1s_2s_3s_4-k^2c_1c_2c_3c_4+d_1d_2d_3d_4.}$$

999. (LAKSHMISHANKAR N. BHATT and K. J. SANJANA) :—Prove that the following function of the four quantities p, q, r, s—

$$2\Sigma p^2q^4+2\Sigma p^3q^2r-2\Sigma p^4qr-4\Sigma p^3q^3-6\Sigma p^2q^2r^2$$

is equal to 1536 $\mathrm{cosec}^4 4\alpha$, when p, q, r, s stand for the tangents or cotangents of the angles α, $\alpha+\frac{\pi}{4}$, $\alpha+\frac{2\pi}{4}$, $\alpha+\frac{3\pi}{4}$.

Prove further that the same function has the value $8\cos^2 2\alpha$, when p, q, r, s stand for the sines or cosines of the angles

$$\alpha,\ \alpha+\frac{\pi}{2},\ \alpha+\frac{2\pi}{2},\ \alpha+\frac{3\pi}{2}.$$

1000. (K. J. SANJANA) :—Prove that

$$(n-1)^{-4}+(n+1)^{-4}+(3n-1)^{-4}+(3n+1)^{-4}+(5n-1)^{-4}+(5n+1)^{-4}+\ldots$$

$$=\frac{\pi^4}{48n^4}\sec^2y\,(3\sec^2y-2),$$

$$(n-1)^{-5}-(n+1)^{-5}+(3n-1)^{-5}-(3n+1)^{-5}+(5n-1)^{-5}-(5n+1)^{-5}+\ldots$$

$$=\frac{\pi^5}{96n^5}\tan y\,\sec^2y\,(3\sec^2y-1),-$$

where $y\equiv\pi/2n$ and n is an integer greater than unity.

1001. (K. J. SANJANA, M.A.) :—Prove that

$$1+(2n-1)^{-4}+(2n+1)^{-4}+(4n-1)^{-4}+(4n+1)^{-4}+\ldots$$
$$=\frac{\pi^4}{48n^4}\operatorname{cosec}^2 y\,(\operatorname{cosec}^2 y-2).$$

$$1+(2n-1)^{-6}+(2n+1)^{-6}+(4n-1)^{-6}+\ldots\ldots$$
$$\frac{\pi^6}{960n^6}\operatorname{cosec}^2 y\,(15\operatorname{cosec}^4 y-15\operatorname{cosec}^2 y+2),$$

$$1+(2n-1)^{-8}+(2n+1)^{-8}+\ldots$$
$$=\frac{\pi^8\operatorname{cosec}^2 y}{80640n^8}(315\operatorname{cosec}^6 y-420\operatorname{cosec}^4 y+126\operatorname{cosec}^2 y-4),-$$

where $y\equiv\pi/2n$ and n is any positive integer.

1002. (N.P. PANDYA) :—Find the lowest prime numbers which can be arranged into a magic square of nine cells. Also find a magic square of nine cells, all numbers in it being perfect squares.

1003. (S. R. RANGANATHAN) :—If $\phi_n(x)$ stands for the *nth.* Bernoulli's polynomial, show that

$$\phi_n\left(\frac{1}{4}\right)=2^{n-1}\left[\phi_n\left(\frac{1}{8}\right)-\phi_n\left(\frac{3}{8}\right)\right],$$

or
$$=\frac{2^{2n-2}+2^{2n-2}-2^{n-1}}{2^{3n-1}+2^{n-1}-1}\left[\phi_n\left(\frac{1}{8}\right)+\left(\frac{3}{8}\right)\right],$$

according as n, which is greater than 1, is odd or even.

1004. (C. KRISHNAMACHARI) :—Show that the effects of the elliptic motion of the earth on aberration in latitude and longitude are given respectively by

$$-C\sin\beta\,[\sin(\odot-\lambda)-e\sin(\varpi-\lambda)]$$

and
$$-C\sec\beta\,[-\cos(\odot-\lambda)+e\cos(\varpi-\lambda)]$$

where $2ra=U.T.C.\sqrt{1-e^2}\sin 1''$ the notation being as in Ball's *Spherical Astronomy.*

1005. (K. B. MADHAVA) :—Show that when $|x|<1$, and the real part of z is positive and less than 1, $\Gamma(1-z)=\lim_{x\to 1}(1-x)^{1-z}\,f(x,z)$, where

$$f(x,z)=\sum_{n=1}^{\infty}{}'\cdot\frac{x^n}{n^z}.$$

1006. (K. B. MADHAVA) :—It is known that the hypergeometric series $F(a, b, c, x)$ diverges if $a+b=c$. Show that as $x\to 1-0$, the series however, becomes infinite like

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\log\left(\frac{1}{1-x}\right).$$

1007. (ALPHA) :—Prove that if

$$S_o=1!-2!+3!-4!+\ldots,$$

then $1(1!)-2(2!)+3(3!)-4(4!)+\ldots=1-2S_o$,
$1^2(1!)-2^2(2!)+3^2(3!)-4^2(4!)+\ldots=5S_o-2$;
and generally $\Sigma(-)^{n-1}n^k(n!)$ is of the form $\alpha S_o+\beta$, where α and β are integers (positive or negative).

(Bromwich : *Infinite Series*, p. 482, Ex. 21)

1008. (Communicated by Mr. Martyn M. Thomas) :—If I_n is written for the integral $\int_0^{\frac{\pi}{2}} \cos 2nx \log(2\cos\frac{1}{2}x)\,dx$,

prove that $\frac{1}{n}I_{n+1}-\frac{1}{n+1}I_n=\frac{(-)^{n+1}}{2n(n+1)(2n+1)}$;

and deduce that

$$2nI_n=\left(1-\frac{1}{3}+\frac{1}{5}-\ldots=\frac{1}{2n-1}\right)-\frac{\pi}{4}.$$

1009. (Communicated by Mr. Hemraj) :—A B C is a triangle ; O, P are its circumcentre and orthocentre. If O P meets AB, AC in B_1, C_1 ; prove that the join of the circumcentre and the orthocentre of AB_1C_1 is parallel to BC.

1010. (K. J. Sanjana, M.A.,):—Solve the equation $x^3+y^3=a^3+b^3$ where a and b are given positive quantities and x and y are required to be postive. Examples :—

$$x^3+y^3=19\equiv\left(\frac{5}{2}\right)^3+\left(\frac{3}{2}\right)^3, x^3+y^3=91=4^3+3^3.$$

1011. (Selected):—In an ellipse the tangent at P cuts the directrices in Z, Z′, and the remaining tangents from Z, Z′ to the ellipse meet at T. Shew that P T is normal to the ellipse and bisected by the minor axis.

1012. (N. Sankara Aiyar) :—Solve

(1) $x^6+12x^5+45x^4+100x^3+120x^2+78x+21=0$
(2) $2x^4+12x^3+30x^2+32x+12=0$

by expressing the left hand members in the forms (A^6+B^6) and (A^4+B^4), respectively.

1013. (S. R. Ranganadhan) :—If $N=a_1a_2a_3a_4$ where the a's are mutually prime integers show that the product of the numbers less than N and prime to it is given by

$$\frac{(a_1a_2a_3a_4-1)!\;\Big|\Big|\,(a_1a_2-1)!}{\Big|\Big|\left\{a_1^{(a_2-1)(a_3-1)(a_4-1)}(a_1a_2a_3-1)!\;(a_1-1)!\right\}}$$

1014. (S. R. Ranganadhan):—Prove this extension of Wilson's Theorem. If m be any nonprime number, then

$\pi d(m)+(1-1)^{\frac{m-1}{2}}\equiv m$ when m iseven $\pi d(m)+(-)^{\frac{m-1}{2}}\equiv m$ when m is odd where $\pi d(m)$ denotes the product of numbers less than m and prime to it.

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254	"closing balance"	297-4-4	2997-4-4
256	last	$\longrightarrow 1$	$x \longrightarrow 1$
257	20	$\frac{1}{1^r}+\frac{1}{3^r}$	$\frac{1}{1^r}+\frac{1}{2^r}+\frac{1}{3^r}$
258	2	$_n-4$	$_n-\delta$
266	20	are right	are at right
266	23	point	points
267	1&2	direction	directions
269	6	$d\overline{\theta+n+1}$	$d\theta+(n+1)^d\phi$
269	11	$(n+2c_n$	$(n+2)c_n$
269	11	$^{n+1}$	$n+1$
270	2	$d\ b\ r\ d\ \alpha$	$a\ b\ r\ d\ \alpha$
270	3	$c-_8$	$c-_2$
270	7	$^{b}_{2}$	b^2
270	17	disrict	district
276	5	given angle	given angle α
283	5 from last	Here	(ii) Here
284	1	,,	'Insert' (3)
287	3	$\Gamma(\alpha+\frac{1}{2}\delta)$	$\Gamma(\alpha+\frac{1}{2})$
287	3	$\delta(\alpha+\frac{1}{2}\psi)$	$\psi(\alpha+\frac{1}{2})$
288	17	form	forms
288	19	ein	sin
288	23	$\frac{b^2}{a^2}.a^2$	$\frac{^4c}{^2a}$
288	last	$\frac{\alpha}{-\alpha^2}$ $_{n+1}$	$\frac{\alpha}{1+\alpha^2}$ $_{2n+1}$
289	6	i_2	i
290	6	were	where
290	3 from last	24	34
292	Do.	Q.922	Q. 942
292	Q. 941	Q. $3<y.\frac{x}{2}$	$3<y\leq\frac{x}{2}$.
302	6 from last	$(z-)$	$(z-a)^2$
309	3	sec cos α	sec β cos α
309	5	(tan β sin α)	tan β sin α)
312	25	mathematics	mathematics

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315	9 from last	priorty	priority
319		Question 329	Q. 829
321	7	$(lx+ly)^2+(mx+ny)^2$	$(lx+my)^2+(mx+ny)^2$
322	7	2π	$\frac{\pi}{2}$
322	11	sec $^{2n}\phi$	$\cos^{2n}\phi$
322	3 from last		Insert "$d\phi$"
323	8	A. H.	A. A.
324	10	in centre	centre
324	14	if	it
326	6	condition	conditions
327	5	9000	9900
327	5	90	99
329	16	p_1	p
330	7	the second ($^{s}{}_{2n}$)	s_{2n-1}
330	8	Bemoullean	Bernoullian
330	4 from last	lipse	ellipse
336	12	$\frac{l+mx_r+nx_r^2}{xr}$	$\frac{l+mx_r+nx_r^2}{x_r}$
342	2 from last	n	in
346	10	x_r	x
347	7	d	dx
351	17	mology	homology
354	10	commun	common
356	16	$-3r^2+3rs$	$-3q^2+3rs$
356	20	$2p+q$	$2q+p$
362	2	$\frac{\mu}{d\mu}$	$\frac{d\mu}{dx}$
363	last	esult	result
375	7 from last	17	1917
383	4	form) $3q$	form ($3q$
399	3 from last	$\left(\log 1-\frac{1}{y}\right)^2$	$\log^2\left(5-\frac{1}{y}\right)$
408	6	term	terms
410	Q·971	esultant	resultant
410	Q·971	*Hydrostatecs*	*Hydrostatics*
423	5 from last	fro	for
431	6 from last	Lagranges	Lagrange's
433	16	$1849=43$	$1849=43^2$

5	16&17	$ab-n^2$	$ab-h^2$
435	23	A B sin $^2\theta\cos^2\theta$	AB sin $^2\theta\cos^2\theta\lambda^2$
435	25	{ }	{ } ^{2}
436	8	K	k
437	6 from last	C	c
441	12	$\frac{n^7}{7}+\frac{x^1}{11}$	$\frac{x^7}{7}+\frac{x^{11}}{11}$
441	16	$\frac{x_2}{\sqrt{2}}$	$\frac{i\,x^2}{\sqrt{2}}$
443	6 from last	$\int\frac{x^2dx}{1+n^4}$	$\int\frac{x^2dx}{1+x^4}$
446	2	pandiagoraqual Bears	pandiagonal squaı es
456	3	8	s
470	last	$\sum\frac{1}{d}=$	$\sum\frac{1}{d}$
476	16	$^l{}_1,m+1$	$=\Sigma l_{1,m+1}$
481	11	3	39
484	4 from last	2^{2n-1}	2^{2n-1}
487	8 from last	$\pi-\alpha$	$\frac{\pi}{2}-\alpha$
489	8 from last	*fou*	*four*
489	7 from last	1 i	*lie*
490	Q 1000	$(3\sec^2y-1),-$	$(3\sec^2y-1),$
491	Q 100+	$\operatorname{cosec}^2y-2$	$3\operatorname{cosec}^2y-2$
491	Q 1001	126 $\operatorname{cosec}^2y-4$	123 $\operatorname{cosec}^2y-2$
492	Q 1008	$=\frac{1}{2n-1}$	$-\frac{1}{2n-1}$



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PROGRESS REPORT.

1. *The Founder of Our Society.*—The honor of being the Founder of our Society belongs to Mr. V. Ramaswami Aiyar (whose photo accompanys the present number of the Journal), now Special Deputy Collector of Ramnad. His first letter proposing the formation of the Society, dated 25th December 1906, is contained in his letter, dated 3rd April 1907, announcing the formation of the Society under the name '*The Analytic Club.*' The latter was published in the leading Madras dailies of the 4th April 1907; and it is but fitting that we place it on record in the pages of our Journal. It will be observed from the dates, that our Society has just completed the twelfth year of its existence. The discovery of Mr. S. Ramanujan, B. A., F. R. S., one of the greatest living Mathematicians of the world, marks this period of its life. The founder of the Society devoutly hopes that the objects with which the Society was formed will be fullfilled to a greater degree in the future periods of its existence, and expects that all members will unite their best endeavours in the realization of this hope. The letter announcing the formation of the Society is printed below.

2 *Mr. S. Ramanujan, F. R. S.* This distinguished Mathematician has returned to Madras in somewhat indifferent health, after a prolonged stay at Cambridge. By his unique mathematical talents and by the amount of useful and original work, he has raised India in the estimation of the outside world. We extend our most cordial welcome to him and most fervently pray that he may be soon restored to his full vigour to prosecute his glorious work in the scientific world. The committee hope to show their mark of appreciation further by a suitable announcement in the next issue of the Journal.

3. *Mr. T. J. Mirchandani B. A., B. Sc.*—Demonstrator, D. J. S. College, Karachi, has been elected a member of our Society.

POONA,
1st April 1919.

D. D. KAPADIA,

Hon. Joint Secretary.

MR. V. RAMASWAMIAIYER'S LETTER, DATED 3RD APRIL 1907.

SIR,—I recently sent a proposal to some gentlemen interested in mathematics suggesting the formation of a small Mathematical Society. The proposal ran as follows :—

SIR,—I believe several friends interested in Mathematics have felt the present lack of facilities for seeing mathematical periodicals and books. This is a very great disadvantage we are suffering.

I propose therefore that a few friends may at once join and form a small Mathematical Society and subscribe for all the important Mathematical periodicals and, as far as possible, for all important books in Higher Mathematics.

We may call the Society "The Analytic Club" for the present, and have it in view to give it a broader basis with a suitable name by and by.

Our work immediately will be to obtain all the important periodicals and new books and circulate them to members. I shall be glad to undertake the duties of Secretary for the first year and do my best to promote the object in view.

If half a dozen members can be counted upon to join immediately and each subscribe Rs. 25 per annum, we shall be able to make a good start. The Annual Subscription may perhaps be somewhat less, say, if a dozen members can be had ; but even a dozen members paying Rs. 25 per annum would not suffice to enable the Club to obtain the more important books appearing every year. I propose therefore that the subscription be Rs. 25 per annum.

It appears to me necessary also that members should be prepared for a further sacrifice, and I propose that each member should send the journals and books he receives on to the next, and the last to the Secretary, at his own cost. This in effect would be to add about Rs. 5 more to one's subscription. I hope friends interested in mathematics will not consider this a too heavy sacrifice—at any rate initially, in giving the Club a start.

Will you kindly write to me if you are in favour of the proposal ? In case you are, I request you will send me your subscription of Rs. 25 for 1907, as early as possible, so that we may make a start at once.

This is only a tentative scheme and we may try it for a year and then introduce necessary changes.

I propose to consider the Club formed as soon as three friends have agreed to the proposal, making with me four members. Thereafter all business requiring determination by the Club can be done by circulation. Requesting the favour of an early reply. I remain, Sir, yours truly,

(Signed) V. RAMASWAMI AIYAR.

2. In response to this proposal (which I was able to send only to a very limited number of persons) the undermentioned have written to me consenting to become members of the proposed Society :

Messrs :

R. N. Apte, M.A., LL. B., F.R A.S., Prof. of Math., Rajaram College, Kolhapur.

M. V. Arunachala Sastri, M. A., Assistant Professor, Nizam's College, Hyderabad.

K. Chinnatambi Pillai, B.A., Assistant Professor, Christian College, Madras.

B. Hanumanta Rao, B.A., Prof. of Mathematics, College of Engineering, Madras.

D. K. Hardikar, B.A., Professor of Mathematics, Nizam's College, Hyderabad.

G. Kasturiranga Aiyangar, M.A., Lecturer, Maharajah's College, Mysore.

B. Krishnamachari, M.A., Asst. Superintendent, Accountant-General's Office, Madras.

A. V. Kuttikrishna Menon, M. A., Teachers' College, Saidapet.

V. Madhava Rao, M.A., Prof. of Mathematics, Maharaja's College, Vizianagaram.

M. T. Narayana Aiyangar, M.A., Prof. of Mathematics, Central College, Bangalore.

R. P. Paranjpye, B. sc., M.A., Principal and Professor of Mathematics, Fergusson College, Poona.

R. Ramachandra Rao, B. A., Collector of Kurnool.

K. J. Sanjana, M A., Principal and Professor of Mathematics, Samaldas College, Bhavnagar, Kathiawar.

P. V. Seshu Aiyar, B.A., Lecturer, Government College, Kumbhakonam.

S. P. Singaravelu Mudaliar, B.A., Asst. Professor of Mathematics, Christian College, Madras.

R. Swaminatha Aiyar, B.A., Treasury Deputy Collector, Coimbatore.

T. R. Venkataswami Nayudu, B.A., Professor of Mathematics, Maharaja's College, Mysore.

K. Krishnan Nayar, B.A., B.C.E., District Board Engineer, Mangalore.

S. A. Subramania Aiyar, B.A, B.C.E., Executive Engineer, Madras P. W. D., Madanapalli.

With me it makes 20 members.

3. I beg to declare on behalf of all those that have joined, that the Society is now formed, under the proposed name "The Analytic Club" for the time being; and I shall be its Secretary provisionally.

4. My thanks are due to the gentlemen who have joined, for the support they have given me in starting the Club. The membership has already exceeded my modest anticipations, and many more, I think, will be joining. This renders some changes and a better organization at once necessary. I shall soon be submitting to members proposals for a simple Constitution for the Society according to which the affairs of the Society will be managed by a committee consisting of a President, a few

office-bearers and some additional members. From the support that I have received in this respect also, I feel we shall have a Committee giving the greatest possible confidence.

5. I think, the Society should now aim at being a combination of all Professors, Assistant Professors and Lecturers, and men of like interest in Mathematics or in the promotion of Mathematical study, in whatever province of India, with a view to the creation of a Central Mathematical Library from which members can obtain all mathematical, including astronomical, books and journals through the post. Every journal should be sent in circulation to members who have made a general application for the same; and every book and journal should be sent to members on special application, as often as applied for. All this, of course, subject to rules that shall be made in this behalf.

6. If this conception of the immediate aims of the Society be approved, our Library should preferably be in a place which is postally a good centre for all India. In this respect, Poona is, next to Bombay, the most central place for all India. It is practically Bombay as regards the rest of India and it is a centre as regards Bombay Presidency itself. Further, I am glad to be able to mention that Prof. Paranjpye will be willing to take charge of the Library, provision being made for the discharge of purely mechanical work through Assistant Librarians. My draft constitution will be found to provide for this. I trust Madras members will endorse my suggestions and show their full appreciation of the reasons which make Poona, which contains that noble institution, the Fergusson College, so felicitous a centre for our Society. Further, by voting Poona as our centre, we Madrassees will convince the rest of India that we do not look at the matter from a purely provincial point of view. And it is my deep hope that a combination formed on these lines may be of fruitful consequences in the future.

7. Further, we shall have to get more than one copy of most books, and journals, and probably several copies of some journals, for the sake of convenience. This being the case, some of the extra copies we get can be placed in Branch Libraries in Madras, Allahabad, &c., as membership extends, so that members in these Provinces may get them quicker on their application.

8. The subscription of Rs. 25 has been felt heavy by some of those whom I addressed. Among the provisions in the draft constitution that I shall submit, will be one enabling the Committee to reduce the subscription in the case of any particular member to Rs. 15, and to continue this concession as long as they deem fit. I trust that this provision will be passed and that it will sufficiently meet all cases where such encouragement may be necessary. The scheme ought to be taken advantage of, more specially by the younger gentlemen interested in mathematics, many of whom will be the future occupants of the mathematical chairs of our colleges,—and I hope none of them will feel deterred from applying for membership at once.

9. Intending members will communicate with the undersigned.

THE INDIAN MATHEMATICAL SOCIETY.

Statement of Accounts of the Society for the year 1918.

Receipts.	Rs.	A.	P.	*Expenditure.*	Rs.	A.	P.
Balance from 1917	2,997	4	4	Ordinary working expenses	231	5	9
Subscriptions from Members	2,457	4	0	Books and Journals	443	3	9
Subscriptions for Journal	212	0	0	Printing Journal	640	2	0
Interest on Investments	111	0	3	Library	286	4	0
Miscellaneous	33	10	0	Closing Balance	4,210	3	1
Total ...	5,811	2	7	Total ...	5,811	2	7

I have examined the Treasurer's books and vouchers, as also the monthly Statements submitted by the Secretary, Assistant Secretary and Assistant Librarian and declare the above accounts to be correct.

N. RANGANATHAN,
Auditor.

5th April 1919.

MADRAS,
22nd March 1919.

S. NARAYANA AIYAR,
Honorary Treasurer.

Conics with vanishing Θ, Θ^1.

R. VYTHYNATHASWAMY.

[*Presented to the Second Indian Mathematical Conference, January, 1919.*]

§ I. *Self Harmonic Quartics.*

Two quartics $(a_0a_1a_2a_3a_4)(x,1)^4$, $(b_0b_1b_2b_3b_4)(x,1)^4$ are said to be harmonic, if

$$a_0b_4-4a_1b_3+6a_2b_2-4a_3b_1+a_4b_0=0.$$

A quartic is self-harmonic, if

$$a_0a_4-4a_1a_3+3a_2^2=0, \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

i e. if its $I-$invariant vanishes.

The relation (1) is of the second degree in the coefficients. *Hence when* $\alpha\ \beta\ \gamma$ *three of the roots of a self harmonic quartic are known, there are two values* $\delta\ \delta'$ *(say) of the fourth root.*

Theorem I. *The roots of the Hessian of the cubic whose roots are* α, β, γ *are* δ, δ'.

Let the co-ordinates of α, β, γ be $1, \omega, \omega^2$, so that the roots of the Hessian are $0,\infty$. If λ be the fourth root, we find, on equating to zero the I—invariant of $(x^3-1)(x-\lambda)$, that $\lambda=0$ or ∞. This proves the theorem.

Note. δ,δ' may be termed the cyclic centres of $\alpha\beta\gamma$. This has already been done in my Note on 'Pascal Hexagon' (J. I. M. S. Vol. X, p. 342.

Theorem II. *If* δ *be a cyclic centre of* α, β, γ, *every one of the numbers* $\alpha, \beta, \gamma, \delta$ *is a cyclic centre of the remaining three.*

For the quartic $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ is self-harmonic and the self-harmonic relation is a symmetric one as among $\alpha\ \beta\ \gamma\ \delta$.

Theorem III. *If* $\alpha\beta\gamma\delta$ *is a self-harmonic quartette, (so that each of the numbers is a cyclic centre of the other three) and if the other cyclic centres of* $\beta\gamma\delta$, $\alpha\gamma\delta$, $\alpha\beta\delta$, $\alpha\beta\gamma$ *be* $\alpha'\beta'\gamma'\delta'$ *respectively, then* (1) $\alpha'\beta'\gamma'\delta$, *is a self-harmonic quartette ;* (2) *the other cyclic centres of* $\beta'\gamma'\delta'$, $\alpha'\gamma'\delta''$ $\alpha'\beta'\delta'$, $\alpha'\beta'\gamma'$ *are* $\alpha\beta\gamma\delta$ *respectively* (3) $\alpha\beta\gamma\delta$ *and* $\alpha'\beta'\gamma'\delta'$ *have the same automorphic linear group.*

As before we choose the co-ordinates of α,β,γ, as $1, \omega, \omega^2$. Let δ be 0 so that δ' is ∞. By forming the Hessians, we readily find $\alpha'\beta'\gamma'$ to be $-\frac{1}{2}, -\frac{\omega}{2}$ and $-\frac{\omega^2}{2}$. Write these values as per following scheme

α	β	γ	δ
1	ω	ω^2	0
$-\frac{1}{2}$	$-\frac{1}{2}\omega$	$-\frac{1}{2}\omega^2$	∞
α'	β'	γ'	δ'.

Since ∞ is obviously a cyclic centre of

$$-\frac{1}{2}, -\frac{\omega}{2}, -\frac{\omega^2}{2},$$

it follows that $\alpha'\beta'\gamma'\delta'$ is self-harmonic. Further since the other cyclic centre of $\alpha'\beta'\gamma'$ *i. e.* $\left(-\frac{1}{2}, -\frac{\omega}{2}, \frac{\omega^2}{2}\right)$ is δ (0), we may either verify directly, or preferably predict from symmetry, that the other cyclic centres of $\alpha'\beta'\delta'$, $\alpha'\gamma'\delta'$, and $\beta'\gamma'\delta'$ are γ, β and α respectively. Finally to prove (3) we remark that the process by which we derive $\alpha'\beta'\gamma'\delta'$ from $\alpha\beta\gamma\delta$ involves only the Hessian and the self-harmonic relation, and is consequently invariantic for linear transformations. Hence, if any linear transformation carries the group $(\alpha\beta\gamma\delta)$ into itself, it cannot but transform $(\alpha'\beta'\gamma'\delta')$ also into itself.

Cor. 1. Associated with every self-harmonic quartette there is another and the two are symmetrically related.

Cor. 2. One cross-ratio of a self-harmonic quartette is an imaginary cube root of -1. *Hence any two self-harmonic quartettes are projective.*

Cor 3. Let $(\alpha\delta)$, $(\alpha'\delta')$ be corresponding pairs selected respectively from a self-harmonic quartette and its associated quartette. If α'' be the harmonic conjugate of α' *w.r.t.* $(\delta\delta')$, the pairs $(\alpha\delta)$, $(\alpha''\delta')$ are harmonic.

This is apparent if we take the special coordinates exhibited in the scheme.

Cor. 4. From the known property of the Hessian, we see that if $(\alpha\alpha')$ be a corresponding pair of two associated self-harmonic quartettes, the homographic transformation of period three whose fixed points are $(\alpha\alpha')$ carries $(\beta\gamma\delta)$ into itself and $(\beta'\gamma'\delta')$ into itself.

Theorem IV. Given the linear group of a quartic (*i.e.* a group consisting of identity and three mutually harmonic involutions) the quartic belongs to a one-parameter family. Three of the quartics of the family are perfect squares. If the parameters of these be the roots of the cubic $F(\lambda)=0$, the roots of the Hessian of F are the two self-harmonic associated quartics of the family.

Proof. Let the involutions of the given group have fixed points $(0,\infty)$, $(1,-1)$, $(i,-i)$, so that they are $-x, 1/x, -1/x$ It is obvious that every quartic belonging to this group must be a reciprocal quadratic in x^2, *i.e.* of the form

$$x^4+2\lambda x^2+1.$$

λ may be conveniently taken as the parameter of the quartic. There are evidently three values of λ for which the quartic is a perfect square, *viz.* ∞, $1,-1$. The values of λ for which the quartic is self-

harmonic are $\pm\sqrt{3}i$. It may be inferred that the quartics determined by these two values are associates since they belong to the same group.

It is easily verified that $\pm\sqrt{3}i$ are the cyclic centres of $(\infty, 1, -1)$.

§ II. *Properties of the Polar Line of a* $\triangle$ *w. r. t. a Circum-conic.*

The cyclic centres of the vertices of an inscribed $\triangle$ are the intersections of the conic with its polar line (vide; 'Pascal Hexagon,' *loc. cit.*) The theorems of the previous paragraph translate themselves into the following geometrical properties.

(1) Let the polar line of ABC be D D′, (all the letters referring to points on the conic). Then the polar lines of BCD, ACD, ABD pass respectively through A, B and C. Let these polar lines be AA′, BB′, CC′.

(2) The polar lines of B′C′D′, A′C′D′, A′B′D′, A′B′C′ are AA′ BB′, CC′, DD′ respectively.

(3) The cross ratios (A B C D), (A′B′C′D′) on the conic are $-\omega$ $-\omega^2$.

(4) There is a conic touching the tangents at A B C D and having A B C D as a self-polar quadrangle. This conic passes through A′B′C′D′ and its Θ, Θ' invariants with the given conic both vanish.

These follow immediately from the geometric interpretation of harmonic quartics quoted in Grace and Young's Algebra of Invariants. These results are proved below by elementary methods.

(5) The conic inscribed in corresponding triangles BCD, B′C′D′ (say) has double contact with the given conic the chord of contact being AA′. (cor 4),

(6) AA′, BB′, CD′, C′D touch a conic having double contact with the given conic. (cor 2).

(7) Let D′L, D′M, D′N be drawn conjugate to DA, DB, DC respectively. LA′, LB′, LC′ are all conjugate to DD′ and are therefore *concurrent* at the pole of DD′ (cor. 3).

(8) ABCD and A′B′C′D′ have the same harmonic triangle (Th. III.)

§ III. *The Four-point System of Conics.*

It can be shewn that every four-point system contains a unique pair of conics for which Θ, Θ' vanish. If these be S_1, S_2, the values of λ for which $S_1+\lambda S_2$ is a line conic are the roots of an equation of the form $\lambda^3+K=0$ since $\Theta\,\Theta'$ vanish. Now the values of λ for which $S_1+\lambda S_2$ is S_1 or S_2 are $0, \infty$. Hence we have the important theorem.

The cyclic centres of the three line-conics of a four-point system are the conics for which Θ, Θ' *vanish.*

Now let us choose a different co-ordinate system for the conics, by taking the cross ratio of the four common points in an assigned order on each of the conics as its parameter. The line-conics are represented in this system by the parameters $0, \infty, 1$. Hence the parameters of the two conics for which Θ, Θ' vanish are $-\omega, -\omega^2$. Hence,

The four points of intersection of two conics for which Θ, Θ' *vanish form a self-harmonic quartette on either ; and conversely.*

Since any two conics can be reciprocated into each other, we have the perfectly general theorem.

The cross ratio of the points of intersection of two conics on one of them is equal to the cross of the points of contact of the common tangents on the other.

Hence the points of contact of common tangents on either of two conics for which Θ, Θ' vanish, form a self-harmonic quartette.

Now if A B C D be the points of intersection A′B′C′D′ the points of contact of common tangents on one of the conics, then evidently A B C D and A′B′C′D′ have the same harmonic triangle. Hence

The points of intersection and the points of contact of common tangents on either of two conics for which Θ, Θ' *vanish, form associated self harmonic quartettes.*

The converse of this may be proved by using the general cross-ratio theorem.

Bernoulli's Polynomials and Fourier's Series.

By S. R. Ranganathan.

[The object of this note is (i) to establish the Fourier expansions of Bernoulli's Polynomials and (ii) to deduce the sums of certain interesting types of series in a finite form and to work out some illustrative examples with the help of these formulae. The results (1·4), (2·4) and (1·6) seem to be particularly interesting.

Of these (i) is suggested as a problem in Whittaker and Watson's *Modern Analysis*; the other references, which are all to Bromwich's *Infinite Series* are given as they occur.]

In what follows, series of the type

$$\sum \frac{\sin 2\pi n x}{n^p} \text{ and } \sum \frac{\cos 2\pi n x}{n^p}$$

occur frequently. Hence, it would be well to observe at the beginning, that both the above series are absolutely convergent for all values of $p>1$; they are also easily shown, with the help of Weierstrass's M-test to be uniformly convergent for all values of x, provided $p>1$ (*vide*: Bromwich, Chap. VII, Art. 44, Ex. 1.). Further,

$$\sum \frac{\sin 2\pi n x}{n}$$

is uniformly convergent in the open interval (0, 1).

Starting with the well-known result (Bromwich, Chapter VIII, Art. 65)

$$\frac{-1}{\pi}\left(\frac{\sin 2\pi x}{1}+\frac{\sin 4\pi x}{2}+\frac{\sin 6\pi x}{3}+\ldots\right)=x-\tfrac{1}{2}, \text{ if } 0<x<1$$

and, integrating term by term, we get

$$\frac{(-1)^2}{2\pi^2}\left(\frac{\cos 2\pi x}{1^2}+\frac{\cos 4\pi x}{2^2}+\frac{\cos 6\pi x}{3^2}+\ldots\right)=\frac{x^2}{2!}-\frac{x}{2}+\frac{B_1}{2!}.$$

Hence the constant of integration is determined to be $\frac{(-1)^2 B_1}{2!}$, by putting $x=0$ on both sides and remembering that

$$\frac{1}{2^{2s-1}\pi^{2s}}\sum\frac{1}{r^{2s}}=\frac{B_s}{(2s)!}$$

where B_s is the sth. Bernoullian number, s, being an integer.

Similarly, integrating the first result $(2n-1)$ times, we get

$$\frac{(-1)^{n+1}}{2^{2n-1}\pi^{2n}}\left\{\frac{\cos 2\pi x}{1^{2n}}+\frac{\cos 4\pi x}{2^{2n}}+\frac{\cos 6\pi x}{3^{2n}}+\ldots\right\}$$

$$= \frac{x^{2n}}{(2n)!} - \frac{x^{2n-1}}{2\,(2n-1)!} + \frac{B_1\,x^{2n-2}}{2!\,(2n-2)!} - \frac{B_2\,x^{2n-4}}{4!\,(2n-4)!}$$
$$+ \frac{B_3\,x^{2n-6}}{6!\,(2n-6)!} - \ldots\ldots (-)^{n+1}\frac{B_n}{(2n)!}$$
$$= \frac{1}{(2n)!}\left\{ x^{2n} - \frac{(2n)\,x^{2n-1}}{2} + \frac{(2n-1)\,(2n)\,B_1\,x^{2n-2}}{2!} - \ldots \right.$$
$$\left. + (-)^{n+1}\,B_n \right\}.$$
$$= \frac{1}{(2n)!}\left\{ \phi_{2n}(x) + (-)^{n+1}\,B_n \right\}.$$

where $\phi_m(x)$ stands for the mth. Bernoulli's Polynomial, (*cf.* Bromwich, Chapter X, Art 94).

Integrating once more, we get

$$\frac{(-1)^{n+1}}{2^{2n}\pi^{2n+1}}\left\{ \frac{\sin 2\pi x}{1^{2n+1}} + \frac{\sin 4\pi x}{2^{2n+1}} + \frac{\sin 6\pi x}{3^{2n+1}} + \ldots \right\}$$
$$= \frac{1}{(2n+1)!}\left\{ \phi_{2n+1}(x) \right\}$$

Thus, we have in the interval (0,1),

$$\frac{\cos 2\pi x}{1^{2n}} + \frac{\cos 4\pi x}{2^{2n}} + \frac{\cos 6\pi x}{3^{2n}} + \ldots$$
$$= \frac{2^{2n-1}\pi^{2n}}{(2n)!}\left\{ B_n + (-)^{n+1}\phi_{2n}(x) \right\} \quad \ldots \quad (1)$$

and

$$\frac{\sin 2\pi x}{1^{2n+1}} + \frac{\sin 4\pi x}{2^{2n+1}} + \frac{\sin 6\pi x}{3^{2n+1}} + \ldots$$
$$= (-)^{n+1}\frac{2^{2n}\pi^{2n+1}}{(2n+1)}\phi_{2n+1}(x) \quad \ldots \quad \ldots \quad (2)$$

Putting $x = \frac{1}{4}$ in (1), we get

$$\frac{2^{n-1}\pi^{2n}}{(2n)!}\left\{ B_n + (-)^{n+1}\phi_{2n}\left(\tfrac{1}{4}\right) \right\}$$
$$= -\frac{1}{2^{2n}} + \frac{1}{4^{2n}} - \frac{1}{6^{2n}} + \frac{1}{8^{2n}} - \quad \ldots \quad \ldots \quad \ldots \quad (1)$$
$$= -\frac{1}{2^{2n}}\left(1 - \frac{1}{2^{2n-1}}\right)\left\{ 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \ldots \right\}$$
$$= -\frac{2^{2n-1}-1}{2^{4n-1}} \cdot \frac{2^{2n-1}\pi^{2n}}{(2n)!}\,B_n.$$

Thus, we have

$$B_n = \frac{(-)^n 2^{4n-1}}{2^{4n-1} + 2^{2n-1} - 1}\,\phi_{2n}\left(\tfrac{1}{4}\right), \quad \ldots \quad \ldots \quad (3)$$

which gives B_n in terms of the preceding B's. With the help of this formula, the first few B's are easily calculated, since the first few $\phi\left(\frac{1}{4}\right)$'s are easily calculated with the help of the substitution suggested in Bromwich, *viz.* $y = x(x-1)$ and $z = (x - \frac{1}{2})$.

Writing in full, we have

$$B_n=\frac{(-)^n}{2^{4n}+2^{2n}-2}\left\{(1-4n)+\frac{2n(2n-1)B_1 2^4}{2!}\right.$$
$$-\frac{2n(2n-1)(2n-2)(2n-3)B_2 2^8}{4!}+\ldots\ldots$$
$$\left.+(-)^n 2^n(2n-1)B_{n-1}2^{4n-4}.\right\}$$

Again, putting $x=\frac{1}{4}$ in (2), we get

$$\frac{(-)^{n+1}2^{2n}\pi^{2n+1}}{(2n+1)!}\phi_{2n}(\tfrac{1}{4})=1-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\frac{1}{7^{2n+1}}+\ldots \qquad (2{\cdot}1)$$
$$=\frac{\pi^{2n+1}E_n}{2^{2n+1}(2n)!},$$

where E_n is the nth. Euler's number. (*Cf.* Bromwich, Chap. X, Ex. 38.)

Thus,

$$E_n=\frac{(-)^{n+1}2^{4n+2}}{2n+1}\phi_{2n+1}(\tfrac{1}{4}), \qquad \ldots \qquad \ldots \quad (4)$$

which gives E_n in terms of the B's. Writing in full

$$E_n=(-)^n\left\{\frac{4n+1}{2n+1}-\frac{2n}{2!}B_1 2^4+\frac{2n(2n-1)(2n-2)}{4!}B_2 2^8\right.$$
$$\left.\ldots+(-)^n B_n 2^{4n}\right\}.$$

Next, putting $x=\frac{1}{3}$ in (1) and multiplying both sides by 2, we get

$$\frac{(2\pi)^{2n}}{(2n)!}\left\{B_n+(-)^{n+1}\phi_{2n}(\tfrac{1}{3})\right\}$$
$$=2\left\{\frac{1}{3^{2n}}+\frac{1}{6^{2n}}+\frac{1}{9^{2n}}+\ldots\right\}-\left\{\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{4^{2n}}+\ldots\right\} \quad \ldots \quad (1{\cdot}2)$$
$$=\frac{1}{3^{2n-1}}\left\{\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\ldots\right\}-\left\{\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\ldots\right\}$$
$$=\frac{1-3^{2n-1}}{3^{2n-1}}\cdot\frac{2^{2n-1}\pi^{2n}}{(2n)!}B_n.$$

Thus,

$$B_n=\frac{(-)^n 2.3^{2n-1}}{3^{2n}-1}\phi_{2n}(\tfrac{1}{3}). \qquad \ldots \qquad \ldots \quad (5)$$

And, putting $x=\frac{1}{3}$ in (2) and multiplying both sides by $\frac{2}{\sqrt{3}}$, we get

$$\frac{1}{1^{2n+1}}-\frac{1}{2^{2n+1}}+\frac{1}{4^{2n+1}}-\frac{1}{5^{2n+1}}+\frac{1}{7^{2n+1}}-\frac{1}{8^{2n+1}}+\frac{1}{10^{2n+1}}-\ldots$$
$$=\frac{(-)^{n+1}(2\pi)^{2n+1}}{\sqrt{3}\,(2n+1)!}\phi_{2n+1}(\tfrac{1}{3}). \quad \ldots \qquad (2{\cdot}2)$$

By means of this formula, the following results are easily got.—

$$\frac{1}{1^3}-\frac{1}{2^3}+\frac{1}{4^3}-\frac{1}{5^3}+\frac{1}{7^3}-\ldots\ldots=\frac{4\pi^3}{81\sqrt{3}} \qquad (2{\cdot}21)$$

$$\frac{1}{1^5}-\frac{1}{2^5}+\frac{1}{4^5}-\frac{1}{5^5}+\frac{1}{7^5}-\ldots\ldots=\frac{4\pi^5}{729\sqrt{3}} \qquad (2{\cdot}22)$$

$$\frac{1}{1^7}-\frac{1}{2^7}+\frac{1}{4^7}-\frac{1}{5^7}+\frac{7}{7^7}-\ldots\ldots=\frac{47\pi^7}{98{,}415\sqrt{3}} \qquad (2{\cdot}23)$$

$$\frac{1}{1^9}-\frac{1}{2^9}+\frac{1}{4^9}-\frac{1}{5^9}+\frac{1}{7^9}-\ldots\ldots=\frac{3236\pi^9}{55{,}801{,}305\sqrt{3}}. \qquad (2{\cdot}24)$$

Putting $x=\frac{1}{6}$ in (1), transposing and multiplying by 2, we get

$$1-\frac{1}{2^{2n}}-\frac{1}{4^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}-\frac{1}{8^{2n}}-\frac{1}{10^{2n}}+\frac{1}{11^{2n}}+\frac{1}{13^{2n}}-\ldots\ldots$$

$$=\frac{(2\pi)^{2n}}{(2n)!}\left\{B_n+(-)^{n+1}\phi_{2n}\left(\tfrac{1}{6}\right)\right\}+\frac{2}{3^{2n}}\left\{1-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\ldots\right\}$$

$$=\frac{2.\pi^{2n}}{3^{2n}(2n)!}\left\{(3^{2n}.2^{2n-1}+2^{2n-1}-1)\,B_n+(-)^{n+1}3^{2n}.2^{2n-1}\phi_{2n}\left(\tfrac{1}{6}\right)\right\}\ldots \qquad (1{\cdot}3)$$

Also (1·2) can be written as

$$1+\frac{1}{2^{2n}}+\frac{1}{4^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\ldots$$

$$=\frac{(2\pi)^{2n}}{3^{2n}(2n)!}\left\{(1-3^{2n})\,B_n+(-)^n3^{2n}\phi_{2n}\left(\tfrac{1}{3}\right)\right\}. \qquad \ldots\ (1{\cdot}2)$$

Subtracting (1·3) from (1·2),

$$\frac{2\pi^{2n}}{3^{2n}(2n)!}\left\{1-2^{2n}.3^{2n})B_n+(-)^n2^{2n-1}3^{2n}\left[\phi_{2n}\left(\tfrac{1}{3}\right)+\phi_{2n}\left(\tfrac{1}{6}\right)\right]\right\}$$

$$=2\left\{\frac{1}{2^{2n}}+\frac{1}{4^{2n}}+\frac{1}{8^{2n}}+\frac{1}{10^{2n}}+\frac{1}{14^{2n}}+\frac{1}{16^{2n}}+\ldots\right\}$$

$$=\frac{2\pi^{2n}}{3^{2n}(2n)!}\left\{1-3^{2n})B_n+(-)^n3^{2n}\phi_{2n}\left(\tfrac{1}{3}\right)\right\},\ \text{by 1·2.}$$

So that, making use of (5), we get

$$\phi_{-n}\left(\tfrac{1}{6}\right)=\frac{1}{2^{2n-1}3^{2n}}\left\{3^{2n}(1-2^{n-1})\,\phi_{2n}\left(\tfrac{1}{3}\right)+(-)^n3^{2n}(2^{2n}-1)B_n\right\}$$

$$=(-)^n\frac{3^{2n-1}(2^{2n-1}+1)+2^{2n-1}-1}{2^{2n}\,3^{2n-1}}B_n. \qquad \ldots \qquad \ldots\ (6$$

Hence, we get from (1·3) and (6)

$$1-\frac{1}{2^{2n}}-\frac{1}{4^{2n}}+\frac{1}{5^{2n}}-\frac{1}{7^{2n}}-\frac{1}{8^{2n}}-\frac{1}{10^{2n}}+\cdots$$
$$=\frac{\pi^{2n}\,(3^{2n}-1)\,(2^{2n-1}-1)}{3^{2n}\,(2n)\,!}\mathrm{B}_n. \qquad \ldots \quad (1\cdot4)$$

This gives for $n=1, 2, 3$ and 4 respetively,

$$1-\frac{1}{2^2}-\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{7^2}-\ldots=\frac{2\pi^2}{27}. \qquad \ldots \quad \ldots \quad (1\cdot41)$$

$$1-\frac{1}{2^4}-\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{7^4}-\ldots=\frac{7\pi^4}{729}. \qquad \ldots \quad \ldots \quad (1\cdot42)$$

$$1-\frac{1}{2^6}-\frac{1}{4^6}+\frac{1}{5^6}+\frac{1}{7^6}-\ldots=\frac{403\pi^6}{393{,}660}. \qquad \ldots \quad \ldots \quad (1\cdot43)$$

$$1-\frac{1}{2^8}-\frac{1}{4^8}+\frac{1}{5^8}+\frac{1}{7^8}-\ldots=\frac{5207\pi^8}{49{,}601{,}160}. \qquad \ldots \quad (1.44)$$

Also, putting $x=\frac{1}{6}$ in (2) and multiplying both side by $\frac{2}{\sqrt{3}}$, we get

$$\frac{1}{1^{2n+1}}+\frac{1}{2^{2n+1}}-\frac{1}{4^{2n+1}}-\frac{1}{5^{2n+1}}+\frac{1}{7^{2n+1}}+\frac{1}{8^{2n+1}} \quad \ldots$$
$$=\frac{(-)^{n+1}(2\pi)^{2n+1}}{\sqrt{3}\,(2n+1)!}\phi_{2n+1}(\tfrac{1}{6}). \qquad \ldots \quad (2.3)$$

Subtracting (2·2) from (2·3)

$$\frac{(-)^{n+1}\,(2\pi)^{2n+1}}{\sqrt{3}\,(2n+1)!}\left\{\phi_{2n+1}(\tfrac{1}{6})-\phi_{2n+1}\,(\tfrac{1}{3})\right\}$$
$$=2\left\{\frac{1}{2^{2n+1}}-\frac{1}{4^{2n+1}}+\frac{1}{8^{2n+1}}-\frac{1}{10^{2n+1}}+\ldots\right\}$$
$$=\frac{(-)^{n+1}\,(2\pi)^{2n+1}}{\sqrt{3}\,(2n+1)!\,2^{2n}}\phi_{2n+1}\,(\tfrac{1}{3}),$$

so that $\phi_{2n+1}(\frac{1}{6})=\frac{2^{2n}+1}{2^{2n}}\,\phi_{2n+1}(\frac{1}{3})$. $\ldots \quad \ldots \quad \ldots \quad (7)$

Hence, we get from (2·3) and (7)

$$\frac{1}{1^{2n+1}}+\frac{1}{2^{2n+1}}-\frac{1}{4^{2n+1}}-\frac{1}{5^{2n+1}}+\frac{1}{7^{2n+1}}+\frac{1}{8^{2n+1}}-\ldots$$
$$=\frac{(-)^{n+1}2\pi^{2n+1}}{\sqrt{3}\,(2n+1)!}\,(2^{2n+1}+1)\,\phi_{2n+1}(\tfrac{1}{3}). \qquad \ldots \quad (2\cdot4)$$

Thus, by calculating the Bernoulli's Polynomials of argument $\frac{1}{3}$, both (2·2) and (2·4) can be evaluated. For example,

$$\frac{1}{1^3}+\frac{1}{2^3}-\frac{1}{4^3}-\frac{1}{5^3}+\ldots=\frac{5\pi^3}{81\sqrt{3}}. \qquad \ldots \quad \ldots \quad (2\cdot41)$$

$$\frac{1}{1^5}+\frac{1}{2^5}-\frac{1}{4^5}-\frac{1}{5^5}+\ldots=\frac{17\pi^5}{2916\sqrt{3}} \text{ (J.I.M.S.Q. 950)} \quad \ldots (2{\cdot}42)$$

$$\frac{1}{1^7}+\frac{1}{2^7}-\frac{1}{4^7}-\frac{1}{5^7}+\ldots=\frac{91\pi^7}{157{,}464\sqrt{3}} \quad \ldots \quad \ldots (2{\cdot}43)$$

$$\frac{1}{1^9}+\frac{1}{2^9}-\frac{1}{4^9}-\frac{1}{5^9}+\ldots=\frac{207{,}913\pi^9}{396{,}809{,}280,\sqrt{3}.} \quad \ldots \quad \ldots (2.44)$$

Again putting $x=\frac{1}{8}$ in (1), transposing and multiplying both sides by $\sqrt{2}$, we get

$$1-\frac{1}{3^{2n}}-\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\frac{1}{9^{2n}}-\frac{1}{11^{2n}}-\frac{1}{13^{2n}}+\ldots$$

$$=\frac{(2\pi)^{2n}}{\sqrt{2}\,(2n)!}\left\{B_n+(-)^{n+1}\phi_{2n}(\tfrac{1}{8})\right\}+$$

$$\frac{2}{\sqrt{2}}\left\{\frac{1}{4^{2n}}-\frac{1}{8^{2n}}+\frac{1}{12^{2n}}-\ldots\right\}.$$

$$=\frac{(2\pi)^{2n}}{\sqrt{2}\,(2n)!}\left\{B_n+(-)^{n+1}\phi_{2n}(\tfrac{1}{8})\right\}+$$

$$\frac{1}{\sqrt{2}.\,2^{4n-1}}\left\{1-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\ldots\right\}$$

$$=\frac{\pi^{2n}}{\sqrt{2}.\,(2n)!\,2^{4n-1}}\left\{(2^{6n-1}+2^{2n-1}-1)B_n+\right.$$

$$\left.(-)^{n+1}2^{6n-1}\phi_{2n}(\tfrac{1}{8})\right\} \quad \ldots (1{\cdot}5)$$

The following particular cases of (1·5) are easily calculated:—

$$1-\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}-\ldots=\frac{\pi^2}{8\sqrt{2}} \ldots \quad \ldots \quad \ldots (1{\cdot}51)$$

$$1-\frac{1}{3^4}-\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}-\ldots=\frac{11\pi^4}{768.\sqrt{2}}$$

(*cf.* Bromwich, Chap., X., Ex. 14) $\quad \ldots (1{\cdot}52)$

$$1-\frac{1}{3^6}-\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}-\ldots=\frac{2161\pi^6}{1{,}474{,}560\sqrt{2}} \quad \ldots \quad \ldots (1{\cdot}53)$$

Similarly, we get from (2)

$$\frac{1}{1^{2n+1}}+\frac{1}{3^{2n+1}}-\frac{1}{5^{2n+1}}-\frac{1}{7^{2n+1}}+\frac{1}{9^{2n+1}}+\frac{1}{11^{2n+1}}-\ldots$$

$$=\frac{(-)^{n+1}(2\pi)^{2n+1}}{\sqrt{2}.\,(2n+1)!}\phi_{2n+1}(\tfrac{1}{8})-\frac{2}{\sqrt{2}}.$$

$$\frac{1}{2^{2n+1}}\left\{1-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\ldots\right\}$$

$$=\frac{\pi^{2n+1}}{\sqrt{2}\,(2n+1)!\,2^{4n+2}}\left\{(-)^{n+1}2^{6n+3}\phi_{2n+1}(\tfrac{1}{8})-(2n+1)E_n\right\} \ldots (2{\cdot}5)$$

which gives, for example,

$$\frac{1}{1^3}+\frac{1}{3^3}-\frac{1}{5^3}-\frac{1}{7^3}+\ldots=\frac{3\pi^3}{64\sqrt{2}} \quad \ldots \quad \ldots (2{\cdot}51)$$

$$\frac{1}{1^5}+\frac{1}{3^5}-\frac{1}{5^5}-\frac{1}{7^5}+\ldots=\frac{19\pi 5}{4096\sqrt{2}}. \quad \ldots \quad \ldots \quad (2{\cdot}52)$$

$$\frac{1}{1^7}+\frac{1}{3^7}-\frac{1}{5^7}-\frac{1}{7^7}+\ldots=\frac{307\pi^7}{455,360\sqrt{2}}. \quad \ldots \quad (2{\cdot}53)$$

$$\frac{1}{1^9}+\frac{1}{3^9}-\frac{1}{5^9}-\frac{1}{7^9}+\ldots=\frac{83,579\pi^9}{1,633,607,680\sqrt{2}}. \quad \ldots \quad (2{\cdot}54)$$

Lastly, adding (1·2) and (1·4) side by side, we get, after slight simplification,

$$1+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\frac{1}{11^{2n}}+\frac{1}{13^{2n}}+\frac{1}{17^{2n}}+\frac{1}{19^{2n}}+\ldots$$
$$=\frac{\pi^{2n}(3^{2n}-1)\,(2^{2n}-1)}{2.\,3^{2n}\,(2n)!}\,B_n. \quad \ldots \quad (1{\cdot}6)$$

Thus,

$$1+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}+\ldots=\frac{\pi^2}{9}, \quad \ldots \quad \ldots \quad (1{\cdot}61)$$

$$1+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{11^4}+\frac{1}{13^4}+\ldots=\frac{5\pi^4}{486}. \quad \ldots \quad \ldots \quad (1{\cdot}62)$$

$$1+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{11^6}+\frac{1}{13^6}+\ldots=\frac{7\pi^6}{7290}. \quad \ldots \quad \ldots \quad (1{\cdot}63)$$

Similarly we get from (2·2) and (2·4),

$$\frac{1}{1^{2n+1}}-\frac{1}{5^{2n+1}}+\frac{1}{7^{2n+1}}-\frac{1}{11^{2n+1}}+\frac{1}{13^{2n+1}}-\frac{1}{17^{2n+1}}+\frac{1}{19^{2n+1}}-\ldots$$
$$=\frac{(-)^{n+1}\,2^{2n}\pi^{2n+1}}{\sqrt{3}(2n+1)!}\left\{\phi_{2n+1}\left(\tfrac{1}{6}\right)+\phi_{2n+1}\left(\tfrac{1}{3}\right)\right\}$$
$$=\frac{(-)^{n+1}\pi^{2n+1}(2^{2n+1}+1)}{\sqrt{3}\,(2n+1)!}\,\phi_{2n+1}\left(\tfrac{1}{3}\right). \quad \ldots \quad \ldots \quad (2{\cdot}6)$$

Thus,

$$\frac{1}{1^3}-\frac{1}{5^3}+\frac{1}{7^3}-\frac{1}{11^3}+\frac{1}{13^3}-\ldots=\frac{\pi^3}{18\sqrt{3}} \quad \ldots \quad \ldots \quad (2{\cdot}61)$$

$$\frac{1}{1^5}-\frac{1}{5^5}+\frac{1}{7^5}-\frac{1}{11^5}+\frac{1}{13^5}-\ldots=\frac{11\pi^5}{1944\sqrt{3}} \quad \ldots \quad \ldots \quad (2{\cdot}62)$$

$$\frac{1}{1^7}-\frac{1}{5^7}+\frac{1}{7^7}-\frac{1}{11^7}+\frac{1}{13^7}-\ldots=\frac{301\pi^7}{524,880\sqrt{3}} \quad \ldots \quad \ldots \quad (2{\cdot}63)$$

$$\frac{1}{1^9}-\frac{1}{5^9}+\frac{1}{7^9}-\frac{1}{11^9}+\frac{1}{13^9}-\ldots=\frac{15371\pi^9}{264,539,520\sqrt{3}}. \quad \ldots \quad (2{\cdot}64)$$

Note :—The calculation of $\phi_{2n+1}\left(\frac{1}{3}\right)$ and $\phi_n\left(\frac{1}{8}\right)$ which are required in the above can be done almost mentally up to ϕ_9 by making use of the substitution given in Bromwich, *viz.*, $y=x(x-1)$ and $z=x-\frac{1}{2}$. Hence it is that the formulae are interesting for application to simple numerical cases.

Legitimacy of the ordinary Complex Numbers.

By G. A. Miller.

It is well known that in 1797 Caspar Wessel presented to the Academy of Copenhagen a memoir in which he established the legitimacy of the ordinary complex numbers by giving a geometric meaning to these numbers and to their combination by addition, subtraction, multiplication and division. Somewhat later W. R. Hamilton (1837), and A. L. Cauchy (1847), established this legitimacy in two additional ways, the former having employed for this purpose the now classic method of real number couples while the latter employed the less commonly used method based on equivalence with respect to modulus i^2+1. C. F. Gauss had stated that he would publish a demonstration of the legitimacy, but he failed to do so.

In view of the great usefulness of the ordinary complex numbers it may be desirable to give here a proof of their legitimacy which differs very materially from those noted above but is based on facts which are very well known. This proof is here based largely on the group concept but it can clearly be given without the explicit use of this concept.

It is evident that the totality of the complex numbers $a+b\,i$, where a and b are real numbers and $i=\sqrt{(-1)}$ can be placed in a (1, 1) correspondence with the translations in the plane represented by the equations

$$x'=x+a$$
$$y'=y+b.$$

In fact, we may say that each of these translations is a complex number and *vice versa*. By doing this, these complex numbers become dynamic (and not static) geometric elements.

The two complex numbers

$$a+bi \text{ and } c+di$$

or the two translations

$$x'=x+a \qquad x'=x+c$$
$$y'=y+b, \qquad y'=y+d,$$

are clearly equivalent to the single translation

$$x'=x+a+c$$
$$y'=y+b+d$$

That is, when the complex numbers are regarded as these translations, two of them are equivalent to their sum and this proves that the ordinary complex numbers can be combined according to addition, reducing to the addition of reals when the numbers are real. The tota-

* Cf. *Encyclopedie des Sciences Mathematiques*, t. 1, vol. 1, p. 339.

ity of the complex numbers thus constitutes a continuous two parameter group, viz., the group of translations in the plane. This group is abelian and its identity is the ordinary zero.

The complex number $a+bi$ may also be said to be the linear transformation represented by the equations*

$$x' = ax - by$$
$$y' = bx + ay$$

If these complex numbers with the exception of 0 are represented by the totality of such linear transformations, it is easy to see that two complex numbers are equivalent to their product. In fact, if

$$z = x + iy \text{ and } z' = x' + iy'$$

then $z' = (a+bi)z$. Hence the ordinary complex numbers, with the exception of the identity of addition, constitute another two parameter continuous group of transformations. This group exhibits the fact that these numbers can be combined by multiplication and gives a perfectly clear geometric significance to such combinations.

Although 0, or the identity of the translations group, does not belong to the latter group of transformations, we may clearly require that when 0 or the transformation

$$x' = 0x - 0y$$
$$y' = 0x + 0y$$

is combined with any of the transformations in this group it shall reduce to this tranformation. If this is done, it results that the ordinary complex numbers become real and their legitimacy has been fully established, since the said transformations are evidently legitimate.

An advantage attained by regarding the ordinary complex numbers as such transformations is that it furnishes a much broader notion of number and it puts life into numbers. The static geometric interpretation of these numbers brought them from the ghost world into the world of reals, while the dynamic geometric interpretation endows them with life and the power to do things. If one complex number is regarded as belonging to one of the two continuous groups of transformations mentioned above, then all the other complex numbers should be regarded temporarily as belonging to the same group and to be affected by the first number. Our emphasis is, however, not on this evident interpretation but on the fact that it establishes the legitimacy of the ordinary complex numbers in an interesting and useful manner.

In the *Encyclopedie des Sciences Mathematiques*, tome 1, volume I, page 362, certain groups of transformations are represented by ordinary complex numbers. The two groups noted above would seem to have

* Cf. Pascal's *Repertorium des hoheren Mathematik*, vol. 2 (1910), p. 167.

close contact with those considered at this place in the *Encyclopedie*. In fact, these groups might be regarded as among the simplest and most interesting illustrative examples of continuous transformation groups, so that the present proof of the legitimacy of the ordinary complex numbers may also serve as an easy introduction to some elements of the group concept.

UNIVERSITY OF ILLINOIS.
October, 1918.

Astronomical Notes.

1. The Orbit of Sirius.

The micrometer measures of the companion now extend over more than a complete circuit. The independent investigations of the Orbit have been made by Dr. Aitken and M. Jonkheere respectively by discussing all the available measures and there is a remarkable agreement between the two results. The elements obtained by Dr. Aitken are as follows :—

Period		50.04 years
Time of Periastron passage	...	1894.133
Eccentricity	...	0.5945
Semimajor axis	...	7″.570
Inclinatiod	...	43°.31
Nodal Point	...	42°.71
Angle between Note & Periastron.		145°.69

2. A Total Eclipse of the Sun.

A total eclipse of the Sun (invisible in India) will occur on May 29. The path of totality passes through Central Africa and crossing the Atlantic, passes through S. America. The duration of totality is 6^m 50^s in mid Atlantic. The eclipse occurs in a region of the sky which is exceptionally rich in bright stars and the Astronomer Royal has drawn attention to the importance of the occasion as it affords a favourable opportunity for testing Einstein's theory of relativity, according to which a ray of light passing near the Sun will be deflected, the deflection being inversely proportional to the distance from the Sun's centre. It is proposed to take photographs at the time of the eclipse showing a good number of stars and if successful photographs are taken, these will be compared with those of the same region already taken for the purpose, which show the same stars in their undisturbed positions. The comparison is expected to throw some light on the validity of Einstein's theory.

3. Photo-Electric observations of the brightness of the Corona were made by Prof. Stebbins during the total eclipse of June 8, 1918. The light was found to be equal to 0.6 of the light of the full Moon.

4. Comet 1786 II :—Miss. Palmer has, in Astronomical Journal No. 744, investigated the orbit of the comet 1786 II which was dis-

covered by Miss. Caroline Herschel and was observed for a period of 82 days by several astronomers. She deduces the most probable period to be 9373 years!

5. *Parallax of Nova Persei* :—From the motion of the nebulosity ronnd this Nova, Prof. Turner has determined its parallax and the value obtained is $+0''.0093$. A new determination by the direct method has been made by A. Van Maanen with the 60-inch reflector of the Mount Wilson Solar Observatory and the relative parallax deduced from 14 exposures is $+0''.007 \pm 0''.004$ which corresponds to an absolute parallax of $+0''.009$. The close agreement of the values obtained by the two different methods is noteworthy.

6. *A New Publication.* Since the late Prof. Watson's classic treatise on "*Theoretical Astronomy*" went out of print, there has been a great need for a text-book in English dealing with determination of orbits and celestial mechanics generally. The publication of Prof. Plummer's "*Introductory Treatise on Dynamical Astronomy*" admirably satisfies this need. The book will serve as a good introduction to the treatise of Tisserand and other standard works on the subject.

7. In the "Observatory" February 1919, attention is drawn to a slip in Sir R. Ball's Spherical Astronomy. The last equation on page 338, giving $\tan \frac{h}{2}$ (Semidiurnal arc) should read

$$\tan \frac{h}{2}$$

$$= \pm \{ \sin \tfrac{1}{2}(z+\phi-\delta) \sin (z-\phi+\delta) \sec \tfrac{1}{2}(z+\phi+\delta) \sec \tfrac{1}{2}(z-\phi-\delta) \}^{\frac{1}{2}}.$$

T. P. Bhaskara Sastri.

Supplementary Notes.

(1) *Statistical Mechanics in Astronomy.* It is well known that statistical mechanics owes its origin to the investigation of the laws of thermodynamics on mechanical principles. Clausius and Maxwell may be said to be the founders of the science. The method was greatly developed by Maxwell and Boltzmann.

On account of the elegance and simplicity of its principles, the science of statistical mechanics received an independent development apart from the field of thermodynamics from contributors since the time of Clausius and Maxwell, but it was chiefly known as the theory of gases.

It was proposed by Poincare to apply the theory of gases to a stellar system. But a difficulty was felt. In gas-dynamics the prominent feature is the collisions of molecules, whereas in the stellar system, collisions are exceedingly rare and cannot be the harmless rebound of the theory of gases.

During the last few years, the general methods of statistical mechanics have been brought into use for astronomical purposes and we have thus a regular progression through rigid dynamics, hydrodynamics, and gas-dynamics to stellar dynamics. Some of the prominent workers in this field of inquiry are Schwarzschild, Eddington, Jeans and Charlier. Charlier's theory lays importance on the small deflections of stars by chance encounters, and he finds that the stellar universe has approached considerably towards statistical equilibrium.

(2) *Question* 841. On p. 79 (J. I. M. S., 1917) there is a note on Question 841, stating an elegant, though rather artificial, Oxford (1898) rider, which runs thus :—

"If the moon were to move in a circular orbit round the Earth, in a plane inclined at an angle ω to the ecliptic, the time of revolution being equal to one sidereal day, show that it appears to an observer on the earth to oscillate in a slender figure of eight, over a point on the equator."

The solution may be given thus :—

Let l' be the orbital co-ordinate and R', δ' the equatorial co-ordinates of the moon referred to the node of the orbit on the equator and let ω' be the inclination of the orbit to the equator.

Since l' is the orbital co-ordinate of the moon, it is also the equatorial co-ordinate of a suitably chosen point on the earth's equator, as the two angular velocities are the same. The R' and δ' of the moon are given by the usual formulae :

$$\tan R' = \tan l' \cos \omega',$$
$$\sin \delta' = \sin l' \sin \omega'. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Hence the hour angle measured westward from the meridian through the fixed point in question is

$$h = l' - R'.$$

From this

$$\tan h = \frac{\tan l' . 2 \sin^2 \frac{\omega'}{2}}{1 + \tan^2 l' \cos \omega'}. \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Equations (1) and (2) give the apparent motion as l' varies through 360°. From (1) δ' varies from 0 to ω', ω' to 0, 0 to $-\omega'$, $-\omega'$ to 0 as l' goes through the four quadrants; and from (2) $\tan h$ vanishes at 0, 90°, 180°, 270° and is of alternate signs in the four quadrants and has a turning value $\pm \sin^2 \frac{\omega'}{2} \sqrt{\sec \omega'}$ (in each quadrant), which is small, so that the figure is that of 8.

R. N. Apte.

SOLUTIONS.

Question 962

(A. C. L. WILKINSON) : —If two circles are such that hexagons can be inscribed in the one which are circumscribed to the other, prove that the sum of the products of the diagonals taken two at a time is constant and equal to

$$\frac{4R^2r^2\{(R^2-c^2)^2+4R^2r^2\}}{(R^2-c^2)^3}.$$

Solution by F. H. B. Gulasekharam, M.A.

Let two fixed circles of radii R and r respectively with their centres distant c apart be such that a hexagon $A_1A_2A_3A_4A_5A_6$ is inscribed in the former and circumscribed to the latter.

(*a*) *Then the triangles* $A_1A_3A_5$ *and* $A_2A_4A_6$ *are circumscribed to the same fixed circle, coaxial with the two given circles.*

For, the variable triangle $A_1A_2A_3$ is inscribed in a fixed circle, while two of its sides touch another fixed circle ; hence, by Poncelet's Theorem, the side A_1A_3 touches a fixed circle coaxial with the two given circles. Hence, the sides A_1A_3, A_3A_5, A_5A_1, A_2A_4, A_4A_6, A_6A_2 all touch a fixed circle coaxial with the two given circles.

(*b*) *The diagonals* A_1A_4, A_2A_5, A_3A_6 *intersect in* L, *the limiting point of the system of circles coaxal with the two given circles.*

For, applying Poncelet's Theorem to the variable quadrilateral $A_1A_2A_3A_4$, the side A_1A_4 touches a fixed circle coaxial with the two given circles. Hence A_1A_4, A_2A_5 and A_3A_6 touch the same fixed circle. But by Brianchon's theorem, the three diagonals A_1A_4, A_2A_5, A_3A_6 meet in a point. Hence their point of intersection L is a limiting point of the system of circles coaxal with the two given circles.

(*c*) If t_r be the length of the tangent to the inner given circle from the vertex A_r ($r=1, 2, 3, 4, 5, 6$), then

(i) $t_1t_4=t_2t_5=t_3t_6=$ a constant,

(ii) $t_1t_3+t_3t_5+t_5t_1=t_2t_4+t_4t_6+t_6t_2=$ a constant,

(iii) $t_1t_2+t_2t_3+t_3t_4+t_4t_5+t_5t_6+t_6t_1=$ a constant.

(i) For, from the property of coaxal circles, (Theor. (4). p. 350, Nixon's *Euc. Revised*),

$$\frac{t_1}{A_1L}=\frac{t_2}{A_2L}=\frac{t_3}{A_3L}=\frac{t_4}{A_4L}=\frac{t_5}{A_5L}=\frac{t_6}{A_6L}=\text{a constant}, \quad \ldots \quad (W)$$

also $A_1L.A_4L = A_2L.A_5L = A_3L.A_6L =$ a constant.

Hence $t_1t_4 = t_2t_4 = t_3t_6 =$ a constant.

(ii) If q_r be the length of the tangent to the incircle of $\triangle A_1A_3A_5$, or of $\triangle A_2A_4A_6$), from the vertex A_r, we have

$$\frac{q_1q_3q_5}{q_1+q_3+q_5} = \rho^2,$$

where ρ is the incircle of $\triangle A_1A_3A_5$ or of $\triangle A_2A_4A_6$; and

$$\frac{(q_1+q_3)(q_3+q_5)(q_5+q_1)}{q_1+q_3+q_5} = 4R\rho.$$

Hence $$\rho^2 + 4R\rho = q_1q_3 + q_3q_5 + q_5q_1.$$

We have shewn that ρ is constant.

Hence $q_1q_3 + q_3q_5 + q_5q_1$ is constant.

Again from the property of coaxal circles,

$$\frac{q_1}{t_1} = \frac{q_2}{t_2} = \frac{q_3}{t_3} = \frac{q_4}{t_4} = \frac{q_5}{t_5} = \frac{q_6}{t_6} = \text{a constant.}$$

$\therefore$ $$t_1t_3 + t_3t_5 + t_5t_1 = \text{a constant.}$$

Similarly $$t_2t_4 + t_4t_6 + t_6t_2 \text{ is constant.}$$

(iii) We have $$\frac{q_1q_3q_5}{q_1+q_3+q_5} = \rho^2 = \frac{q_2q_4q_6}{q_2+q_4+q_6}.$$

$\therefore$ $$\frac{q_1q_2q_3q_4q_5q_6}{(q_1+q_3+q_5)(q_2+q_4+q_6)} = \rho^4 \text{ (a constant).}$$

Hence $$\frac{t_1t_2t_3\ t_4t_5t_6}{(t_1+t_3+t_5)(t_2+t_4+t_6)} \text{ is constant.}$$

Hence from c (i),

$$(t_1+t_3+t_5)t_2 + (t_4+t_6) \text{ is constant.}$$

Hence $$t_1t_2 + t_2t_3 + t_3t_4 + t_4t_5 + t_5t_6 + t_6t_1 \text{ is constant.}$$

(d) $A_1A_4 \cdot A_2A_5 + A_2A_5 \cdot A_3A_6 + A_3A_6 \cdot A_1A_4$ is constant.

For, from the equations (W) above,

$$A_1A_4 \propto (t_1+t_4);\ A_2A_5 \propto (t_2+t_5)\ A_3A_6 \propto (t_3+t_6).$$

Hence $$A_1A_4 \cdot A_2A_5 + A_2A_5 \cdot A_3A_6 + A_3A_6 \cdot A_1A_4$$
$$\propto \{ (t_1+t_4)(t_2+t_5) + (t_2+t_5)(t_3+t_6) + (t_3+t_6)(t_1+t_4) \}$$

which is constant, from c (ii) and c (iii) above.

(e) The constant values in the foregoing are easily determined by taking the case in which the hexagon is symmetrical with respect to the line of centres of the two given circles. But before determining the

constants, let us establish the invariant relation connecting r, R and c for poristic hexagons.

Let $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$ be the hexagon in the symmetrical position, the axis of symmetry being $\alpha_1\alpha_4$, the diameter of the given outer circle.

Write $\overset{\wedge}{\alpha_1\alpha_4\alpha_3}=\phi$; $\overset{\wedge}{\alpha_1\alpha_4\alpha_2}=\theta$. Then since the sides $\alpha_1\alpha_2$, $\alpha_4\alpha_3$ $\alpha_2\alpha_3$ touch the inner circle, it is easy to see that

$$r=(R+c)\cos\theta=(R-c)\sin\phi=R\cos(\theta-\phi)+c\cos(\theta+\phi).$$

Hence. $\cos\theta\cos\phi(R+c)+\sin\theta\sin\phi(R-c)=r.$

$$\therefore \quad \cos\phi+\sin\theta=1. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Now writing $\dfrac{r}{R+c}=\cos\theta=p,$

$$\frac{r}{R-c}=\sin\phi=q,$$

the relation (i) is equivalent to

$$3-2(p^2+q^2)=(p^2-q^2)^2,$$

or $$\{1+(p-q)^2\}\{1+(p+q)^2\}=4, \quad \ldots \quad \ldots \quad (2)$$

giving the invariant relation between r, R, c, for poristic hexagons.

(*f*) *To determine the consant value of*

$$A_1A_4\cdot A_2A_5+A_2A_5\cdot A_3A_6+A_3A_6\cdot A_1A_4 = \alpha_1\alpha_4\cdot\alpha_2\alpha_5+\alpha_2\alpha_5\cdot\alpha_3\alpha_6+\alpha_3\alpha_6\cdot\alpha_1\alpha_4.$$

We have $\alpha_1\alpha_4=2R$

$$\alpha_2\alpha_5=\alpha_3\alpha_6=2R\sin\theta+\phi.$$

Again, since from (*e*) (i)

$$\cos\phi+\sin\theta=1, \text{ and } \sin\phi+\cos\theta=p+q,$$

we have $$2+2\sin(\theta+\phi)=\{1+(p+q)^2\}.$$

$$\therefore \quad \alpha_1\alpha_4+\alpha_2\alpha_5=R[1+(p+q)^2].$$

Hence, the required constant value

$$\begin{aligned}
&=2\alpha\alpha_4\cdot\alpha_2\alpha_5+(\alpha_2\alpha_5)^2\\
&=(\alpha_1\alpha_4+\alpha_2\alpha_5)^2-(\alpha_1\alpha_4)^2\\
&=R^2[1+(p+q)^2]^2-4R^2\\
&=R^2[1+(p+q)^2]\left[1+(p+q^2)-\frac{4}{1+(p+q)^2}\right]\\
&=R^2[1+(p+q)^2][1+(p+q)^2-1-(p-q)^2], \text{ from } e(2)\\
&=4R^2pq[1+(p+q)^2]\\
&=\frac{4R^2r^2[(R^2-c^2)^2+4R^2r^2]}{(R^2-c^2)^3}.
\end{aligned}$$

Question 963.

(A. C. L. WILKINSON) :—If two circles are such that quadrilaterals can be inscribed in the one which are circumscribed to the other, prove that the ratio of these quadrilaterals is constant.

Solution by F. H. V. Gulasekharam, M.A.

To avoid ambiguity, the latter part of the question should read " prove that the area of any such quadrilateral bears a constant ratio to the area of the corresponding quadrilateral whose angular points are the four points of contact of the inner circle and the original quadrilateral."

Let there be two circles such that a quadrilateral can be inscribed in the one and circumscribed to the other. Let $A_1A_2A_3A_4$ be one such quadrilateral; and T_1, T_2, T_3, T_4 be the points of contact of the sides A_1A_2, A_2A_3, A_3A_4, A_4A_1 respectively with the inner circle.

If ϕ be the angle between the diagonals A_1A_3 and A_2A_4, the area of the quadrilateral $A_1A_2A_3A_4 = \frac{1}{2} A_1A_3 . A_2A_4 \sin\phi$.

Again since T_1T_3 and T_2T_4 intersect at right angles, the area of the quadrilateral $T_1T_2T_3T_4 = \frac{1}{2} T_1T_3 . T_2T_4$.

(*vide* : Nixon, *Euclid Revised*, p. 187. Ex. 87)

Hence we are required to prove that

$$\frac{A_1A_3 . A_2A_4 . \sin\phi}{T_1T_3 . T_2T_4} \text{ is constant.}$$

Again we see that T_1T_3 and T_2T_4 bisect the angles between A_1A_3 and A_2A_4; and the point of intersection L of A_1A_3 and A_2A_4 is the limiting point of the system of circles coaxal with the two given circles. (Nixon : *Euclid Revised*, pp. 370, 375 and 424.)

Hence if t_r denotes the length of the tangent from the vertex A_r $(r = 1, 2, 3, 4)$ to the inner circle, we have

$$\frac{t_1}{A_1L} = \frac{t_2}{A_2L} = \frac{t_3}{A_3L} = \frac{t_4}{A_4L} = \text{a constant (say) } 1/\lambda;$$

also $A_1L . A_3L = A_2L . A_4L = \text{a constant.}$

$\therefore$ $t_1t_3 = t_2t_4 = \text{a constant (say) } \mu$

Also $A_1A_3 = \lambda t_1 + t_3)$; $A_2A_4 = \lambda(t_2 + t_4)$.

By Ptolemy's theorem,

$$A_1A_3 . A_2A_4 = A_1A_2 . A_3A_4 + A_1A_4 . A_2A_3.$$

$\therefore$ $\lambda^2 (t_1+t_3) (t_2+t_4) = (t_1+t_2) (t_3+t_4) + (t_1+t_4) (t_2+t_3)$

$$= (t_1+t_3) (t_2+t_4) + 4\mu.$$

Hence $(t_1+t_3) (t_2+t_4)$ is constant.

Hence $A_1A_3 . A_2A_4$ is constant. (1)

Again since LT_1 bisects the angle A_1LA_2,

$$LT_1 = 2\frac{A_1L . A_2L}{A_1L + A_2L}\cos\frac{\phi}{2}$$ (Hobson: *Plane Trig.* p. 196.)

Similarly $$LT_3 = 2.\frac{A_3L . A_4L}{A_3L + A_4L}\cos\frac{\phi}{2}$$

Hence $$T_1T_3 \text{ varies as } \left\{\frac{t_1t_2}{t_1+t_2} + \frac{t_3t_4}{t_3+t_4}\right\}\cos\frac{\phi}{2}.$$

But $$t_1t_3 = t_2t_4 = \text{a constant.}$$

$$\therefore \quad T_1T_3 \propto \frac{(t_1+t_2+t_3+t_4)}{(t_1+t_2)(t_3+t_4)}\cos\frac{\phi}{2}.$$

Similarly $$T_2T_4 \propto \frac{(t_1+t_2+t_3+t_4)}{(t_1+t_4)(t_2+t_3)}\cos\frac{\pi-\phi}{2}.$$

$$\therefore \quad T_1T\ T_2T_4 \propto \sin\phi \quad ... \quad ... \quad ... \quad (2)$$

$$\left(\text{For } \frac{\{t_1+t_2+t_3+t_4\}^2}{(t_1+t_2)(t_2+t_3)(t_3+t_4)(t_4+t_1)} = \frac{1}{r^2}.\right)$$

(Hobson's *Plane Trig.*, p. 208)

Hence from (1) and (2)

$$\frac{A_1A_3 . A_2A_4 . \sin\phi}{T_1T_3 . T_2T_4} \text{ is constant,}$$

whence the theorem.

Question 964.

(H Br.) :—If $f(x, y)$ is a rational algebraic function of x and y such that $f(-x,y) = (-)^{k+1} f(x,y)$, show that

(i) $$\int_0^\infty f(\sin 2\theta, \cos 2\theta)\frac{d\theta}{\theta^{k+1}}$$

$$= \frac{(-)^k}{k!}\int_0^\infty f(\sin 2\theta, \cos 2\theta)\tan\theta\frac{d^k}{d\theta^k}(\cot\theta)\frac{d\theta}{\theta}$$

$$= \frac{(-)^k}{k!}\int_0^{\frac{1}{2}\pi} f(\sin 2\theta, \cos 2\theta)\frac{d^k}{d\theta^k}(\cot\theta)\,d\theta;$$

and (ii) $$\int_0^\infty f(\sin\theta, \cos\theta)\frac{d\theta}{\theta^{k+1}}$$

$$= \frac{(-)^k}{k!}\int_0^{\frac{1}{2}\pi} f(\sin\theta, \cos\theta)\frac{d^k\lambda}{d\theta^k}\,d\theta;$$

where $\lambda = \cot\theta$, or $\operatorname{cosec}\theta$, according as

$f(-x,-y) = f(x,y)$, or $-f(x, y.)$

[The integrals are assumed to be finite].

Solution by K. B. Madhava.

These results may be supposed to be more general than the general case considered by me in solution to Q. 702 (published on p. 190 of Vol. VIII, J.I.M. S.,) but the same method of procedure is also available here.

Since the integrals may be assumed to be finite, the range of integration may be split up into parts so that

$$\int_0^{\infty} = \int_0^{\frac{1}{2}\pi} + \int_{\frac{1}{2}\pi}^{\pi} + \int_{\pi}^{\frac{3}{2}\pi} + \int_{\frac{3}{2}\pi}^{2\pi} \quad \dots + \quad \dots \quad \dots \quad (1)$$

and the following substitutions be made :

in the second interval, put $\phi = \pi - \theta$

third „ put $\phi = \theta - \pi$

fourth „ put $\phi = 2\pi - \theta$, and so on (2)

We shall therefore have in the first case,

$$\int_0^{\infty} f(\sin 2\theta, \cos 2\theta) \frac{d\theta}{\theta^{k+1}}$$

$$= \int_0^{\frac{1}{2}\pi} d\phi \left[\frac{f(\sin 2\phi, \cos 2\phi)}{\phi^{k+1}} + \frac{(-)^{k+1} f(\sin 2\phi, \cos 2\phi)}{(\pi - \phi)^{k+1}} \right.$$

$$\left. + \frac{f(\sin 2\phi, \cos 2\phi)}{(\pi + \phi)^{k+1}} + \dots \right]$$

$$= \int_0^{\frac{1}{2}\pi} f(\sin 2\phi, \cos 2\phi)\ d\phi \left[\frac{1}{\phi^{k+1}} + \left\{ \frac{1}{(\phi - \pi)^{k+1}} \right.\right.$$

$$\left.\left. + \frac{1}{(\phi + \pi)^{k+1}} \right\} + \dots \right]$$

$$= \frac{(-)^k}{k!} \int_0^{\frac{1}{2}\pi} f(\sin 2\phi, \cos \phi)\ \frac{d^k}{d\phi^k}(\cot \phi)\, d\phi,$$

which proves the first part.

Again, for the second part, proceeding in the same manner we notice that f repeats itself, multiplied by unity in every odd interval and by $(-)^{k+1}$ in every event one ; and that $\frac{d^k}{d\theta^k}(\cot \theta)$ repeats itself similarly, so that in every interval we shall have $f(\sin 2\theta, \cos 2\theta) \frac{d^k}{d\theta^k}(\cot \theta)$ repeats unaltered by the substitutions (2), $\tan \theta$ however changes into $(-)^n \tan \theta$.

Therefore

$$\int_0^\infty f(\sin 2\theta, \cos 2\theta) \tan\theta \frac{d^k}{d\theta^k}(\cot\theta) \frac{\theta}{d\theta}.$$

$$= \int_0^{\frac{1}{2}\pi} f(\sin 2\theta, \cos 2\theta).\, d\theta \left[\frac{\tan\theta}{\theta} - \frac{\tan\theta}{\pi-\theta} + \frac{\tan\theta}{\pi+\theta} - \frac{\tan\theta}{2\pi-\theta} + \cdots\right] \frac{d^k}{d\theta^k}(\cot\theta)$$

$$= \int_0^{\frac{1}{2}\pi} f(\sin 2\theta, \cos 2\theta).\, d\theta \left[\frac{\tan\theta}{\theta} + \frac{\tan\theta}{\theta-\pi} + \frac{\tan\theta}{\theta+\pi} + \frac{\tan\theta}{\theta+2\pi} \cdots + \right] \frac{d^k}{d\theta^k}(\cot\theta)$$

$$= \int_0^{\frac{1}{2}\pi} f(\sin 2\theta, \cos 2\theta).\, d\theta.\, \tan\theta.\, \cot\theta.\, \frac{d^k}{d\theta^k}(\cot\theta).$$

$$= \int_0^{\frac{1}{2}\pi} f(\sin 2\theta, \cos 2\theta) \frac{d^k}{d\theta^k}(\cot\theta);$$

which establishes the first case completely.

For case (ii), we still proceed in the same manner; thus

$$\int_0^\infty f \sin\theta, \cos\theta) \frac{d\theta}{\theta^{k+1}}$$

$$= \int_0^{\frac{1}{2}\pi} d\theta \left[\frac{f(\sin\theta, \cos\theta)}{\theta^{k+1}} + \frac{f(\sin\theta, -\cos\theta)}{(\pi-\theta)^{k+1}} + \frac{f(-\sin\theta, -\cos\theta)}{(\pi+\theta)^{k+1}} + \frac{f(-\sin\theta, \cos\theta)}{(2\pi-\theta)^{k+1}} + \frac{f(\sin\theta, \cos\theta)}{(2\pi+\theta)^{k+1}} + \cdots\right]$$

$$= \int_0^{\frac{1}{2}\pi} d\theta.\, f(\sin\theta, \cos\theta) \left[\frac{1}{\theta^{k+1}} - \left(\frac{1}{(\theta+\pi)^{k+1}} + \frac{1}{(\theta+\pi)^{k+1}}\right) + \left(\frac{1}{(\theta-2\pi)^{k+1}} + \frac{1}{(\theta+2\pi)^{k+1}} - + \cdots\right]\right.$$

if $f(-x, -y) = -f(x, y)$;
the integral

$$= \int_0^{\frac{1}{2}\pi} d\theta.\, f(\sin\theta, \cos\theta) \left[\frac{1}{\theta^{k+1}} + \left(\frac{1}{(\theta-\pi)^{k+1}} - \frac{1}{(\theta+\pi)^{k+1}}\right) + \frac{1}{(\theta-2\pi)^{k+1}} - \frac{1}{(\theta+2\pi)^{k+1}}\right) + \cdots\right]$$

if $f(-x, -y) = f(x, y)$;

so that the second part $= \int_0^{\frac{1}{2}\pi} d\theta . f(\sin\theta, \cos\theta) \frac{d^k \lambda}{d\theta^k}$

where $\lambda = \cot\theta$, or cosec θ according as $f(-x,-y) = f(x, y)$ or $-f(x, y)$, as in the problem.

Question 965.

(HEMRAJ):—ABC is a triangle. A point J is taken on AI produced, such that IJ subtends at B or C an angle α, where $\cot\alpha = 3 \cot\frac{1}{2}C$ or $3\cot\frac{1}{2}B$. Show that there exists a circle through A and J which cuts AB and AC in D and E, so that $BD = CE = BC$. [Suggested by Q. 959]

Solution by K. J. Sanjana and N. G. Leather.

Take the figure and notation of the solution to Q. 959.

Then $\alpha = \angle JBI_1$; let $\cot a = \frac{l+m+1}{l+m-1}\cot\frac{C}{2}$, more generally. It is easily seen that $\angle BI_1I = \frac{1}{2}C$, so that $\angle BJI_1 = \frac{1}{2}C - \alpha$.

Hence $$\frac{JI_1}{I_1B} = \frac{\sin\alpha}{\sin(\frac{1}{2}C-\alpha)} = \frac{1}{\sin\frac{1}{2}C\cot\alpha - \cos\frac{1}{2}C} = \frac{l+m-1}{2\cos\frac{1}{2}C},$$

$$\therefore \quad JI_1 = \frac{l+m-1}{2\cos\frac{1}{2}C} . 4R\cos\frac{C}{2}\sin\frac{A}{2} = 2R(l+m-1)\sin\frac{A}{2}.$$

Adding I_1A, $JA = R\{2(l+m-1)\sin\frac{1}{2}A + 4\cos\frac{1}{2}B\cos\frac{1}{2}C\}$

$$= \frac{R}{\cos\frac{1}{2}A}\left\{(l+m-1)\sin A + \sin A + \sin B + \sin C\right\}$$

$$= \{(l+m)a + b + c\}/2\cos\tfrac{1}{2}A.$$

Let any circle through A and J cut AB in D and AC in E. From

$$JA . DE = JD . AE + JE . AD,$$

we get $$\frac{(l+m)a+b+c}{2\cos\frac{1}{2}A} . 2R'\sin A = 2R'\sin\frac{A}{2}(b'+c'),$$

$$\therefore \quad b'+c' = (l+m)a + b + c, \text{ or } (b'-b)+(c'-c) = (l+m)a.$$

Thus $BD + CE = (l+m).BC$. If $l = m = 1$, $\cot\alpha = 3\cot\frac{1}{2}C$, and $BD + CE = 2a$, so that for some position of the circle $BD = CE = a$.

The same result follows from $\cot\alpha = 3\cot\frac{1}{2}B$.

Question 965 (*a*).

(A. C. L. WILKINSON) :—Prove that

$$\frac{1-2\,(2+k^2)\,s_1^2+5k^2s_1^4}{2s_1c_1d_1}+\frac{1-2\,(1+k^2)\,s_2^2+3k^2s_2^4}{2s_2c_2d_2}$$
$$-\frac{1-2\,(1+2k^2)\,s_3^2+5k^2s_3^4}{2s_3c_3d_3}=-3k^2s_1s_2s_3\,;$$

where $s_1=\mathrm{sn}\left(\frac{k}{3}\right)$, $s_2=\mathrm{sn}\left(\frac{ik'}{3}\right)$, $s_3=\mathrm{sn}\left(\frac{k+ik'}{3}\right)$.

Solution by F. H. V. Gulasekharam, M. A.

From the addition formula for the E functions, we have

$$3\,\mathrm{E}\left(\frac{k+ik'}{3}\right)-3\,\mathrm{E}\left(\frac{k}{3}\right)-3\,\mathrm{E}\left(\frac{ik'}{3}\right)$$
$$=-3k^2s_1s_2s_3. \qquad \ldots \qquad \ldots \quad (\mathrm{A})$$

We shall now establish the following relations :—

(i) $3\,\mathrm{E}\left(\frac{k+ik'}{3}\right)=\mathrm{E}+i\,(k'-\mathrm{E}')-\frac{1-2\,(1+2k^2)\,s_3^2+5k^2s_3^4}{2s_3c_3d_3}$

(ii) $3\,\mathrm{E}\left(\frac{k}{2}\right)=\mathrm{E}-\frac{1-2\,(2+k^2)\,s_1^2+5k^2s_1^4}{2s_1c_1d_1}$

(iii) $3\,\mathrm{E}\left(\frac{ik'}{3}\right)=i\,(k'-\mathrm{E}')-\frac{1-2\,(1+k^2)\,s_2^2+1k^2s_2^t}{2s_2c_2d_2}$

To prove (i).

It is easy to reduce the relation

$$3\,\mathrm{E}u=\mathrm{E}(3u)+k^2\mathrm{snu}\ \mathrm{sn}\ 2u\ (\mathrm{snu}+\mathrm{sn}\ 3u)$$

Now write $3u=k+ik'$, Then $\mathrm{E}(3u)=\mathrm{E}+i(k'-\mathrm{E}')$, $\mathrm{sn}\,3u=\frac{1}{k}$,

and, $\mathrm{sn}\,2u=\mathrm{sn}\left[\frac{2}{3}(k+ik')\right]=\mathrm{sn}\left[(k+ik')-\frac{1}{3}(k+ik)'\right]=\frac{1}{k}\frac{d_3}{c_3}$.

Hence $3\,\mathrm{E}\left(\frac{k+ik'}{3}\right)=\mathrm{E}+i(k'-\mathrm{E}')+\frac{s_3d_3}{c_3}(1+ks_3)$

$$=(\mathrm{E}+i(k'-\mathrm{E}')+\frac{2s_3^2d_3^2}{2s_3c_3d_3}\,(1+ks_3).$$

Again, since $\mathrm{sn}\left[\frac{2}{3}(k+ik')\right]=\frac{1}{k}\frac{d_3}{c_3}$,

$$\frac{2s_3c_3d_3}{1-k^2s_3^4}=\frac{1}{k}\frac{d_3}{c_3},$$

i.e. $1-2ks_3+2k\,s_3^2-k^2s_3^4=0.$

Making use of the foregoing relation

$$\begin{aligned} 2s_3^2d_3^2(1+ks_3) &= 2s_3^2+2ks_3^3-2k^2s_3^4-2k^3s_3^5 \\ &= 2s_3^2+(k^2s_3^4+2ks_3-1)-2k^2s_3^4 \\ &\qquad -2s(1-2ks_3+2ks_3^3) \\ &= -[1-2(1+2k^2)s_3^2+5k^2s_3^4]. \end{aligned}$$

Hence relation (i) is established.

To prove relation (ii).

As before, starting with the identity

$$3\,\mathrm{E}u = \mathrm{E}\,3u + k^2\,\mathrm{sn}u\,\mathrm{sn}\,2u\,(\mathrm{sn}u+\mathrm{sn}\,3u),$$

write $3u=k$; so that $\mathrm{E}\,3u=\mathrm{E}$, $\mathrm{sn}\,3u=1$,

$$\text{and, } \mathrm{sn}\,2u=\mathrm{sn}\frac{2k}{3}=\mathrm{sn}\left(k-\frac{k}{3}\right)=\frac{c_1}{d_1}$$

$$\text{Hence } 3\,\mathrm{E}\left(\frac{k}{3}\right)=\mathrm{E}+\frac{k^2s_1c_1\,(1+s_1)}{d_1}$$

$$=\mathrm{E}+\frac{2k^2s_1^2c_1^2\,(1+s_1)}{2s_1c_1d_1}$$

$$\text{Again, since } \mathrm{sn}\frac{2k}{3}=\frac{c_1}{d_1},\quad \frac{2s_1c_1d_1}{1-k^2s_1^4}=\frac{c_1}{d_1},$$

$$i.e. \qquad 1-2s_1+2k^2s_1^3-k^2s_1^4=0.$$

From the foregoing relation,

$$\begin{aligned} 2\,k^2s_1^2c_1^2\,(1+s_1) &= 2\,k^2s_1^2+2\,k^2s_1^3-2\,k^2s_1^4-2\,k^2s_1^5 \\ &= 2\,k^2s_1^2+(k^2s_1^4+2\,s_1-1)-2\,k^2s_1^4 \\ &\qquad -2s_1\,(1-2s_1+2\,k^2s_1^3) \\ &= -[1-2\,(2+k^2)s_1^2+5\,k^2s_1^4] \end{aligned}$$

Hence relation (ii) is established.

To prove relation (iii).

$$\text{We have } 3\mathrm{E}\,(iu,\,k)=3i\left[\frac{\mathrm{sn}(u,k')(\mathrm{dn})u,k')}{\mathrm{sn}(u,k')}+u-\mathrm{E}(u,k')\right]$$

(*Vide* Dixon. *E. F. p.* 54.)

$$\text{write } u=\frac{\mathrm{K}'}{3};\ \mathrm{sn}\left(\frac{\mathrm{K}'}{3},k'\right)=\mathrm{S};\ \mathrm{cn}\left(\frac{\mathrm{K}'}{3};\,k'\right)=\mathrm{C} \text{ and } \mathrm{du}\frac{k'}{3}k'-0$$

$$\text{Hence } 3\mathrm{E}\left(\frac{i\mathrm{K}'}{3},k\right)=i\left[\frac{3\mathrm{SD}}{\mathrm{C}}+k'-3\,\mathrm{E}\left(\frac{\mathrm{K}'}{3},k'\right)\right]$$

Again from relation (iii),

$$2\,\mathrm{E}\left(\frac{\mathrm{K}'}{2},k'\right)=\mathrm{E}'-\frac{1-2(2+k'^2)\mathrm{S}^2+5k'^2\mathrm{S}^4}{2\,\mathrm{SCD}}$$

Hence, after simplification,

$$3\,\mathrm{E}\left(\frac{i\mathrm{K}'}{3},k,\right) = i\,(\mathrm{K}'-\mathrm{E}') + \frac{i}{2\mathrm{SCD}}\,[1+2k^2\mathrm{S}^2-k'^2\mathrm{S}^4]$$

$$= i(\mathrm{K}'-\mathrm{E}') - \frac{1}{2s_2c_2d_2}\,[c_2^4-2k^2s_2^2c_2^2-k'^2\mathrm{S}_2^4]$$

$$\left[\text{for } s_2 = sn\left(\frac{ik'}{3},\,k\right) = i\frac{\mathrm{S}}{\mathrm{C}};\; c_2 = \frac{1}{\mathrm{C}} \text{ and } d_2 = \frac{\mathrm{D}}{\mathrm{C}}\right]$$

$$= i\,(\mathrm{K}'-\mathrm{E}') - \frac{1-2\,(1+k^2)\,s_2^2+3\,k^2s_2^4}{2\,s_2c_2d_2}.$$

Hence from the relations (A), (i), (ii) and (iii) the proposed identity easily follows.

Question 966

(A. C. L. Wilkinson) :—Prove that

$$\mathrm{E}-3\,\mathrm{E}\left(\frac{\mathrm{K}}{3}\right) = -\frac{k^2}{k'}\,\mathrm{cn}^3\left(\frac{\mathrm{K}}{3}\right)$$

Solution by F. H. V. Gulasekharam, M.A., and K. R. Rama Iyer,

In the solution to Q. 965 (*a*), it was shewn that

$$3\,\mathrm{E}\left(\frac{\mathrm{K}}{3}\right) = \mathrm{E} + k^2\,\frac{s_1c_1(1+s_1)}{d_1},$$

where $s_1 = \mathrm{sn}\left(\frac{\mathrm{K}}{3}\right)$, $c_1 = \mathrm{cn}\left(\frac{\mathrm{K}}{3}\right)$ and $d_1 = \mathrm{dn}\left(\frac{\mathrm{K}}{3}\right)$.

It was also shewn that

$$1-2s_1+2k^2s_1^3-k^2s_1^4 = 0.$$

$$\therefore\quad (1-s_1)^2(1-k^2s_1^2) = (1-k^2)s_1^2$$

$$\therefore\quad (1-s_1)^2d_1^2 = k'^2s_1^2$$

$$\therefore\quad (1-s_1)d_1 = k's_1$$

$$\therefore\quad c_1^2d_1 = k's_1\,(1+s_1)$$

$$\therefore\quad 3\,\mathrm{E}\left(\frac{\mathrm{K}}{3}\right) = \mathrm{E} + \frac{k^2}{k'}\,c_1^3$$

$$\therefore\quad \mathrm{E}-3\,\mathrm{E}\left(\frac{\mathrm{K}}{3}\right) = -\frac{k^2}{k'}\,c_1^3.$$

Additional Solution (1) *by F. H. V. Gulasekharam.* (2) *by the Proposer.*

(1) When $u_1+u_2+u_3+u_4=0$,

$$\sum_{r=1}^{4} \mathrm{E}u_r = \frac{k^2}{k^2-k'^2(c_1c_2c_3c_4)^{-1}} \sum_{r=1}^{4} \frac{s_r d_r}{c_r}.$$

where $s_r=\mathrm{sn}\ u_r$, $c_r=\mathrm{cn}\ u_r$ etc.

[*Vide.* J. I. M. S., Vol. X, pp. 338, 336.]

Write $u_4=-\mathrm{K}$, then when $u_1+u_2+u_3=\mathrm{K}$,

$$\sum_{r=1}^{3} \mathrm{E}u_r = \mathrm{E}+\frac{k^2}{k'}c_1c_2c_3 \quad \ldots \quad \ldots \quad (1)$$

Now writing $u_1=u_2=u_3=\frac{\mathrm{K}}{3}$, we get the required result.

Note.—(*a*) By applying Jacobi's imaginary transformation to (1), we can prove that when $u_1+u_2+u_3=\mathrm{i}k'$,

$$\sum_{r=1}^{3} \mathrm{E}u_r = i(\mathrm{K}'-\mathrm{E}')-\frac{ek'^2}{k.c_1c_2c_3} + \sum_{r=1}^{3} \frac{s_r c_r}{d_r} \quad \ldots \quad (2)$$

Now, writing $u_1=u_2=u_3=\left(\frac{i\mathrm{K}'}{3}\right)$, we get a result for $\mathrm{E}\left(\frac{\mathrm{K}'}{3}\right)$.

(*b*) Again, we have when $u_1+u_2+u_3+u_4=0$,

$$\sum_{r+1}^{4} \mathrm{E}u_r = \frac{k^2}{1+k'^2(d_1d_2d_3d_4)^{-1}} \sum_{r=1}^{4} \frac{s_r c_r}{d_r}$$ (*Ibid.* pp. 338,336).

Write $u_4=-(\mathrm{K}+i\mathrm{K}')$

Hence when $u_1+u_2+u_3=\mathrm{K}+i\mathrm{K}'$,

$$\sum_{r=1}^{3} \mathrm{E}u_r = \mathrm{E}+i(\mathrm{K}'-\mathrm{E}')-\frac{i}{k'}d_1d_2d_3. \quad \ldots \quad (3)$$

Now writing $u_1=u_2=u_3=\left(\frac{\mathrm{K}+i\mathrm{K}'}{3}\right)$, we get a result for $\mathrm{E}\left(\frac{\mathrm{K}+i\mathrm{K}'}{3}\right)$.

(2) Consider the intersections of the curves

$y^3=x(1-x)(1-k^2x)$ (1)

$y=(1-x)(ax+b)$ (2)

The abscissae of the points of intersection are given by

$$F(x)=(1-x)(ax+b)^2-x(1-h^2x)=0, \quad \ldots \quad \ldots \quad (3)$$

and we have immediately $x_1x_2x_3=\frac{b^2}{a^2}$. $\ldots \quad \ldots \quad \ldots \quad (4)$

Applying Abel's Theorem to (3), we have

$$\frac{\partial F}{\partial x_1}\,dx_1=-2y_1(x_1da+db),$$

whence as in Abel's theorem

$$\frac{dx_1}{y_1}+\frac{dx_2}{y_2}+\frac{dx_3}{y_3}=0,$$

or, $u_1+u_2+u_3=$constant, where u_1, u_2, u_3 are the arguments of the three points of intersection.

From (2) $as_1^2+b=\frac{s_1d_1}{c_1}$, $as_2^2+b=\frac{s_2d_2}{c_2}$,

whence $$a=\frac{s_1d_1c_2-s_2d_1c_2}{c_1c_2(s_1^2s_2^2)}+\frac{du(u_1+u_2)}{c_1c_2\ \mathrm{sn}\ (u_1+u_2)},$$

$$b=\frac{s_1s_2\ \mathrm{cn}\ (u_1+u_2)}{c_1c_2\ \mathrm{sn}\ (u_1+u_2)};$$

combining these values with (4), we have

$$u_1+u_2+u_3=\mathrm{K} \text{ and } a=\frac{k'}{c_1c_2c_3}.$$

Consider now $$\sum\frac{x_1dx_1}{y_1}=-2\sum\frac{x_1^2da+x_1db}{\frac{\partial F}{\partial x_1}}=\frac{2}{a^2}da,$$

whence $$\Sigma k^2\ \mathrm{sn}^2u_1du_1=\frac{k^2da}{a^2} \text{ and since } \Sigma du_1=0.$$

We find on integratiug and taking $u_1=u_2=u_3=\frac{k}{3}$ for the lower limit,

$$\mathrm{E}(u_1)+\mathrm{E}(u_2)+\mathrm{E}(u_3)-3\mathrm{E}\left(\frac{\mathrm{K}}{3}\right)=\frac{k^2}{k'}\mathrm{cn}\ u_1\ \mathrm{cn}\ u_2\ \mathrm{cn}\ u_3-\frac{k^2}{k'}\ \mathrm{cn}^3\ \frac{\mathrm{K}}{3},$$

where $u_1+u_2+u_3=\mathrm{K}$.

Writing $u_1=\mathrm{K}$, $u_2=u_3=0$, we have the given result.

Question 967.

(M. K. Kewalramani, M.A.) :—Prove that

$$1^{-5}+2^{-5}-4^{-5}-5^{-5}+7^{-5}+8^{-5}-\ldots=\frac{35\pi^6}{2916\sqrt{3}}$$

Solution by (1) *K. B. Madhava* (2) *by L. S. Vaidyanathan, M. A., and* (3) *by N. G. Leather.*

(1) Several problems of this nature have been solved in the pages of the J.I.M.S. (*vide* : Vol. 7.)

It may therefore be interesting to work out very general cases.

Write $$f(y)=\sum_{-\infty}^{\infty}\frac{e^{iny}}{x-n}$$

which satisfies the *formal* equation

$$(f'y)-ixf(y)=-i\sum_{\infty}e^{iny}$$

which, therefore, gives the *real* equation

$$[f'(y)-ixf(y)](1-e^{iy})=0.$$

Hence $f(y)=Ce^{ixy}$ where C is independent of y and y ranges from 0 to 2π.

Putting $y=\pi$, we have $C=2\pi i/(e^{2\pi ix}-1)$,
so that

$$\frac{2\pi ie^{ixy}}{e^{2\pi ix}-1}=\sum_{-\infty}^{\infty}\frac{e^{iny}}{x-n}(0<y<2\pi).$$

change y into $2\pi y$ and $2\pi ix$ into θ; we shall then have after some manipulation

$$\text{(A)}\qquad \frac{\theta e^{\theta y}}{e^{\theta}-1}=1+\sum_{\infty}{}'\frac{\theta e^{2n\pi iy}}{\theta-2n\pi i}0(<y<1$$

where the dash on the signa indicates the omission of $n=0$.

If we now remember, that the coefficient of $\frac{t^n}{n!}$ in $t\frac{e^{xt}-1}{e^t-1}$ is Bernoulli's Polynomial $\phi_n(x)$, and the coefficient of $\frac{t^n}{n!}$ in $\frac{t^2e^{xt}}{e^t-1}$, the Polynomial $\phi'_n(x)$
we shall have

$$\phi'_{2m}(x)=2m\,\phi_{2m-1}\,\phi(x)\ (m>1)$$

and $$\phi'_{2m+1}(x)=(2m+1)[\phi_{2m}(x)+(-)^{m-1}B_m](m\geq 1)$$

where B_m in the *mth* Bernoullian number.

These results are given in § 94 of Bromwich: *Theory of Infinite Series*; vide: also J. I. M. S., Vol. 8., p. 148.

Expanding the right side of (A) and equating the coeffs. of

$$\frac{\theta^{2k}}{2k!}\text{ and }\frac{\theta^{2k+1}}{(\theta k+1)!}$$

we have

$$\left.\begin{aligned} 2\sum_{n=1}^{\infty}\frac{\sin 2n\pi y}{(2n\pi)^{2r+1}} &= -\frac{1}{(2r+1)!}\phi_{2r+1}(y) \\ 2\sum_{n=1}^{\infty}\frac{\cos 2n\pi y}{(2n\pi)^{2r}} &= (-)^{r-1}\frac{1}{(2r)!}\phi_{2r}(y)+B_r \end{aligned}\right\} \quad \ldots \quad (B)$$

and other expressions.

The values of $\phi_r(y)$ up to $r=6$ are given in Bromwich, p. 236; the rest may be easily calculated.

From (B) we get a variety of results giving particular values to y and r. For the purposes of the present example $y=\frac{1}{6}$ and $r=2$. The result given by the Proposer seems to be incorrect; it ought to be.

$$\frac{17\pi^5}{2916\sqrt{3}}$$

(2) We know that

$$\tfrac{1}{2}(\pi-x)=\sin x+\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x+\ldots\ldots 0<x<2\pi$$

Integrating the above between π and x. we have

$$\frac{1}{2}\left(x\pi-\frac{x^2}{2}\right)-\frac{\pi^2}{6}=-\left[\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\ldots\right] \quad \ldots \quad (1)$$

Since $$\sum\frac{1}{n^2}=\frac{\pi^2}{6}.$$

Integrating (1) between the same limits thrice more we have the following equations.

$$\frac{1}{2}\left(\frac{x^2\pi}{2}-\frac{x^3}{6}\right)\frac{-\pi^2 x}{6}=-\left[\frac{\sin x}{1^3}+\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}+\ldots\right] \quad \ldots \quad (2)$$

$$\frac{1}{2}\left(\frac{x^3\pi}{6}-\frac{x^4}{24}\right)-\frac{x^2\pi^2}{12}+\frac{\pi^4}{90}=\left[\frac{\cos x}{1^4}+\frac{\cos 2x}{2^4}+\frac{\cos 3x}{3^4}+\ldots\right] \ldots \quad (3)$$

$$\frac{1}{2}\left(\frac{x^4\pi}{24}-\frac{x^5}{120}\right)-\frac{x^3\pi^2}{36}+\frac{\pi^4 x}{90}=\left[\frac{\sin x}{1^5}+\frac{\sin 2x}{2^5}+\frac{\sin 3x}{3^5}+\ldots\right] \quad (4]$$

putting $x=\frac{\pi}{3}$, we have

$$\frac{\pi^5}{3^5}\cdot\frac{17}{24}=\frac{\sqrt{3}}{4}.S$$

after simplifying, where S stands for the series whose sum we are seeking.

$$\therefore \qquad S=\frac{\pi^5\,17}{3^5\times 12\times\sqrt{3}}=\frac{17\,\pi^5}{2916\sqrt{3}}.$$

(3) $\frac{\pi}{2n}\tan\frac{m\pi}{2n}=\frac{1}{n-m}-\frac{1}{n+m}+\frac{1}{3n-m}-\frac{1}{3n+m}+\ldots$ (Hobson, § 294)

Differentiating with respect to m,

$$\frac{1}{(n-m)^2}+\frac{1}{(n+m)^2}+\frac{1}{(3n-m)^2}+\frac{1}{(3n+m)^2}+\ldots=\left(\frac{\pi}{2n}\right)^2\sec^2\frac{m\pi}{2n}.$$

Differentiating again

$$=\left[\frac{1}{(n-m)^3}-\frac{1}{(n+m)^3}+\frac{1}{(3n-m)^3}-\frac{1}{(3n+m)^3}+\ldots\right]$$

$$=\left(\frac{\pi}{2n}\right)^3\sec^2\frac{m\pi}{2n}\tan\frac{m\pi}{2n}.$$

Differentiating again

$$3\left[\frac{1}{(n-m)^4}+\frac{1}{(n+m)^4}+\frac{1}{(3n-m)^4}+\frac{1}{(3n+m)^4}+\ldots\right]$$

$$=\left(\frac{\pi}{2n}\right)^4\left[2\sec^2\frac{m\pi}{2n}\tan^2\frac{m\pi}{2n}+\sec^4\frac{m\pi}{2n}\right]$$

$$=\left(\frac{\pi}{2n}\right)^4\left[3\sec^4\frac{m\pi}{2n}-2\sec^2\frac{m\pi}{2n}\right].$$

Differentiating again

$$12\left[\frac{1}{(n-m)^5}-\frac{1}{(n+m)^5}+\frac{1}{(3n-m)^5}-\frac{1}{(3n+m)^5}\ldots\right]$$

$$=\left(\frac{\pi}{2n}\right)^5\left[12\sec^4\frac{m\pi}{2n}-4\sec^2\frac{m\pi}{2n}\right]\tan\frac{m\pi}{2n}.$$

$$\therefore\quad\frac{1}{(n-m)^5}-\frac{1}{(n+m)^5}+\frac{1}{(3n-m)^5}-\frac{1}{(3n+m)^5}-\ldots\ldots$$

$$=\left(\frac{\pi}{2n}\right)^5\left[\sec^4\frac{m\pi}{2n}-\frac{1}{3}\sec^2\frac{m\pi}{2n}\right]\tan\frac{m\pi}{2n}.$$

Put $n=3$, $m=2$

$$\frac{1}{1^5}-\frac{1}{5^5}+\frac{1}{7^5}-\frac{1}{11^5}+\quad=\left(\frac{\pi}{6}\right)^5\left[16-\frac{4}{3}\right]\sqrt{3}$$

$$=\frac{\pi^5.\ 44}{6^5.\ \sqrt{3}}=\frac{\pi^5.\ 11}{2^3.\ 1^5.\ \sqrt{8}}.$$

Put $n=3$, $m=1$

$$\frac{1}{2^5}-\frac{1}{4^5}+\frac{1}{8^5}-\frac{1}{10^5}+\ldots=\left(\frac{\pi}{6}\right)^5\left[\frac{16}{9}-\frac{4}{9}\right]\frac{1}{\sqrt{3}}$$

$$=\frac{\pi^5.\ 4}{6^5.\ 3\sqrt{3}}=\frac{\pi^5}{2^3.\ 3^6.\ \sqrt{3}}.$$

Adding, we get

$$\frac{1}{1^5}+\frac{1}{2^5}-\frac{1}{4^5}-\frac{1}{5^5}+\frac{1}{7^5}+\frac{1}{8^5}-\frac{1}{10^5}-\frac{1}{11^5}-\ldots$$

$$=\frac{\pi^5\sqrt{[33+1]}}{2^3.\ 3^6.\ \sqrt{3}}=\frac{17\pi^5}{2916\sqrt{3}}.$$

QUESTIONS FOR SOLUTION.

1027. (SELECTED) :—If p is a prime number and equal to $2n+1$, show that

$$(2n)!\equiv(-1)^n 2^{4n}.(n!)^2, \text{ (mod, } p^2)$$

1028. (N. P. PANDYA) :—Two tangents to a parabola a are fixed, and a third one is variable. Find the locus of the centroid of a triangle inscribed in the parabola, having its sides parallel to the three tangents.

1029. (SELECTED) :—Prove that

$$\left(\frac{d}{dx}e^x\right)^n uv =$$

$$\sum_{r=0}^{n} \frac{n\,1}{r\,!\,(n-r)\,!}\left[\left(\frac{d}{dx}e^x\right)^r e^{-rx}u\right]\left[\left(\frac{d}{dx}^1 e^x\right)^{n-r}\left(\frac{d}{dx}e^x\right)^{n-r} e^{rx}v\right].$$

1030. (SELECTED) :—If $x^{2n-1}+y^{2n-1}+a_1{}^{2n-1}+a_2{}^{2n-1}+\ldots a_{2r}{}^{2n-1}=0$, for integral values of n from 1 to r inclusive, then

$$(x+a_1)(x+a_2)\ldots(x+a_{2r})=(y+a_1)(y+a_2)\ldots(y+a_{2r}).$$

1031. (C. KRISHNAMACHARI) :—With the usual notation in elliptic functions, show that if $u+v+w=0$,

$$p(u)p(v)p(w)=\tfrac{1}{4}g_3+\tfrac{1}{4}\left\{\frac{p'(u)p(v)-p'(v)p(u)}{p(u)-p(v)}\right\}^2$$

Deduce that

$$p(z)\{p(z+\omega_1)+p(z+\omega_2)+p(z+\omega_3)\}+\tfrac{1}{4}g_2$$
$$=\tfrac{1}{4}p'(z)\left\{\frac{1}{e_1-p(z)}+\frac{1}{e_2-p(z)}+\frac{1}{e_3-p(z)}\right\}$$

Note.—where $p(u)$ denotes the Weierstrassian function.

1032. (N. P. PANDYA) :—The month of April, 1919, coincides with the month of Chaitra of the Samvat year 1975. Find other pairs of coincident months in the next 300 years.

1033. (V. RAMASWAMI AIYAR, M.A.) :—If two figures in a plane, one consisting of lines, and the other, of points, be in ortho-polar correspondence, (that is with respect to some triangle of reference ABC) prove that they will continue to be in ortho-polar correspondence under any relative motion of translation.

Show also that the circum circle of the triangle of reference continues invariable in the figure of lines and the nine-points circle, in the figure of points.

1034. (V. Ramaswami Aiyar, M.A.) :–Prove that the locus of the double foci of a system of confocal bicircular quarties consists of the two circular cubics which form the extreme members of the system.

1035. (K. Ananda Rao):—Suppose that

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n$$

is a polynomical with integral (positive, negative or zero) co-efficients, and that is *irreducible* that is to say it cannot be expressed as the product of two polynomials whose co-efficients are also integral. Then show that $f(x)$ cannot have a root of multiplicity greater than $\frac{1}{2}\, n$?.

1036. (A. C. L. Wilkinson).—A variable circle touches an ellipse at a fixed point. Show that the locus of the point of intersection of the other two common tangents of the circle and ellipse is the confocal hyperbola passing through the fixed point.

1037. (A. C. L. Wilkinson) :—If one angle of a triangle is 60^0, prove that the Euler line of the triangle is equally inclined to the two sides containing the angle of 60^0.

1038. (A. C. L. Wilkinson) :—The equation of a conic referred to tangential areal coordinates is

$$al^2 + bm^2 + cn^2 + 2hlm + 2gnl + 2fmn = o,$$

show that the magnitudes of its semi-axes are given by the equation

$$I^2 x^4 + 4\, I\, J\, R^2 x^2 + 16\, R^4 \Delta \sin^2 A \sin^2 B \sin^2 C = 0,$$

where R is the radius of the circumcircle of ABC, the triangle of reference, and

$$I = a + b + c + 2f + 2g + 2h$$

$$J = A \sin^2 A + B \sin^2 B + C \sin^2 C - 2\, F \cos A \sin B \sin C - 2G \cos B \sin C \sin A - 2\, H \cos C \sin A \sin B.$$

$$S = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} \text{ and } A = \frac{\partial \Delta}{\partial a},\ F = \tfrac{1}{2}\frac{\partial \Delta}{\partial f} \text{ etc.},$$

Hence, or otherwise, prove that if a conic is inscribed in a triangle, the product of the squares of its semiaxes is $4\, R\alpha\beta\gamma$, where R is the radius of the circumcircle of the triangle and α, β, γ are the perpendiculars from the centre of the conic on the lines joining the middle points of the sides of the triangle.

1039. (A. NARASINGA RAO) :—The integer 142857 is such that every cyclic permutation of its digits is a multipleof itself. Find other numbers of the same type.

1040. (A. NARASINGA RAO) :—Show that any small displacement of a rigid body may be resolved into 6 rotations about the edges of an arbitrary tetrahedron A B C D. If the motion be one of pure rotation about an instantaneous axis, prove that

$$^{w}ab.\ ^{w}cd + ^{w}bc.\ ^{w}ad + ^{w}ca.\ ^{w}bd = 0,$$

^{w}ab being angular velocity about the edge A B, with a similar notation for the others.

Determine the angular velocity about the axis ; also find the relation between the w's when the motion is one of pure translation.

1041. (A. NARASINGA RAO) :—Prove the following analogue of Ceva's Theorem in three dimensions.

A, B, C, D, are 4 points on the faces of a tetrahedron A B C D. If AA_1 BB_1 CC_1 & DD_1 are the generators of a hyperboloid, we have four relations of the type

$$\frac{\triangle A_1CD}{\triangle B_1CD} \cdot \frac{\triangle B_1AD}{\triangle C_1AD} \cdot \frac{\triangle C_1BD}{\triangle A_1BD} = 1.$$

Are these conditions all independant? If not, determine the relation between them.

Some Definite Integrals.

By S. Ramanujan.

I have shown elsewhere * that the definite integrals

$$\phi_w(t) = \int_0^\infty \frac{\cos \pi t x}{\cosh \pi x} e^{-\pi w x^2} dx,$$

$$\psi_w(t) = \int_0^\infty \frac{\sin \pi t x}{\sinh \pi x} e^{-\pi w x^2} dx$$

can be evaluated in finite terms if w is any rational multiple of i.

In this paper I shall show, by a much simpler method, that these integrals can be evaluated not only for these values but also for many other values of t and w.

Now we have

$$\phi_w(t) = 2 \int_0^\infty \int_0^\infty \frac{\cos 2\pi x z}{\cosh \pi z} \cos \pi t x . e^{-\pi w x^2} dx\, dz$$

$$= \frac{e^{-\frac{\pi t^2 w'}{4}}}{\sqrt{w}} \int_0^\infty \frac{\cosh \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx$$

where w' stands for $1/w$.

It follows that

$$\phi_w(t) = \frac{1}{\sqrt{w}} e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(i t w') \qquad (1)$$

Again

$$\phi_w(t+w) = \frac{1}{\sqrt{w}} e^{-\frac{\pi (t+w)^2}{4} w'} \times$$

$$\int_0^\infty \frac{\cosh \frac{\pi t}{w} x \cosh \pi x + \sinh \frac{\pi t}{w} x \sinh \pi x}{\cosh \pi x} e^{-\frac{\pi}{w} x^2} dx$$

$$= \frac{1}{\sqrt{w}} e^{-\frac{\pi (t+w)^2}{4w}} \left\{ \tfrac{1}{2} \sqrt{w}\, e^{\pi t^2/4w} + 2 \int_0^\infty \int_0^\infty \frac{\sin 2\pi x z}{\sinh \pi z} \sinh \frac{\pi t x}{w} e^{-\frac{\pi}{w} x^2} dx\, dz \right\}$$

* *The Messenger of Mathematics*, New Series, No. 522, Oct. 1914.

$$= \frac{1}{\sqrt{w}}\, e^{-\frac{\pi(t+w)^2}{4w}} \left\{ \tfrac{1}{2}\sqrt{w}\, e^{\pi t^2/4w} + \sqrt{w} e^{\pi t^2/4w} \int_0^\infty \frac{\sin \pi t x}{\sinh \pi x}\, e^{-\pi w x^2} dx. \right.$$

In other words

$$e^{\pi t^2/4w} \{ \tfrac{1}{2} + \psi_w(t) \} = e^{\frac{\pi(t+w)^2}{4w}} \phi_w(t+w). \tag{2}$$

It is obvious that

$$\begin{cases} \phi_w(t) = \phi_w(-t) \\ \psi_w(t) = -\psi_w(-t) \end{cases} \tag{3}$$

From (1), (2) and (3) we easily find that

$$\tfrac{1}{2} + \psi_w(t+i) = \frac{i}{\sqrt{w}} e^{-\frac{\pi t^2}{4w}} \left\{ \tfrac{1}{2} - \psi_{w'} \left(\frac{i\,t}{w} + i \right) \right\} \tag{4}$$

It is easy to see that

$$\phi_w(i) = \frac{1}{2\sqrt{w}};\ \psi_w(i) = \frac{i}{2\sqrt{w}};\ \phi_w(w) = \tfrac{1}{2}\, e^{-\frac{\pi w}{4}};$$

$$\tfrac{1}{2} - \psi_w(w) = e^{-\frac{\pi w}{4}} \phi_w(0);\ \phi_w(w \pm i) = \left(\frac{1}{2\sqrt{w}} \pm \frac{i}{2} \right) e^{-\frac{\pi w}{4}};$$

$$\psi_w(w \pm i) = \tfrac{1}{2} \pm \frac{i}{2\sqrt{w}} e^{-\frac{\pi w}{4}};\ \phi_w(\tfrac{1}{2}w) + \psi_w(\tfrac{1}{2}w) = \tfrac{1}{2}.$$

Again we see that

$$\phi_w(t+i) - \phi_w(t-i) = \frac{1}{\sqrt{w}}\, e^{-\frac{\pi t^2}{4w}}; \tag{5}$$

and

$$\psi_w(t+i) - \psi_w(t-i) = \frac{i}{\sqrt{w}}\, e^{-\frac{\pi t^2}{4w}}. \tag{6}$$

From (1) and (5) we deduce that

$$e^{\frac{\pi(t+w)^2}{4w}} \phi_w(t+w) + e^{\frac{\pi(t-w)^2}{4w}} \phi_w(t-w) = e^{\frac{\pi t^2}{4w}}. \tag{7}$$

Similarly from (4) and (6) we obtain

$$e^{\frac{\pi(t+w)^2}{4w}} \left\{ \frac{1}{2} - \psi_w(t+w) \right\} = e^{\frac{\pi(t-w)^2}{4w}} \left\{ \frac{1}{2} + \psi_w(t-w) \right\}. \tag{8}$$

It is easy to deduce from (5) that if n is a positive integer, then

$$\phi_w(t)+(-1)^{n+1}\phi_w(t\pm 2ni) \tag{9}$$

$$=\frac{1}{\sqrt{w}}\left\{e^{-\frac{\pi}{4w}(t\pm i)^2}-e^{-\frac{\pi}{4w}(t\pm 3i)^2}+e^{-\frac{\pi}{4w}(t\pm 5i)^2}\right.$$

$$\left.-\ldots\text{to } n \text{ terms}\right\}.$$

Similarly from (6) we have

$$\psi_w(t)-\psi_w(t\pm 2ni) \tag{10}$$

$$=-\frac{1}{\sqrt{w}}\left\{e^{-\frac{\pi}{4w}(t+i)^2}+e^{-\frac{\pi}{4w}(t+3i)^2}+e^{-\frac{\pi}{4w}(t+5i)^2}\right.$$

$$\left.-\ldots\text{to } n \text{ terms}\right\}$$

Again from (7) we have

$$e^{\frac{\pi t^2}{4w}}\phi_w(t)+(-1)^{n+1}e^{\frac{\pi(t+2nw)^2}{4w}}\phi_w(t+2nw)$$

$$=e^{\frac{\pi}{4w}(t+w)^2}-e^{\frac{\pi}{4w}(t+3w)^2}+e^{\frac{\pi}{4w}(t+5w)^2} \tag{11}$$

$$-\ldots\text{to } n \text{ terms};$$

and from (9)

$$e^{\frac{\pi t^2}{4w}}\left\{\frac{1}{2}+\psi_w(t)\right\}+(-1)^{n+1}e^{\frac{\pi(t+2nw)^2}{4w}}\times \tag{12}$$

$$\left\{\frac{1}{2}+\psi_w(t+2nw)\right\}$$

$$=e^{\frac{\pi}{4w}(t+2w)^2}-e^{\frac{\pi}{4w}(t+4w)^2}+e^{\frac{\pi}{4w}(t+6w)^2}$$

$$-\ldots\text{to } n \text{ terms}.$$

Now combining (9) and (11) we deduce that if m and n are positive integers and $s=t+2mw\pm 2ni$, then

$$\phi_w(s)+(-1)^{(m+1)(n+1)}e^{-\frac{1}{2}\pi m(s+t)}\phi_w(t) \tag{13}$$

$$=e^{-\frac{\pi s^2}{4w}}\left\{e^{\frac{\pi}{4w}(s-w)^2}-e^{\frac{\pi}{4w}(s-3w)^2}+e^{\frac{\pi}{4w}(s-5w)^2}\right.$$

$$\left.\ldots\text{ to } m \text{ terms}\right\}$$

$$+\frac{(-1)^{(m+1)(n+1)}}{\sqrt{w}}e^{-\frac{1}{2}\pi m(s+t)}\left\{e^{-\frac{\pi}{4w}(t\pm i)^2}\right.$$

$$\left.-e^{-\frac{\pi}{4w}(t\pm 3i)^2}+e^{-\frac{\pi}{4w}(t\pm 5i)^2}-\ldots\text{to } n \text{ terms}\right\}.$$

Similarly combining (10) and (12) we obtain

$$\frac{1}{2}-\psi_w(s)+(-1)^{mn+m+1}\,e^{-\frac{1}{2}\pi m(s+t)}\left\{\frac{1}{2}-\psi_w(t)\right\} \tag{14}$$

$$=e^{-\frac{\pi s^2}{4w}}\left\{e^{\frac{\pi}{4w}(s-2w)^2}-e^{\frac{\pi}{4w}(s-4w)^2}\right.$$

$$\left.+e^{\frac{\pi}{4w}(s-6w)^2}-\ldots\text{to } m \text{ terms}\right\}$$

$$\pm(-1)^{mn+m+1}\frac{i}{\sqrt{w}}e^{-\frac{1}{2}\pi m(s+t)}\left\{e^{-\frac{\pi}{4w}(t\pm i)^2}\right.$$

$$\left.+e^{-\frac{\pi}{4w}(t\pm 3i)^2}+e^{-\frac{\pi}{4w}(t\pm 5i)^2}+\ldots\text{to } n \text{ terms}\right\}$$

where s and t have the same relation as in (13).

Suppose now that $s=t$ in (13) and (14). Then we see that

if $w=\frac{in}{m}$, then

$$\phi_w(t)\left\{1+(-1)^{(m+1)(n+1)}e^{-\pi mt}\right\} \tag{15}$$

$$=e^{-\frac{\pi t^2}{4w}}\left\{e^{\frac{\pi}{4w}(t-w)^2}-e^{\frac{\pi}{4w}(t-3w)^2}\right.$$

$$\left.+e^{\frac{\pi}{4w}(t-5w)^2}-\ldots\text{to } m \text{ terms}\right\}$$

$$+\frac{(-1)^{(m+1)(n+1)}}{\sqrt{w}}\cdot e^{-\pi mt}\left\{e^{-\frac{\pi}{4w}(t-i)^2}\right.$$

$$\left.-e^{-\frac{\pi}{4w}(t-3i)^2}+\ldots\text{to } n \text{ terms}\right\};$$

$$\left\{\frac{1}{2}-\psi_w(t)\right\}\left\{1+(-1)^{mn+m+1}e^{-\pi mt}\right\} \tag{16}$$

$$=e^{-\frac{\pi t^2}{4w}}\left\{e^{\frac{\pi}{4w}(t-2w)^2}-e^{\frac{\pi}{4w}(t-4w)^2}+\ldots\text{to } m \text{ terms}\right\}$$

$$+(-1)^{mn+m}\frac{i}{\sqrt{w}}e^{-\pi mt}\left\{e^{-\frac{\pi}{4w}(t-i)^2}\right.$$

$$\left.+e^{-\frac{\pi}{4w}(t-3i)^2}+\ldots\text{to } n \text{ terms}\right\}$$

where $\sqrt{w}$ should be taken as

$$e^{\frac{i\pi}{4}}\sqrt{\left(\frac{n}{m}\right)}.$$

In (15) and (16) there is no loss of generality in supposing that one of the two numbers m and n is odd.

Now equating the real and imaginary parts in (15), we deduce that if m and n are positive integers of which one is odd, then

$$2\cosh nt\int_0^\infty \frac{\cos 2tx}{\cosh \pi x}\cos\left(\frac{\pi m x^2}{n}\right)dx \tag{17}$$

$$=[\cosh\{(1-n)t\}\cos(\pi m/4n)-\cosh\{(3-n)t\}\cos(9\pi m/4n)$$

$$+\sqrt{\left(\frac{n}{m}\right)}\left[\cosh\left\{\left(1-\frac{1}{m}\right)nt\right\}\cos\left(\frac{\pi}{4}-\frac{nt^2}{\pi m}+\frac{\pi n}{4m}\right)\right.$$

$$\left.-\cosh\left\{\left(1-\frac{3}{m}\right)nt\right\}\cos\left(\frac{\pi}{4}-\frac{nt^2}{\pi m}+\frac{9\pi n}{4m}\right)+\ldots\text{to } m \text{ terms}\right];$$

and

$$2\cosh nt\int_0^\infty \frac{\cos 2tx}{\cosh \pi x}\sin\left(\frac{\pi m x^2}{n}\right)dx \tag{18}$$

$$=-[\cosh\{(1-n)t\}\sin(\pi m/4n)$$

$$-\cosh\{(3-n)t\}\sin(9\pi m/4n)$$

$$+\cosh\{(5-n)t\}\sin(25\pi/4n)-\ldots\text{to } n \text{ terms}]$$

$$+\sqrt{\left(\frac{n}{m}\right)}\left[\cosh\left\{\left(1-\frac{1}{m}\right)nt\right\}\sin\left(\frac{\pi}{4}-\frac{nt^2}{\pi m}+\frac{\pi n}{4m}\right)\right.$$

$$\left.-\cosh\left\{\left(1-\frac{3}{m}\right)nt\right\}\sin\left(\frac{\pi}{4}-\frac{nt^2}{\pi m}+\frac{9\pi n}{\pi m}\right)\right.$$

$$\left.+\ldots\text{to } n \text{ terms}\right].$$

Equating the real and imaginary parts in (16), we can find similar expressions for the integrals

$$\int_0^\infty \frac{\sin tx}{\sinh \pi x}\sin\left(\frac{\pi m x^2}{n}\right)dx,\quad \int_0^\infty \frac{\sin tx}{\sinh \pi x}\cos\left(\frac{\pi m x^2}{n}\right)dx.$$

From these formulae we can evaluate a number of definite integrals, such as

$$\int_0^\infty \frac{\cos 2\pi tx}{\cosh \pi x}\cos \pi x^2\,dx=\frac{1+\sqrt{2}\sin \pi t^2}{2\sqrt{2}\cosh \pi t},$$

$$\int_0^\infty \frac{\cos 2\pi tx}{\cosh \pi x}\sin \pi x^2\,dx=\frac{-1+\sqrt{2}\cos \pi t^2}{2\sqrt{2}\cosh \pi t},$$

$$\int_0^\infty \frac{\sin 2\pi tx}{\sinh \pi x}\cos \pi x^2\,dx=\frac{\cosh \pi t-\cos \pi t^2}{2\sinh \pi t},$$

$$\int_0^\infty \frac{\sin 2\pi tx}{\sinh \pi x}\sin \pi x^2\,dx=\frac{\sin \pi t^2}{2\sinh \pi t},$$

and so on.

Again supposing that $s=-t$ in (13), we deduce that if $t=m\,w\pm n\,i$ where m and n are positive integers of which one at least is odd, then

$$\phi_w(t)=\frac{1}{2}e^{-\frac{\pi t^2}{4w}}\left\{e^{\frac{\pi}{4w}(t-w)^2}-e^{\frac{\pi}{4w}(t-3w)^2}+\ldots\text{to } m \text{ terms}\right\}$$
$$+\frac{1}{2\sqrt{w}}\left\{e^{-\frac{\pi}{4w}(t\pm i)^2}-e^{-\frac{\pi}{4w}(t\pm 3i)^2}+\ldots\text{to } n \text{ terms}\right\}. \quad (19)$$

This formula is not true when both m and n are even.

If $t=m\,w\pm n\,i$ where m and n are both even, then

$$\phi_w(t)+(-1)^{(1+\frac{1}{2}m)\,(1+\frac{1}{2}n)}\,e^{-\frac{1}{4}\pi m t}\,\phi_w(0) \quad (20)$$
$$=e^{-\frac{\pi t^2}{4w}}\left\{e^{\frac{\pi}{4w}(t-w)^2}-e^{\frac{\pi}{4w}(t-3w)^2}+\ldots\text{to } \tfrac{1}{2}\ m \text{ terms}\right\}$$
$$+\frac{(-1)^{(1+\frac{1}{2}m)\,(1+\frac{1}{2}n)}}{\sqrt{w}}\,e^{-\frac{1}{4}\pi m\,t}$$
$$\left\{e^{\frac{\pi}{4w}}-e^{\frac{9\pi}{4w}}+e^{\frac{25\pi}{4w}}-\ldots\text{to } \tfrac{1}{2}\ n \text{ terms}\right\}.$$

This is easily obtained by putting $t=0$ and then changing s to t in (13). Similarly from (14) we deduce that if $t=m\ w\pm n\ i$ where m and n are both even, or both odd, or m is even and n is odd, then

$$\psi_w(t)=-\frac{1}{2}e^{-\frac{\pi t^2}{4w}}\left\{e^{\frac{\pi}{4w}(t-2w)^2}-e^{\frac{\pi}{4w}(t-4w)^2}\right. \quad (21)$$
$$\left.+\ldots\text{to } \tfrac{1}{4}m \text{ terms}\right\}$$
$$\pm\frac{i}{2\sqrt{w}}\left\{e^{-\frac{\pi}{4w}(t\mp i)^2}+e^{-\frac{\pi}{4w}(t\mp 3i)^2}+\ldots\text{to } \ n \text{ terms}\right\}.$$

If $t=mw\pm ni$ where m is odd and n is even, then

$$\tfrac{1}{2}-\psi_w(t)+\left\{(m-1)^{1+\frac{1}{4}(m-1)(n+2)}\right. \quad (22)$$
$$e^{-\frac{1}{4}\pi\{(m-1)t+mw\}}\,\phi_w(o)$$
$$=e^{-\frac{\pi t^2}{4w}}\left\{e^{\frac{\pi}{4w}(t-2w)^2}-e^{\frac{\pi}{4w}(t-4w)^2}\right.$$
$$\left.\ldots+\text{to } \tfrac{1}{2}\ (m-1) \text{ terms}\right\}$$

$$\pm(-1)^{1+\frac{1}{4}(m-1)(n+2)} \frac{i}{\sqrt{w}} e^{-\frac{1}{4}\pi(m-1)(t+w)}$$

$$\left\{ e^{-\frac{\pi}{4w}(w\mp i)^2} - e^{-\frac{\pi}{4w}(w\mp 3i)^2} + \ldots \text{to } \tfrac{1}{2}n \text{ terms.} \right\}$$

This is obtained by putting $t=w$ in (14). A number of definite integrals such as the following can be evaluated with the help of the above formulae.

$$\int_0^\infty \frac{\cos \pi t x}{\cosh \pi x} e^{-\pi(t+i)x^2} dx = \frac{1+i}{2\sqrt{2}} e^{-\frac{\pi t}{4}} \left\{ 1 - \frac{i}{\sqrt{(t+i)}} \right\},$$

$$\int_0^\infty \frac{\sin \pi t x}{\sinh \pi x} e^{-\pi(t+i)x^2} dx = \frac{1}{2} - \frac{1+i}{2\sqrt{2}} \cdot \frac{e^{-\frac{\pi t}{4}}}{\sqrt{(t+i)}},$$

and so on.

Fractional Differentiation.*

By M. T. Naraniengar.

[This paper relates to a method of fractional differentiation which is apparently a generalization of the customary processes of differentiation of the Calculus. The idea here developed is akin to that of expressing factorials of fractional numbers by gamma functions and the results arrived at are not altogether uninteresting. So far as the author is aware, no reference to the ideas of this paper is to be found in any previous paper in the English language. In other languages also, such references are few and are of a fragmentary nature. M Paul Dienes in his "*Leçons sur les singularities des functions analytiques*" (one of the collections of Borel's pamphlets) draws casual attention to the genralized derivate of Riemann and of the analogous process of integration of a fractional or incommensurable index by Hadamard. The reference to the work of Riemann is to "*Versuch allg. Auffasung der Integ. and Differ.*" (1847) and "*Mathematische Werke*;" but the articles by Voss and Pringsheim in the *Encyclopaedie der Math Wissenschaften* do not make any mention of fractional differentiation. Forsyth makes a casual reference to the operator $\left(\frac{d}{dt}\right)^{\frac{1}{2}}$ at p. 393, (*Diffl. Eqns.*, 1885,) but abstains from developing any properties of the operator.

The International Catalogue of Scientific Literature gives the following paper, which may have some bearing on the subject matter; but the present writer regrets he has not been able to get access to it,

R. E. Moritzer: "Generalization of the Differentiation Process," *Amer. Jour. of Math.* 24 (1902) 257—302.]

(i) Ordinary Differentiation.

1. The laws of combination of the operator D or $\frac{d}{dx}$ are extended by considering fractional indices in the index law as expressed by the equation

$$D^p D^q u = D^{p+q} u. \tag{1}$$

Thus we may write

$$D^{\frac{1}{2}} D^{\frac{1}{2}} u = Du, \tag{2}$$

$$D^{\frac{1}{3}} D^{\frac{1}{3}} D^{\frac{1}{3}} u = Du, \tag{3}$$

and so on. The meaning of $D^{\frac{1}{2}}$ or $D^{\frac{1}{3}}$ in the above has therefore to be inferred from these functional equations.

* Presented to the Indian Mathematical Conference, Bombay, January 1919.

Let us consider the function x^n. We know that the operator D as applied to it signifies that the function has to be multiplied by the index n and unity subtracted from the index. Hence the symbol $D^{\frac{1}{2}}$ should mean an operation which when repeated twice in succession will be equivalent to the above. In other words, if $D^{\frac{1}{2}}$ is equivalent to multiplication by R (n) and diminution of the index by $\frac{1}{2}$,

i.e., if $$D^{\frac{1}{2}}(x^n) = R(n)x^{n-\frac{1}{2}} \qquad (4)$$

then $$D^{\frac{1}{2}}D^{\frac{1}{2}}(x^n) = R(n)R(n-\tfrac{1}{2})x^{n-1}; \qquad (5)$$

and therefore $$R(n)R(n-\tfrac{1}{2}) = n, \qquad (6)$$

which is the functional equation defining the new symbol R (n).

2. To solve the above equation, we may proceed thus:—

We have $$R(n).\ R(n-\tfrac{1}{2}) = n,$$

$$R(n-\tfrac{1}{2}).\ R(n-1) = n-\tfrac{1}{2}.$$

$$\therefore \quad R(n)/R(n-1) = n/(n-\tfrac{1}{2}).$$

But $$\Gamma(n+1) = n\Gamma(n),$$

and $$\Gamma(n+\tfrac{1}{2}) = (n-\tfrac{1}{2})\,\Gamma(n-\tfrac{1}{2});$$

so that $$\frac{R(n)}{R(n-1)} = \frac{\Gamma(n+1)\,\Gamma(n-\frac{1}{2})}{\Gamma(n)\,\Gamma(n+\frac{1}{2})} = \frac{\Gamma(n+1)/\Gamma(n+\frac{1}{2})}{\Gamma(n)/\Gamma(n-\frac{1}{2})}.$$

Hence $$R(n) \propto \Gamma(n+1)/\Gamma(n+\tfrac{1}{2}) = \lambda\,\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}, \text{ say.}$$

Then $$R(n-\tfrac{1}{2}) = \lambda\Gamma(n+\tfrac{1}{2})\Gamma(n)$$

$$\therefore \quad R(n).\ R(n-\tfrac{1}{2}) = \lambda^2\Gamma(n+1)/\Gamma(n) = \lambda^2 n = n, \text{ by (6)}$$

Thus $$\lambda^2 = 1 \quad \text{or} \quad \lambda = 1;$$

so that $$R(n) = \Gamma(n+1)/\Gamma(n+\tfrac{1}{2}), \qquad \ldots \quad (7)$$

and $$\frac{1}{R(n)} = \frac{1}{\sqrt{\pi}}\left\{\frac{1}{n+\frac{1}{2}} + \frac{\frac{1}{2}}{n+\frac{3}{2}} + \frac{\frac{1}{2}\cdot\frac{3}{2}}{(n+\frac{5}{2})^2_1} + \ldots\right.$$

[Bromwich: *Inf. Series*, p. 477, Ex. 51.]

3. Next consider $D^{\frac{1}{3}}(x^n)$.

Proceeding as before, we have

$$R(n)\ R(n-\tfrac{1}{3})\ R(n-\tfrac{2}{3}) = n,$$

and $$R(n-\tfrac{1}{3})\ R(n-\tfrac{2}{3})\ R(n-1) = n-\tfrac{1}{3}.$$

$$\therefore \quad R(n)/R(n-1) = n/(n-\tfrac{1}{3}) = \frac{\Gamma(n+1)\,\Gamma(n-\frac{1}{3})}{\Gamma(n)\,\Gamma(n+\frac{2}{3})} = \frac{\Gamma(n+1)/\Gamma(n+\frac{2}{3})}{\Gamma(n)/\Gamma(n-\frac{1}{3})}.$$

$$\therefore \quad R(n) = \lambda.\ \Gamma(n+1)/\Gamma(n+\tfrac{2}{3}).$$

It is easily proved that $\lambda = 1$ as in § 2, and we have finally

$$R(n) = \Gamma(n+1)/\Gamma(n+\tfrac{2}{3}),$$

or
$$\frac{1}{R(n)} = \frac{1}{\Gamma(\frac{1}{3})}\left\{\frac{1}{n+\frac{2}{3}} + \frac{2}{3(n+\frac{5}{3})} + \frac{2\cdot 5}{3\cdot 6(n+\frac{8}{3})} + \dots\right\}$$

4. Generally $D^{\frac{1}{p}}(x^n) = R(n)\,x^{n-\frac{1}{p}}$,

where
$$R(n) = \Gamma(n+1)/\Gamma(n+\frac{p-1}{p})$$

and
$$\frac{1}{R(n)} = \frac{1}{\left(\Gamma\frac{1}{p}\right)}\left\{\frac{1}{(n+1-\frac{1}{p})} + \frac{p-1}{p\left(n+2-\frac{1}{p}\right)} + \frac{(p-1)(2p-1)}{p\cdot 2p(n+3-\frac{1}{p})} + \dots\right\}$$

5. The negative factional index is readily interpreted as follows :—

$$D^{\frac{1}{2}}(x^n) = x^{n-\frac{1}{2}}\Gamma(n+1)/\Gamma(n+\tfrac{1}{2}),$$

since
$$D^{-\frac{1}{2}}(x^{n-\frac{1}{2}}) = x^n\Gamma(n+\tfrac{1}{2})/\Gamma(n+1),$$

or
$$D^{-\frac{1}{2}}(x^n) = x^{n+\frac{1}{2}}\Gamma(n+1)/\Gamma(n+\tfrac{3}{2});$$

and so on.

6. In the following we shall use the notation $x!$ instead of $\Gamma(x+1)$.

Theorem I. *The generalised Leibnitz's Theorem.*

If u, v be any two polynomials in x, then

$$D^p(u\,v) = (D_1 + D_2)^p uv$$

whatever p may be, integral or fractional, positive or negative.

The proof of the theorem depends upon the well known property of the hypergeometric function given in Forsyth.

Thus

$$(D_1+D_2)^p(x^m x^n)=D_1^p(x^m)+\frac{p}{1!}D_1^{p-1}(x^m)D_2(x^n)+\dots$$

$$=\frac{m!}{(m-p)!}x^{m-p}x^n+\frac{p}{1!}\frac{m!}{(m-p+1)!}\frac{n!}{(n-1)!}x^{m-p+1}x^{n-1}+\dots$$

$$=\frac{m!}{(m-p)!}x^{m+n-p}\left\{1+\frac{p.n}{m-p+1}+\right.$$

$$\left.\frac{p(p-1)}{2!}\frac{n(n-1)}{(m-p+1)(m-p+2)}+\dots\right\}$$

$$=\frac{m!}{(m-p)!}x^{m+n-p}\,F_1\left\{\begin{matrix}-p,-n,\\ m-p+1\end{matrix}\right\}$$

$$=x^{m+n-p}\frac{(m+n)!}{(m+n-p)!},\quad\left[\text{Forsyth, § 126}\right]$$

which is the same as $D^p(x^m x^n)$

Hence writing $(u.v)$ in the form $\Sigma A(x^m.\ x^n)$the result follows.

The following further extension is obvious :—

$$D^p(u,v.w.\dots\dots)=(D_1+D_2+D_3+\dots)^p\{u.v.w\dots\}$$

6. **Theorem II.** *Let* $f(x)=\Sigma(a_n x^n)$ *and* $F(x)=\Sigma(b_n x^n)$ *where* n *may be integral or fractional ; then the constant term in* $f(D)\,F(x)$ *is equal to the constant term in* $F(D)\,f(x)$.

For,
$$f(D)\,F(x)=\Sigma(a_p D^p).\ \Sigma(b_q x^q)$$
$$=\Sigma a_q b_q\frac{q!}{(q-p)!}x^{q-p}.$$

Hence the constant term is $\Sigma(a_n b_n.\ n!)$ and its symmetry shows that it is the same as the constant term in $F(D)\,f(x)$.

8. *The generalised Binomial Series.*

Consider
$$z=n_{\frac{1}{2}}\frac{x^{\frac{1}{2}}}{(\frac{1}{2})!}+n_{\frac{3}{2}}\frac{x^{\frac{3}{2}}}{(\frac{3}{2})!}+\dots$$

where n_r denotes $\dfrac{n!}{(n-r)!}$.

Then
$$D(z)=n_{\frac{1}{2}}\frac{x^{-\frac{1}{2}}}{(\frac{1}{2})!}+n_{\frac{3}{2}}\frac{x^{\frac{1}{2}}}{(\frac{1}{2})!}+\dots$$

But $n_{\frac{3}{2}}=\dfrac{n!}{(n-\frac{3}{2})!}=\dfrac{n!}{(n-\frac{1}{2})!}.(n-\frac{1}{2})=n_{\frac{1}{2}}(n-\frac{1}{2})$, &c., &c.

$$D(z)=n_{\frac{1}{2}}\frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})!}+n_{\frac{1}{2}}z-\left\{\frac{1}{2}n_{\frac{1}{2}}\frac{x^{\frac{1}{2}}}{(\frac{1}{2})!}+\frac{3}{2}n_{\frac{3}{2}}\frac{x^{\frac{3}{2}}}{(\frac{3}{2})!}+\dots\dots\right\}$$

$$\therefore \qquad \mathrm{D}(z) = n_{\frac{1}{2}} \frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})!} + nz - x\mathrm{D}z$$

$$\therefore \qquad (x+1)\ \mathrm{D}z - nz = n_{\frac{1}{2}} \frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})!} = n_{\frac{1}{2}} \frac{x^{-\frac{1}{2}}}{\pi^{\frac{1}{2}}}.$$

$$\therefore \qquad z = \frac{2n_{\frac{1}{2}}(z+x)^n}{\pi^{\frac{1}{2}}} \int \frac{dx^{\frac{1}{2}}}{(1+x)^{n+1}}.$$

In other words, we have the following generalised binomial series

$$1 + n_{\frac{1}{2}} \frac{x^{\frac{1}{2}}}{(\frac{1}{2})!} + n_1 \frac{x^1}{(1)!} + n_{\frac{3}{2}} \frac{x^{\frac{3}{2}}}{(\frac{3}{2})!} + n_2 \frac{x^2}{(2)!} + \ldots$$

$$= (1+x)^n \left\{ 1 + \frac{2\, n_{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \int \frac{dx^{\frac{1}{2}}}{(1+x)^{n+1}} \right\}.$$

9. *The generalised Exponential Series.*

Let $$z = \frac{x^{\frac{1}{2}}}{(\frac{1}{2})!} + \frac{x^{\frac{3}{2}}}{(\frac{3}{2})!} + \ldots$$

Proceeding as in § 8, we easily get

$$\mathrm{D}(z) - z = x^{\frac{1}{2}} / \pi^{\frac{1}{2}}.$$

$$\therefore \qquad z = \frac{2e^x}{\pi^{\frac{1}{2}}} \int e^{-x} dx^{\frac{1}{2}},$$

so that the series

$$1 + \frac{x^{\frac{1}{2}}}{(\frac{1}{2})!} + \frac{x^1}{1!} + \frac{x^{\frac{3}{2}}}{(\frac{3}{2})!} + \ldots = e^x \left\{ 1 + \frac{2}{\pi^{\frac{1}{2}}} \int e^{-x}\, dx^{\frac{1}{2}} \right\}.$$

10. *The generalised Taylor Series.*

Here we put $$z = f^{\frac{1}{2}}(h)\ \frac{x^{\frac{1}{2}}}{(\frac{1}{2})!} + f^{\frac{3}{2}}(h)\ \frac{x^{\frac{3}{2}}}{(\frac{3}{2})!} + \ldots$$

and obtain $$\mathrm{D}_x z - \mathrm{D}_h z = \frac{x^{-\frac{1}{2}}}{\pi^{\frac{1}{2}}}\ \mathrm{D}_h{}^{\frac{1}{2}} f(h)$$

$$\therefore \qquad z = \frac{1}{\pi^{\frac{1}{2}}} \int^x d\xi\, .\, \xi^{\frac{1}{2}} \mathrm{D}_\xi{}^{\frac{1}{2}} f(x+h-\xi),$$

so that the generalised Taylor Series can be written

$$f(x+h) + \frac{1}{\pi^{\frac{1}{2}}} \int^x d\xi \left[\xi^{\frac{1}{2}} \mathrm{D}_h{}^{\frac{1}{2}} f(x+h-\xi) \right]$$

11. Consider the series,

$$z = \frac{x^a}{a\,!} + \frac{x^{a+1}}{(a+1)\,!} + \cdots$$

where a may be any quantity except a negative integer.

We have $$Dz = \frac{x^{a-1}}{(a-1)\,!} + \frac{x^a}{a\,!} + \cdots$$

$$= \frac{x^{a-1}}{(a-1)\,!} + z.$$

$$\therefore \quad (D-1)z = \frac{x^{a-1}}{(a-1)\,!}$$

$$\therefore \quad z = \frac{e^x}{(a-1)\,!} \int^x x^{a-1} e^{-x} dx$$

$$= \frac{x^a}{a\,!} \int e^{-x} d(x^a).$$

The series z is obviously always convergent. Further, if we remember that

$$x^{1-a} z \cdot (a-1)\,! = F(1, \beta, a, x/\beta), \text{ where } \beta \longrightarrow \infty$$

z can be represented by Barnes' integral as follows :

$$z = \frac{x^{a-1}}{2\pi i} \int_{-i\infty}^{i\infty} \frac{y\,!\,(\beta+y-1)!\,(-1-y)!}{(a+y-1)\,!\,(\beta-1)\,!} \left(-\frac{x}{\beta}\right)^y dy$$

where $\beta = \infty$;

12. *Properties of* $\left(x \frac{d}{dx}\right)$:

Let $D_{\frac{1}{2}}$ denote $\left(x \frac{d}{dx}\right)^{\frac{1}{2}}$, then

$$D_{\frac{1}{2}}(x^n) = \left(x \frac{d}{dx}\right)^{\frac{1}{2}} x^n = x^{\frac{1}{2}} n^{\frac{1}{2}} x^{n-\frac{1}{2}}$$

$$= n^{\frac{1}{2}} x^n.$$

Hence, we have

(i) $D^r_{\frac{1}{2}}(x^n) = (n^{\frac{1}{2}})^r x^n$;

(ii) $\phi\left(D_{\frac{1}{2}}\right) x^n = \phi\left(n^{\frac{1}{2}}\right) x^n$, from (i),

where ϕ is a polynomial.

13. Let λ_m denote the operation

$$\left[x^{\frac{1}{2}-m} D^{\frac{1}{2}} x^m\right];$$

then $$\lambda_m(x^r) = x^{\frac{1}{2}-m}\, D^{\frac{1}{2}}\, x^{m+r}$$
$$= x^{\frac{1}{2}-m}\,(m+r)_{\frac{1}{2}}\, x^{m+r-\frac{1}{2}}$$
$$= (m+r)_{\frac{1}{2}}\, x^r.$$

Hence any number of operations λ_m, λ_n etc, are commutative.

[*Cf.* Edwards: *Diffl. Calc*, p. 486, Ex. 2.]

14. We can prove that when n is an integer,

$$D^n x^{n+\frac{1}{2}} D^{n+1} \equiv (x^{\frac{1}{2}} + \frac{1}{2^{2n+1}}\phi^{2n+1}(x^{\frac{1}{2}}). \qquad [\textit{Ibid}, \text{Ex. } 3]$$

For, the left hand side is equal to

$$D^n x^{n+\frac{1}{2}} D^{n+1} \Sigma(a_r\, x^{\frac{r}{2}})$$
$$= D^n x^{n+\frac{1}{2}} \Sigma \left\{ \left(\frac{r}{2}\right)_{n+1} a_r x^{(r/2-n-1)} \right\}$$
$$= D^n \Sigma \left(\frac{r}{2}\right)_{n+1} a_r x^{\frac{1}{2}(r-1)}$$
$$= \left(\frac{r}{2}\right)_{n+1} \left(\frac{r-1}{2}\right)_n a_r\, x^{\frac{r-1}{2}-n}$$

Also $$\frac{1}{2^{2n+1}}\phi^{2n+1}(x^{\frac{1}{2}}) = \frac{1}{2_{2n+1}} \Sigma(r)_{2n+1}\, a_r\, x^{\frac{r-1}{2}-n};$$

and $$\left(\frac{r}{2}\right)_{n+1}\left(\frac{r-1}{2}\right)_n = \left(\frac{r}{2}\right)\left(\frac{r}{2}-1\right)\cdots\left(\frac{r}{2}-n\right) \times \left(\frac{r-1}{2}\right)\left(\frac{r-1}{2}-1\right)\cdots\left(\frac{r-1}{2}-n+1\right)$$
$$= \frac{r(r-2)\ldots(r-2n)\times(r-1)(r-3)\ldots(r-2n+1)}{2^{2n+1}},$$
$$= \frac{1}{2^{2n+1}}\,(r)_{2n+1}.$$

Hence we obtain the result stated.

More generally, we can verify that

$$\frac{\left(\frac{r}{2}\right)!}{\left(\frac{r}{2}-n-1\right)!}\cdot\frac{\left(\frac{r-1}{2}\right)!}{\left(\frac{r-1}{2}-n\right)!} = \frac{1}{2^{2n+1}}\cdot\frac{r!}{(r-2n-1)!},$$

by the duplication formula. [Whittaker: p. 234, 12.15 Cor.]

The result may also be written in the form

$$2^r\left(\frac{r}{2}\right)!\left(\frac{r-1}{2}\right)! = \pi^{\frac{1}{2}}\,(r)!.$$

Hence, as before, we prove that the result quoted above is universally true, whatever n may be.

2° Partial Differentiation.

15. We shall, in this section, discuss the properties of partial differential symbols $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ &c, as applied to functions of more than one independant variable.

The following properties are immediate :

$$\text{(i)} \quad \frac{d^{m+n}}{dx^m dy^n} = \frac{d^{m+n}}{dy^n dx^m},$$

$$\text{(ii)} \quad \frac{d^{m+n+p}}{dx^m dy^n dz^p} = \frac{d^{m+n+p}}{dy^n dx^m dz^p} = \&c. ;$$

where m, n, p may be any quantities whatever.

16. *Euler's Theorems on Homogeneous Functions.*

These may be written (with our notation)

$$\text{(1)} \quad \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \dots \right) u = n_1 u ;$$

$$\text{(2)} \quad \left(x^2\frac{\partial^2}{\partial x^2} + 2xy\frac{\partial^2}{\partial x \partial y} + y^2\frac{\partial^2}{\partial y^2} \dots \right) u = n_2 u,$$

and so on.

Thus, the generalized theorem would be

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial x} + \dots \right)^r u = n_r u$$

whatever r may be, the left hand member being expanded as an ordinary multinomial.

These results may be further generalized as follows :—

(i) Let $u = x^m y^n$; then $D_1^a D^{\ b} u = m_a n_b u$, for all values of a if D_1 operates on x^m and D_2 on y^n.

Hence $$\phi\,(D_1, D_2)\, u = \phi(m, n)\, u,$$

where we are to understand that $(m^a n^b)$ is to be replaced by $(m_a n_b)$ in the expanded form of $\phi\,(m, n)$.

Again, if $$u = x^m y^n z^p,$$

we have $$\phi\,(D_1, D_2, D_3)\, u = \phi\,(m, n, p).\, u ;$$

and so on.

(ii) If $u =$ a homogeneous expression of the rth degree, Euler's theorems will take the form

$$(D_1 + D_2 + \dots)\, u = (m + n + \dots) u = r\, u.$$

$$(D_1 + D_2 +)^2 u = r_2 u.$$

For fractional indices

$$(D_1 + D_2 + \dots)^{\frac{1}{2}}\, u = r_{\frac{1}{2}}.\, u = \frac{r\,!}{(r - \frac{1}{2})\,!}\, u,$$

and so on.

"On a system of three similar triangles, connected with a given triangle."

By S. Narayanan, Madura College.

1. Let A,B,C,D be four given points in a plane. It is known that the pedal triangles of each taken with respect to the triangle formed by the other three, are all directly similar, the similarity arising from the fact that the sides of each of the triangles are as the rectangles BC.AD, CA.BD, AB.CD. But it is perhaps not known that these four pedal triangles have the same double point. It will be shewn, in this paper, that they have a common double point which is the centre of the rectangular hyperbola ABCD.

From the property of the rectangular hyperbola, it is known that the pedal circles of each of the points A,B,C,D taken with respect to the triangle formed by the other three all pass through a common point. In this paper, quite apart from the geometry of conics, it is shown that these circles meet in a point and that point is the double point in question. These follow from a consideration of three similar triangles, formed in a particular manner, as explained below.

2. Let ABC be a given triangle and A_1 the foot of the perpendicular from A on BC. Drawing any line through A let BB_1, CC_1 be perpendiculars on it from B and C. Then it is readily seen from the figure that the triangle $A_1B_1C_1$ is inversely similar to ABC. Thus for every line drawn through A, there exists a triangle $A_1B_1C_1$, inversely similar to ABC. And a like construction gives, for every line through B as well as for every line through C, triangles inversely similar to ABC. We proceed to consider the triad of similar triangles arising when the lines through A, B, C *all pass through a given point* D.

3. Let the feet of the perpendiculars from B and C on A D be B_1 and C_1; those from C and A on BD be C_2 and A_2; and those from A and B on CD be A_3 and B_3. And let us denote the feet of the perpendiculars from A, B, C on B C, C A, AB respectively by A_1, B_2 and C_3. Then the triangles in question, corresponding to AD, BD' C D are $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$. These are directly similar to one another, each being inversely similar to ABC.

It may be noted at this stage, that $A_1A_2A_3$ is the pedal triangle of A with respect to the triangle BCD, $B_1B_2B_3$ is the pedal triangle of B, with respect to CDA and $C_1C_2C_3$ is that of C with respect to DAB.

4. **Theorem.** *In the similar triangles* $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, *each set of corresponding sides is concurrent.*

To prove that B_1C_1, B_2C_2, B_3C_3 are concurrent.

Let B_2C_2, B_3C_3 meet in H_1. Let D′ be the isogonal conjugate of D in A B C. Now, B_2 and C_3 are the feet of the perpendiculars from B and C on the opposite sides in A B C. Hence the triangle $A\,B_2C_3$ is inversely similar to A B C. Also H_1 in $A\,B_2\,C_3$ corresponds to D′ in A B C. For, the angle $A\,C_3H_1 = B\,C_3B_3 = B\,C\,B_3$ (for B, B_3, C_3, C are concyclic) = the angle $B\,C\,D = A\,C\,D'$. Similarly $A\,B_2\,H_1 = A\,B_2\,C_2 = CBC_2$ (for C, B_2, C_2, B are concyclic) $= C\,B\,D = A\,B\,D'$. Hence H_1 in $A\,B_2C_3$ corresponds to D′ in A B C as stated. Therefore the angle $H_1AC_3 = D'\,A\,C = B\,A\,D$, which shews that H_1 lies on A D or $B_1\,C_1$. Thus B_1C_1, B_2C_2, B_3C_3 are concurrent at H_1.

In the same manner, the other sets of corresponding sides C_1A_1, C_2A_2, C_3A_3 concur at at H_2 and A_1B_1, A_2B_2, A_3B_3 concur at H_3.

5. Let us now specially consider the fact that, in the triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, the corresponding sides intersecting *at the vertices* A_1, A_2, A_3 are concurrent, *viz.*, at H_2 and H_3. It follows that A_1, A_2, A_3, H_2, H_3 are all concyclic; for, as is seen from the figure, H_2H_3 subtends equal angles at A_2 and A_3 on one side and a supplementary angle at A_1 on the other side. *Hence we have the theorem that the pedal circle* $A_1A_2A_3$ *of* A *with respect to the triangle* BCD, *pass through the points* H_2 *and* H_3.

In like manner, the pedal circle $B_1B_2B_3$ of B with respect to the triangle CDA will pass through H_3 and H_1; and the pedal circle $C_1C_2C_3$ of C with respect to DAB will pass through H_1 and H_2.

Thus we see that H_1, H_2, H_3 constitute one set of the points of intersections of the three pedal circles taken two by two. We can *now* shew that the other points of intersections of these pedal circles taken in pairs are all one and the same point w; and that w is the common double point of the similar triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$.

6. Let us take w to be the double point of the two triangles $A_2B_2C_2$ and $A_3B_3C_3$. Then wA_2 and wA_3 are corresponding lines of these triangles. They therefore divide the angles $H_2A_2H_3$ and $H_2A_3H_3$ similarly. Therefore their intersection w lies on the pedal circle of A with respect to the triangle BCD; and therefore $w\,A_1$ divides the angle $B_1A_1C_1$ similarly.

In like manner $w\,B_2$ and $w\,B_3$ being corresponding lines divide the angles B_2 and B_3 similarly and therefore w lies on the circle $H_3H_1B_1B_2B_3$;

that is w lies on the pedal circle of B with respect to the triangle CDA; and therefore $w\,B_1$ divides the angle B_1 similarly.

In like manner, again, w lies on the pedal circle $C_1C_2C_3\,H_1H_2$ of C with respect to the triangle DAB; and $w\,C_1$ divides the angle C similarly.

We have thus shewn that the pedal circles all cointersect in w and that w holds, in respect of the triangle $A_1B_1C_1$ the same position that it holds in respect of the triangles $A_2B_2C_2$ and $A_3B_3C_3$; that is, *w is the common double point of the triangles* $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$.

7. From the fact that the similar triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ have a common double point w, it follows that the triangles $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ which are formed by taking corresponding points of these similar triangles are themselves directly similar and have w for their common double point. And we proved that the circumcircles of these triangles all pass through w. These latter triangles are, as we already observed, the pedal triangles of A with respect to BCD, of B with respect to CDA, and of C with respect to DAB.

8. Taking now all the four pedal triangles of A, B, C, D, *viz*, of each taken with respect to the triangle formed by the other three, it follows that every three of them have a common double point and therefore all the four have a common double point w. And the circumcircles of all these pedal triangles pass through w. Hence w is the centre of the rectangular hyperbola ABCD.

9. We next proceed to shew that w lies on the nine-points circle of ABC; whence, it would follow that it lies also on the nine-points circles of each of the triangles BCD, CDA, DAB. To prove this, we revert to a consideration of the triangle $A_1B_1C_1$, a study of which, as the line through A varies, is interesting. For our present purpose, we just require one property of the triangle.

10. **Theorem.** *As the line through A varies, the locus of O_1, the circumcentre of $A_1\,B_1\,C_1$ is the nine-points circle of ABC.*

For, as the line through A varies, A_1 remains fixed and B_1 moves on circle AA_1B. the The triangle $A_1B_1C_1$ which may be considered to be described on A_1B_1 is of a known species. Hence any assigned point connected with it moves on a circle. In particular, the circumcentre of $A_1B_1C_1$ moves on a circle.

Now when the line through A coincides with AA_1, the triangle $A_1\,B_1\,C_1$ is evanescent and its circumcentre therefore coincides with A_1, a point on the nine-points circle of ABC; when the line through A

takes the position AB, the variable triangle $A_1 B_1 C_1$ assumes the position $A_1 B C_3$ and the circumcentre in this case is the middle point of B H (where H is the orthocentre of ABC) and hence in this case also the circumcentre of $A_1 B_1 C_1$ lies on the nine-points circle of ABC. Lastly, when the line through A coincides with AC, the variable triangle $A_1 B_1 C_1$ becomes $A_1 B_2 C$ whose circumcentre is similarly the middle point of C H and therefore lies on the nine points circle of A B C. *Hence the locus of the circumcentre of the triangle* $A_1B_1C_1$ *is the nine-points circle of* A B C.

Let O_2, O_3 be the circumcentres of $A_2B_2C_2$ and $A_3B_3C_3$. It follows that these also lie on the nine-points circle of A B C.

Theorem. *Any set of corresponding circum-diameters of* $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ *are concurrent, the point of concurrence lying always on the nine-points circle of* A B C.

For, the angle $B A_2 A_1 = B A A_1 = \frac{1}{2}\pi - B = C_2A_2O_2$. Therefore $A_1A_2O_2$ are collinear. Similarly $A_1A_3O_3$ are collinear. Hence O_1A_1, O_2A_2, O_3A_3 concur at A_1.

Similarly O_1B_1, O_2B_2, O_3B_3 concur at B_2 and O_1C_1 O_2C_2, O_3C_3 concur at C_3.

Now l_1, l_2, l_3 being any three corresponding diameters, the angles which A_1O_1, A_2O_2, A_3O_3 make with l_1, l_2, l_3 are equal. Hence l_1, l_2, l_3 concur at a point ϕ on the nine-points circle of A B C.

But w having been shewn to be the common double point of the triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, the lines wO_1 wO_2, wO_3 are corresponding diameters. Therefore their intersection w lies on the nine-points circle of A B C.

In the same manner, w lies on the ninepoints circles of each of the other triangles BCD, CDA, DAB. Thus, all the nine-points circles cointersect in w.

12. Finally let m_1, m_2, m_3, denote *any* three lines drawn arbitrarily through A, B, C forming a triangle $D_1D_2D_3$; constructing the triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ as before, let w_1 w_2 w_3 denote the double points of these three similar triangles taken two by two. Then it follows that they lie on the nine-points circle of ABC, being the points corresponding to D_1 D_2 D_3, in the same manner as w corresponding to D in the foregoing investigation. *Thus whatever the lines* m_1, m_2, m_3 *may be, the circle of similitude of these triangles is always the nine-points circle of ABC.*

It can further be shewn that the *invariable points* of these triangles are their *circumcentres* O_1, O_2, O_3.

The Theory of Linear Systems of Points.

R. VYTHYNATHASWAMY.

[The present theory is not exactly new. In its root principles it will be found identical with the Invariant theory and the theory of Polynomial forms. But the view-points are different. While in the theory of invariants and of binary forms, it is the forms themselves and the relations between forms which are the subjects of study, the centre of interest, here, is the system of forms (or what is the same thing, of groups of points) all possessing a common property. That this displacement of the point of view does bring about a difference of methods and conceptions will be seen in the course of the paper.

With reference to the harmonic property, the chapter on 'Apolarity' in Grace and Young's *Algebra of Invariants* might be consulted with advantage. The difference in nomenclature is explained by the fact that this paper was written before I had any knowledge of the the Apolar Theory.

The principle of the geometrical representation is the same as in the representation of point-pairs in a line by points in a plane explained in my Note on 'Involution and 1, 1 Correspondence'.]

1. *Definition*: The groups of points in a line whose coordinates satisfy k lineo-linear equations of the type

$$F \equiv a_0 x_1 x_2 \ldots x_n + a_1 \Sigma(x_1 x_2 \ldots x_{n-1}) + a_2 \Sigma(x_1 x_2 \ldots x_{n-2}) \ldots = 0$$

will be said to form a linear system of order n and class $n-k+1$.

A system of order n and class n (*i.e.* a system defined by a single equation) will be called a *complete* system. Linear systems of order n and class other than n, will be termed *incomplete*. The involution of point-pairs, for example, is a complete linear system of order 2.

We may give another definition of a linear system which is descriptive in character and which will be found more useful in practice. If we have a system of groups of n points which is such that there is an absolute symmetry among the points composing any group, and that if any $r-1$ points of a group are given, the rest of the group is *in general* uniquely determined, then the system is a linear system of order n and class r. In particular, if, when all but one of the points of any group are known, the group is determined, the system is a comple system. Thus groups of points on a nodal cubic which are its intersections with a line, a conic and a cubic form complete systems of orders 3, 6 and 9 respectively.

2. *Geometrical Interpretation.*

Represent every group of n points by a point in space of n dimensions. If the group be $(x_1\ x_2 \ldots x_n)$, the co-efficients of the equation whose roots are $x_1, x_2 \ldots x_n$ may be taken to be the homogenous co-ordinates of the representative point. It may be noted that the representative point depends only on the group and not on the order of points group.

In this representation a complete system would correspond to a *plane*. Here and in what follows a plane in n dimensions means a flat space of $n-1$ dimensions. An incomplete system of class r would be represented by a flat space of $r-1$ dimensions. An immediate inference from this geometrical representation is that a unique system of class r can be found to contain r given n—ads. In particular, a complete system of order n is determined by n groups.

3. There are n n—ple points belonging to the complete system

$$F \equiv \sum a_r\ \Sigma x_1\ x_2 \ldots x_{n-r} = 0$$

of order n, these points being evidently the roots of the equation

$$\phi \equiv a_0\, x^n + n\ a_1\, x^{n-1} \ldots\ldots\ldots + a_n = 0.$$

These points will be termed the foci of the system. It is obvious that an incomplete system will not in general possess any foci.

A complete system is determined by its foci.

(1) If all the foci of a complete system coalesce at K, the system consists of groups composed of K and any other $n-1$ points.

For, if ϕ is an exact nth power $a_0\,(k-k)^n$, then

$$F \equiv a_0\,(x_1-k)\,(x_2-k)\ldots\ldots(x_n-k).$$

(2) If the complete system whose foci are $x_1\ x_2 \ldots x_n$ contains the group $y_1\ y_2 \ldots y_n$, then the complete system whose foci are $y_1\ y_2 \ldots y_n$, contains the group $x_1\ x_2 \ldots x_n$.

Let $x_1\ x_2 \ldots x_n$, $y_1\ y_2 \ldots y_n$ be respectively roots of the equations

$$a_0\, x^n + {}_nC_1\ a_1\, x^{n-1} + {}_nC_2\ a_2\, x^{n-2} \ldots + a_n = 0, \quad \ldots \quad \ldots \quad (1)$$

$$b_0\, y^n + {}_nC_1\ b_1\, y^{n-1} \ldots\ldots\ldots\ldots + b_n = 0. \quad \ldots \quad \ldots \quad (2)$$

The system whose foci are the roots of (1) contains the roots of (2). Hence

$$(-)^n (a_0\, b_n - {}_nC_1\ a_1\, b_{n-1} + {}_nC_2\ a_2 b_{n-2} \ldots) = 0.$$

The symmetry of the a's and b's in this relation proves the theorem.

The point-groups $x_1 x_2 \ldots x_n$, $y_1 y_2 \ldots y_n$ of this theorem may be said to be *harmonic*. Thus a complete system would consist of all groups which are harmonic to the focal group.

(3) A complete system of odd order contains its focal group.

Proceeding as above, we find the condition that the complete system whose foci are $x_1 x_2 \ldots x_n$, should contain the group $x_2 x_2 \ldots x_n$ to be

$$(-)^n (a_0 a_n - {}^nC_1 a_1 a_{n-1} + \ldots\ldots) = 0$$

This equation is *identically* true if n is odd. Hence the theorem.

This theorem is tantamount to saying that every odd group is self-harmonic.

(4) The harmonic relation is invariant for linear transformations.

For, a complete system being defined by a lineo-linear equation, is transformed into another such system and the focal group is transformed into the focal group.

(5) If r points of a group of the nth order coalesce at K, the group is harmonic to a group consisting of $n-r+1$ points at K and *any other* $r-1$ points.

Make a homographic transformation reducing K to zero. If now the points of the two groups are roots of the equations

$$a_0 x^n + {}^nC_1 a_1 x^{n-1} \ldots\ldots + {}^nC_{n-r} a_{n-r} x^r = 0,$$

$$b_0 x^n + {}^nC_1 a_1 x^{n-1} \ldots\ldots + {}^nC_{r-1} a_{r-1} x^{n-r+1} = 0,$$

using the condition of (2), the two groups are seen to be harmonic for any values of a's and b's.

(6) An even group is self-harmonic if more than half of its points coalesce.

This follows directly from (5).

4. *The Indeterminate Systems and the Extension of the Harmonic Relation.*

Any two groups of points forming together a group of a given complete system may be said to be conjugate. All groups conjugate to a given group of r points will form a complete system of order $n-r$, which may be termed the polar system of the given group. The polar system of the same group in this latter system is its second polar system and so on.

It is obvious on inspection that the first, second.....systems of the point at infinity *w. r. t.* the complete system

$$F \equiv a_0 x_1 x_2 \ldots x_n + a_1 \Sigma x_1 x_2 \ldots x_{n-1} + \ldots\ldots + a_n = 0,$$

are given by $D(F)=0,\ D^2(F)=0,\ldots\ldots D^r(F)=0,$

where D denotes differentiation *w. r. t.* one of the x's.

(1) The foci of the systems $D^k(F)$ are the successive harmonic centres of the foci of F. (*vide*: Mr. Naraniengar's *Note on the Harmonic Centres*, J.I.M.S., Feb. 1913.)

For let $\phi = a_0 x^n + {}^nC_1 a_1 x^{n-1} + \ldots\ldots\ldots + a_n.$

It is evident on inspection that the foci of D(F) are the roots of

$$\frac{d\phi}{dx}=0, \text{ or } \Sigma \frac{1}{x-\rho_r}=0,$$

where $\rho_1\ \rho_2$.....are the roots of $\phi=0$.

Thus the foci of D(F) are the harmonic centres of the foci of F. Repeating the proof, the foci of $D^2(F)$ are the harmonic centres of the foci of D(F) and so on.

(2) If F_1, F_2.....represent the successive polar systems of the origin, then the polar system of the group $t_1 t_2 \ldots t_r$ is given by

$$F_r + D(F_{r-1})\Sigma(t_1) + D^2F_{r-2}\Sigma(t_1t_2) \ldots\ldots + D^r(F).\ t_1t_2\ldots t_r = 0.$$

For $$F \equiv F_1 + x_1 D(F).$$
$$\equiv F_2 + x_2 DF_1 + x_1(DF_1 + x_2 D^2F)$$
$$\equiv F_2 + DF_1(x_1 + x_2) + x_1 x_2 D^2F;$$

and so on.

(3) The polar system of all r-ads of points *w. r. t.* a complete system of order n, have in common a system I_r of order $n-r$ and class $n-2r$, this system evidently consisting of the point-groups common to the complete systems

$$F_r,\ D(F_{r-1}),\ D^2F_{r-2}\ldots\ldots D^r(F).$$

The conjugate group of any group of the system I_r is indeterminate and may be supposed to be any r points whatever. For this reason the systems I_r will be termed the *indeterminate systems* of F.

The system I_r will reduce to a single group if $n-2r=1$. Thus the number of the systems I_r is the greatest integer in $\frac{1}{2}(n-1)$ and if n is odd there will be a single group of $\frac{1}{2}(n+1)$ points whose conjugate group is indeterminate. Since the groups of a complete system are harmonic to the focal group, it follows that a group of the system I_r together with *any* r points is harmonic to the focal group of F.

We might agree to express this by saying that the groups of the system I_r are harmonic to the focal group of F. This is the natural *extension* of the harmonic relation to groups of unequal order.

(5) *Indeterminate Groups of Incomplete Systems.*

An incomplete system of order n and class δ—or, as it will be written, of type $[n, \delta]$—may be regarded as the intersection of $n-\delta+1$ complete systems of order n. An indeterminate group of order $n-r$ of $[n, \delta]$ is therefore a common group of the $n-\delta+1$ systems I_r of the complete systems. Since I_r is of class $n-2r$, it is evident that the $n-\delta+1$ systems I_r have in common a system I_r' of order $n-r$ and class $\{n-r-(n-\delta+1)(r+1)+1\}$ or $[\delta-r(n-\delta+2)]$ The number of the systems I_r' is therefore the greatest integer in $\frac{\delta-1}{n-\delta+2}$. If $\frac{\delta-1}{n-\delta+2}$ is an exact integer μ, we will have a group of $n-\mu$ points whose conjugate is absolutely interminate.

The following is a table showing the indeterminate groups of systems of orders up to 10. A single indeterminate group of order, say 4, is written $4^{(1)}$, while the existence of a system of indeterminate groups of, say order 4 and class 2, is denoted by $4^{(2)}$.

Incomplete systems of order less than 5 have no indeterminate groups.

Table of Indeterminate Systems.

δ	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
4	$4^{(1)}$					
5	$4^{(3)}$, $3^{(1)}$	$5^{(2)}$	$6^{(1)}$			
6		$5^{(4)}$, $4^{(2)}$	$6^{(2)}$	$7^{(2)}$	$8^{(1)}$	
7			$6^{(5)}$, $5^{(3)}$ $4^{(1)}$	$6^{(1)}$, $7^{(4)}$	$8^{(3)}$	$9^{(2)}$
8				$7^{(6)}$, $6^{(4)}$ $5^{(2)}$	$7^{(2)}$, $8^{(5)}$	$9^{(4)}$
9					$8^{(7)}$, $7^{(5)}$ $6^{(3)}$, $5^{(1)}$	$8^{(3)}$, $9^{(6)}$

6. We proceed to the discussion of a very important class of incomplete systems called *focal systems.*

(1) *A system of class r cannot have more than r foci.* If possible, let K be a system of class r and order n ($> r$) containing more than r foci. We can find a unique complete system containing K and any other $n-r$ given n—ads (Compare the geometric interpretation of § 2) not belonging to K. In particular, we can find a complete system containing K and having $n-r$ points for foci. If K has more than r foci, this complete system will have more than n foci, which is impossible.

Def. A system of class r containing r foci is a *focal system.*

A system of class r is completely determined by r groups and therefore by its r foci.

Note that every complete system is a focal system.

(2) *The focal group of a focal system is harmonic to every group of the system.*

For, we can find a complete system containing K and having any other, $n-r$ points as foci. Therefore every group of K is harmonic to the group formed by any $(n-r)$ points and the focal group of K. Thus the focal group of K is an indeterminate group of every complete system whose focal group is a group of K. From our extended definition of the harmonic relation, it follows that the focal group is harmonic to every group of the eystem.

Cor. A unique focal system of class r can be found to contain a given group of $2r-1$ points. For, by § 3 (3) there is a uniqne indeterminate group of r points belonging to the complete system whose foci are the $2r-1$ points.

It will be evident that either no focal system or an infinite number of focal systems of class r can be found to contain a given group of n points according as $n >$ or $< 2r-1$.

7. *Focal systems of class* 2.

Theorem. *The focal system of class 2 and order n whose foci are $f_1 f_2$ consists of all cyclic groups associated with the homographic correspondence of period n whose fixed points are $f_1 f_2$.*

Since a focal system is completely determined by its foci, it suffices to prove the converse, *viz.* that the cyclic groups of such a homographic correspondence do form a system of class 2 with foci at f_1, f_2.

Let f_1, f_2 be $0, \infty$. Then the homographic correspondence of period n with these as fixed points will reduce to $x' = \omega x$ where $\omega^n = 1$. Thus the cyclic groups are roots of the equation $x^n = \lambda$ where λ is a parameter. Hence such groups form a linear system of class 2. Evidently here are values of λ (*viz* 0, ∞), for which all the roots of the above

equation become equal. Hence this linear system is a focal system and has foci at $0, \infty$. This proves the theorem.

It is not difficult to see that this theorem is not a special one but only a particular case of a more general theorem which connects every group of a focal system with the focal group by means of a periodic *rational*, or more probably, *algebraic transformation*. But it is impossible to get even a glimpse of this general theorem in our present state of knowledge of Rational and Algebraic Functions.

8. *Expression of a n-tic as a linear combination of nth powers* (*Vide*:—Grace and Young : *Algebra of Invariants*, Chapter on 'Apolarity'.)

Since a system of class r is represented by a *flat* space of $r-1$ dimensions, it follows that any group of the system (*i.e.* the polynomial whose roots are the points of the group) can be expressed as a linear combination of r fixed groups of the system. In particular, if a focal system with foci at $\alpha, \beta, \gamma \ldots$ can be found to contain a given group of order n, then the polynomial whose roots are the points of the group can be expressed as a linear combination $(x-\alpha)^n$, $(x-\beta)^n$, $(x-\gamma)^n$ etc. Now by § 6, (2) Cor., if n is of the form $2r-1$, there is a unique focal system of class r which contains the group, and if n is of the form $2r$, there is an one-fold infinity of focal systems of class $(r+1)$ containing the group.

Hence a $(2r-1)$—tic can be expressed uniquely as the sum of r terms of the $(2r-1)$ th degree and a $2r$—tic can be expressed in a singly infinite number of ways as the sum of $(r+1)$ powers of the $2r$th degree.

9. *Geometrical Interpretation.*

We represented every group of n points in a line by a point in n-dimensional space and we found that a complete system was represented by a plane. We may now enquire what points in our n—space will correspond to a special type of n—groups, *viz.*, those groups in which all the points coalesce. The locus of such points in n-space will, in the first place, be a curve and this curve should cut every plane in n points, since there are n coalesced groups or foci in every complete system.

Thus the locus of points corresponding to coalesced n—groups is a curve of nth degree in n dimensions.

This curve will be called the twisted n—tic and is the natural extension of the conic and the twisted cubic.

If $X_1 X_2 \ldots X_n$ are points on the twisted n—tic corresponding to the points $x_1 x_2 \ldots x_n$ on the line, the complete system whose foci are $x_1 x_2 \ldots x_n$ is represented by the plane $X_1 X_2 \ldots X_n$.

We have pointed out (§ 3 (5)) that if all the foci of a complete system coalesce, the system consists of groups composed of the point of coalition and any other $n-1$ points.

Hence the locus of all groups which contain a point x_1 is the osculating plane at X_1.

Thus the group $x_1 x_2 \ldots x_n$ is represented by the point of intersection of the osculating planes at $X_1 X_2 \ldots\ldots X_n$. (We see incidentally that the class of the twisted n—tic is n).

If P is any point in n—space, we may define the polar plane of P as the plane determined by the points of contact of the n osculating planes which can be drawn from P. Hence, from what has been stated above it follows that

(1) the complete system whose focal group corresponds to P, is represented by the polar plane of P ;

(2) the focal group of any complete system is represented by the polar point of the corresponding plane.

We have proved that a complete system of odd order contains its focal group and a complete system of even order contains its focal group, only if a certain relation of the second degree be satisfied. *Hence in spaces of odd dimensions, every plane contains its polar point (or the polar plane of any point P passes through P). In spaces of even dimensions the polar plane of P does not pass through P unless P lies on a certain quadric surface which will be called the special quadric associated with the twisted n—tic.* It is easily seen that the special quadric contains the twisted n—tic.

Again, it was proved that if more than half of the points of an even group coalesce the group is self-harmonic. *Hence in space of $2m$ dimensions osculating regions of $m-1$ dimensions are generating regions of the special quadric.*

Again, consider an even group $x_1 x_2 \ldots\ldots x_{2m}$. The polar plane of the corresponding point *w. r. t.* the *special quadric*, is a plane containing all self-harmonic groups harmonic to the given group. But $\{ x_r x_r \ldots\ldots$ $2m$ times $\}$ is a self-harmonic group which is harmonic to $x_1 x_2 \ldots x_{2m}$. Hence the polar plane *w. r. t.* the special quadric contains the points $X_1 X_2 \ldots\ldots X_{2m}$ which are points on the n—tic corresponding to $x_1 x_2 \ldots x_{2m}$

In other words, *in space of even dimensions, polarisation w. r. t the twisted n-tic is identical with polarisation w. r. t. the special quadric.*

It must be remembered in connection with this result that the special quadric exists only in even dimensions. In space of odd dimensions, we may guess that polarisation *w. r. t.* the twisted n—tic would be identical with polarisation *w. r. t.* something which is similar in nature to the linear complex in three dimensions.

Lastly, the symmetry of the harmonic relation shews that: *if the polar plane of A passes through B, the polar plane of B passes through A*— a result valid for all dimensions.

Representation of groups of order less than n in n dimensions.

The group $x_1x_2...x_r$ $(r<n)$ can be represented in n-space only as the region of intersection of the osculating planes at $X_1X_2...X_r$. This is, in effect, equivalent to regarding the group as identical with the linear system of groups composed of $x_2x_3...x_r$ and any other $n-r$ points. A region of this type we shall term '*a region in r planes*'.

The indeterminate system I_r of a complete system K will correspond to the aggregate of regions in $(n-r)$ planes contained by the plane corresponding to K.

In particulur, if n is odd, every plane in n dimensions contains a a unique region in $\frac{1}{2}(n+1)$ planes.

10. *Focal Systems.*

A focal system of the rth class is completely determined by its r foci. Such systems will therefore correspond to regions of $r-1$ dimensions which cut the twisted n—tic in r points.

If n is odd, then through any point P there can be drawn a unique region of $\frac{1}{2}(n-1)$ dimensions cutting the n—tic in $\frac{1}{2}(n+1)$ points [§ 6 (2) cor.]

As an example take a triad A B C and let p be the corresponding point in three-space. The triad A B C has a unique indeterminate pair which is the focal pair of the unique focal system containing ABC and which is also (§ 7.) the pair of fixed points of the homographic correspondence $\begin{vmatrix} A & B & C \\ B & C & A \end{vmatrix}$. This pair I have elsewhere termed the cyclic centres of ABC. (*Vide* : Note on Pascal Hexagon, J.I.M.S., June, 1918). This pair is represented in three-space as the points of intersection with the twisted cubic of the unique chord of the curve which can be drawn through P.

SHORT NOTES.

Addition Formulae for sn $(u_1+u_2+u_3)$ &c.

The symmetrical fractional expressions for the sn, cn, dn functions of $(u_1+u_2+u_3)$ given by me in the J. I. M. S., Vol. VIII p. 211, when reduced to their lowest terms, are identical with Professor Cayley's expressions quoted in Ex. 20, page 523 of Whittaker and Watson's *Modern Analysis* (Revised Edition). I give below the analysis required for this reduction, which supplies an elegant proof of Prof. Cayley's results and must interest those who, like myself, have not had the opportunity of seeing the original proof.

If s_r, c_r, d_r stand as usual for sn u_r, cn u_r, dn u_r; s_r', c_r' d_r' stand for

$$\frac{\partial}{\partial u_r}(s_r), \frac{\partial}{\partial u_r}(c_r), \frac{\partial}{\partial u_r}(d_r)$$

respectively and P stand for

$$\text{sn}\,(u_2-u_3)\ \text{sn}\,(u_3-u_1)\ \text{sn}\,(u_1-u_2),$$

we can prove by Liouville's theorem that

$$\text{sn}\,(u_1-u_4)\ \text{sn}\,(u_2-u_3)+\text{sn}\,(u_2-u_4)\ \text{sn}\,(u_3-u_1)+\text{sn}\,(u_3-u_4)\ \text{sn}\,(u_1-u_2)$$
$$\equiv -\text{P}.\ k^2\ \text{sn}\,(u_1-u_4)\ \text{sn}\,(u_2-u_4)\ \text{sn}\,(u_3-u_4). \quad \ldots \quad \text{(A)}$$

Now writing successively $u_4=0$, iK', K, $K+iK'$, we have

$$\Sigma s_1\ \text{sn}\,(u_2-u_3)=-\text{P}.\ k^2\ s_1\ s_2\ s_3, \quad \ldots \quad \ldots \quad \text{(A·1)}$$

$$\Sigma s_2\ s_3\ \text{sn}\,(u_2-u_3)=-\text{P} \quad \ldots \quad \ldots \quad \text{(A·2)}$$

$$\Sigma c_1\ d_2\ d_3\ \text{sn}\,(u_2-u_3)=-\text{P}\ k^2\ c_1\ c_2\ c_3 \quad \ldots \quad \ldots \quad \text{(A·3)}$$

$$\Sigma d_1\ c_2\ c_3\ \text{sn}\,(u_2-u_3)=-\text{P}.\ d_1\ d_2\ d_3 \quad \ldots \quad \ldots \quad \text{(A·4)}$$

the summation extending to the suffixes 1, 2, 3.

Again writing in (A.1), $u_1+u_2+u_3=3x$, $u_1-u_2=3y$, $u_1-u_3=3z$, and consequently writing $u_2-u_3=3(z-y)$, $u_1=x+y+z$, $u_2=x-2y+z$, $u_3=x+y-2z$, and differentiating with respect to x, we have

$$\Sigma c_1 d_1\ \text{sn}\,(u_2-u_3)=-\text{P}\ k^2\Sigma s_2 s_3 s_1'. \quad \ldots \quad \text{(A. 5.)}$$

The last result gives rise to three others when u_r+K, u_r+iK' or u_r+K+iK' is substituted for u_r. For instance, writing u_r+K+iK' for u_r, we have

$$\Sigma s_1 c_2^2 c_3^2\ \text{sn}\,(u_2-u_3)=-\text{P}\ \Sigma s_1 s_2' s_3' \quad \ldots \quad \ldots \quad \text{(A. 6)}$$

2. Again since $s_1^{n}=s_1^{n-2}\Sigma s_1^2-s_1^{n-2}\,[1+s_2^2 s_3^2-c_2^2 c_3^2]$, we have

$$\Sigma s_1^3\ \text{sn}\,(u_2-u_3)=\Sigma s_1 c_2^2 c_3^2\ \text{sn}\,(u_2-u_3)$$
$$-(1-\Sigma s_1^2)\ \Sigma s_1\ \text{sn}\,(u_2-u_3)-s_1 s_2 s_3\ \Sigma s_2 s_3\ \text{sn}\,(u_2-u_3)$$
$$=-\text{P}\ [\Sigma s_1 s_2' s_3'-s_1 s_2 s_3\ (1+k^2-k^2\Sigma s_1^2)]$$

from (A. 6), (A. 1) and (A. 2) $\quad \ldots \quad \ldots \quad \ldots \quad$ (B. 1)

Now writing $u_r + i\,K'$ for u_r, we have

$$\Sigma s_2^r s_3^s \text{ sn } (u_2-u_3) = -\mathrm{P}[s_1 s_2 s_3 \Sigma s_1 s_2' s_3' - s_1^2 s_2^2 s_3^2 . (1+k^2) + \Sigma s_2^2 s_3^2].$$

It may be noted here that $\Sigma s_1^n \text{ sn } (u_2-u_3)$ is divisible by P when n is odd, and that if $\mathrm{P}.f(n) \equiv \Sigma s_1^n \text{sn } (u_2-u_3)$, we can deduce the following reduction formula for $f(n)$, *for odd values of n greater than 3*:

$$f(n)+(1-\Sigma s_1^2) f(n-2)+s_1^2 s_2^2 s_3^2 f(n-4)$$
$$= -\{\Sigma s_1 s_2' s_3' + c_1^2 c_2^2 c_3^2 [f(1)+f(3)+\ldots+f(n-4)]\}. \quad \text{(B. 3)}$$

3. If X, Y, Z, stand for the different numerators and W for the common denominator of the expressions for sn, cn, dn of $(u_1+u_2+u_3)$ as given in J. I. M. S. (Vol. VIII, p. 211,), we have

$$\begin{aligned}
\mathrm{X} &= \Sigma s_1' s_2 s_3 (s_2^2 - s_3^2) = \Sigma s_1^3 (s_2 s_3' - s_3 s_2') \\
&= \Sigma [s_1^3 (1 - k^2 s_2^2 s_3^2) \text{ sn } (u_2-u_3)] \\
&= \Sigma s_1^3 \text{ sn } (u_2-u_3) - k^2 s_1^2 s_2^2 s_3^2 \Sigma s_1 \text{ sn } (u_2-u_3) \\
&= -\mathrm{P}[\Sigma s_1 s_2' s_3' - s_1 s_2 s_3 (1 + k^2 - k^2 \Sigma s_1^2 + k^4 s_1^2 s_2^2 s_3^2)] \\
&\qquad \text{from (A.1) and (B.1)} \qquad \ldots \quad \text{(C.1)} \\
\mathrm{Y} &= -\Sigma c_1' c_2 c_3 (c_2^2 - c_3^2) = -c_1^3 (c_2 c_3' - c_3 c_2') \\
&= -\Sigma [c_1^3 (d_2 d_3 - k^2 s_2 s_3 c_2 c_3) \text{ sn } (u_2-u_3)] \\
&= -\Sigma \left\{ \left[\left(\frac{d_1^2}{k^2} - \frac{k'^2}{k^2} \right) c_1 d_2 d_3 - k^2 c_1 c_2 c_3 s_2 s_3 (1 - s_1^2) \right] \right. \\
&\qquad \left. \text{sn } (u_2-u_3) \right\} \\
&= -\frac{1}{k^2} d_1 d_2 d_3 \Sigma c_1 d_1 \text{ sn } (u_2-u_3) + \frac{k'^2}{k^2} \Sigma c_1 d_2 d_3 \text{ sn}_4^r (u_2-u_3) \\
&\quad + k^2 c_1 c_2 c_3 \Sigma s_2 s_3 \text{ sn } (u_2-u_3) \\
&\quad - k^2 s_1 s_2 s_3 \, c_1 c_2 c_3 \Sigma s_1 \text{ sn } (u_2-u_3) \\
&= -\mathrm{P}[c_1 c_2 c_3 (1 - k^4 s_1^2 s_2^2 s_3^2) - d_1 d_2 d_3 \Sigma s_2 s_3 s_1' \\
&\qquad \text{from (A.5), (A.3), (A.2), (A.1)}] \ \ldots \quad \text{(C.2)} \\
\mathrm{Z} &= -\frac{1}{k^2} \Sigma d_2 d_3 d_1' (d_2^2 - d_3^2) \\
&= -\frac{1}{k^2} \Sigma [d_1^2 (c_2 c_3 - s_2 s_3 d_2 d_3) \text{ sn } (u_2-u_3)] \\
&= -\frac{1}{k^2} \Sigma \left[d_1 c_2 c_3 (k'^2 + k^2 c_1^2) \text{ sn } (u_2-u_3) \right] \\
&\quad + \frac{1}{k^2} \Sigma \left[d_1 d_2 d_3 \, s_2 s_3 (1 - k^2 s_1^2) \text{ sn } (u_2-u_3) \right] \\
&= -\frac{k'^2}{k^2} \Sigma d_1 c_2 c_3 \text{ sn } (u_2-u_3) - c_1 c_2 c_3 \Sigma c_1 d_1 \text{ sn } (u_2-u_3) \\
&\quad + \frac{1}{k^2} d_1 d_2 d_3 \Sigma s_2 s_3 \text{ sn } (u_2-u_3)
\end{aligned}$$

$$-d_1 d_2 d_3 s_1 s_2 s_3 \Sigma s_1 \operatorname{sn}(u_2-u_3)$$

$$= -P\left[d_1 d_2 d_3 (1-k^2 s_1^2 s_2^2 s_3^2) - k^2 c_1 c_2 c_3 \Sigma s_1 s_2 s_3\right]$$

from (A. 4), (A. 5), (A. 2), (A. 1) ... (C.3)

$$W = -\Sigma [s_1 s_1^3 (s_2^2 - s_3^2)]$$
$$= \Sigma [s_2 s_3 (1-k^2 s_2^2 s_3^2) \operatorname{sn}(u_2-u_3)]$$
$$= \Sigma s_2 s_3 \operatorname{sn}(u_2-u_3) - k^2 \Sigma s_2^3 s_3^3 \operatorname{sn}(u_2-u_3)$$
$$= -P[1-k^2 \Sigma s_2^2 s_3^2 + (k^2+k^4) s_1^2 s_2^2 s_3^2 - k^2 s_1 s_2 s_3 \Sigma s_1 s_2' s_3'']$$

from (A. 2) and (B. 2) ... (C.4).

From (C. 1), (C. 2), (C. 3), (C. 4), are derived Prof. Cayley's results.

F. H. V. Gulasekharam.

On Question 963.

In this question Prof. Wilkinson has given a very interesting property connected with quadrilaterals which are inscribed in one circle and circumscribed to another. The property is stated however in a manner which leads one to surmise that Prof. Wilkinson, at the moment, must have been in the fighting line of thought, leaving *expression* to take care of itself. Mr. F. H. V. Gulasekharam has, in the last number of the Journal, supplied the commentary required to understand the theorem, and also a proof *bringing out other noteworthy properties of the figure*. On reading it, I was tempted to see if the proof could not be simplified,—for a neat theorem is generally capable of a neat proof. In my attempt, I find that the property noticed by Prof. Wilkinson holds for polygons P, *of any number of sides*, which are inscribed in one given circle X and circumscribed to another Y. If Q denote the polygon whose successive vertices are the points of contact of the successive sides of P, then, the areas of P and Q remain in a constant ratio, as P varies under Poncelet's theorem.

In particular, the theorem is true in the case of *triangles*, a matter which I had not known previously, but which is readily verified.

2. To prove the general theorem: Let ABC...KL be the polygon P inscribed in the circle X; and let LA, AB, BC,...KL touch the circle Y in A′, B′, C′,...L′. Let I denote the centre of the circle Y. Then I A′, I B′,...are perpendiculars to the corresponding sides of P; and

therefore every adjoining two of them (as I A′, I B′) include an angle which is the supplement of the corresponding angle (A) of the polygon P. Now the polygon is the sum of n isosceles quadrilaterals like IA′ AB′. If $t_a, t_b, \ldots t_l$ denote the length of the tangents from A, B, ...L to circle Y and r denote its radius, then the area of the quadrilateral is evidently $r.\, t_a$. The area of the polygon P is therefore $r\ \Sigma t_a$. ... (1)

The area of the polygon Q is the sum of n isosceles triangles like I A′B′. Hence an expression for its area is $\frac{1}{2}\, r^2 \Sigma \sin$ A, where A denotes the angle at A of the polygon P. Now sin A = chord LB/2 R, where R is the radius of the circle X. Hence the area of the polygon Q is $\frac{r^2}{4\,\mathrm{R}}\ \Sigma$ (chord LB), where the summation extends to all chords joining two *alternate* vertices of P. ... (2)

By coaxal circle theory all such chords are tangents to a third circle, call it Y′, which is coaxal with X and Y ; and, it may be observed is the same for every polygon P. Denoting the tangents to this circle Y′ from the vertices A, B,...L by $t'_a, t'_b, \ldots t'_c$, we obviously have chord $\mathrm{LB} = t'_c + t'_b$.

Hence the area of

$$\mathrm{Q} = \frac{r^2}{4\,\mathrm{R}} \sum (t'_c + t'_b) = \frac{r^2}{2\,\mathrm{R}} \Sigma t'_a \qquad \ldots \qquad (3)$$

Again by coaxal circle theory, if t, t' be the tangents from any point on the circle X to the circles Y, Y′ the ratio t/t' is a constant, say k, which is dependent merely on the circles.

Hence $$\frac{t_a}{t'_a} = \frac{t_b}{t'_b} = \ldots = \frac{t_l}{t'_l} = k,$$

which gives $$\frac{\Sigma\, t_a}{\Sigma\, t'_a} = k, \qquad \ldots \qquad \ldots \qquad \ldots \qquad (4)$$

$$\therefore \qquad \frac{\text{area of P}}{\text{area of Q}} = \frac{r.\ \Sigma\, t'_a}{r^2/2\ \mathrm{R}.\Sigma\, t'_a} = \frac{2\mathrm{R}}{r}.\ k, \text{ which is constant.}$$

(Ramnad,)
13*th May* '19.

V. Ramaswami Aiyar.

Astronomical Notes.

1. *Change in the commencement of the Astronomical day* :—

It has been decided to substitute in nautical publications a day commencing at midnight, for the astronomical day which begins at noon. The necessary alterations will be made in the Nautical Almanac from the year 1926.

2. *Planet Notes* :—

Venus is a conspicuous object in the Western sky during these months. It will be at greatest elongation on July 5 and attain greatest brilliancy on August 7.

The positions of planets on July 1 will be as below :—

		R. A. h.	m.	Declination	Constellation
Mercury	...	8	6	22°.0 N	Cancer.
Venus	...	9	47	14.8 N	Leo
Mars	...	5	39	23.9 N	Gemini
Jupiter	...	7	39	21.8 N	Cancer
Saturn	...	9	52	14.5 N	Leo
Uranus	...	22	15	11.7 S	Pisces
Neptune	...	8	41	18.2 N	Cancer

It is interesting to note that all the planets except Uranus will occupy the same region of the sky between R.A. 5 h. and 10 h.

Venus will be in conjunction with Saturn on July 2.

Rev. T. E. R. Phillips has found that some remarkable changes have recently occurred on the surface of Jupiter. The South Tropical Disturbance and the Hollow in the South Equatorial Belt disappeared in April but the Red Spot remains quite distinct.

3. *Absolute Magnitudes of Stars.*

In the *Observatory* May 1919 there is a note giving a clear explanation of the term. The *absolute magnitude* is defined as the magnitude which the star would have if seen from a distance of 10 *parsecs*, that is, from a distance equivalent to a parallax of a tenth of a second in arc. The Sun's absolute magnitude on this scale is $5^m.1$, that is, when removed to a distance of 10 *parsecs* the sun will shine as a star of 5.1 magnitude.

The relation between absolute magnitude M and luminosity L of a star is given by $\log L = 0.4\ (5.1 - M)$,
where the luminosity of the sun is chosen as unit.

4. *Cepheid Variables* :—

These Variables are so named after δ Cephei the typical star of the class. They exhibit variation in brightness which is regular and continuous, the rise to maximum being much more rapid than the decline to minimum, and they show variation of radial velocity with the period of light changes. The problem of these Variables has attracted much attention recently. It is supposed that the Cepheids are gaseous bodies expanding and contracting alternately and that such pulsations are the cause of the variation in the star's brightness. The dynamics of this Pulsation Theory is investigated in two papers on the subject by Eddington (See: *Monthly Notices* of the Royal Astronomical Society, *November* 1918 *and January* 1919).

Nizamiah Observatory,
Hyderabad, (*Deccan*).

T. P. Bhaskara Sastri.

SOLUTIONS

Question 985

(M. K. Kewalramani) :—If A′,B′,C′ are points in the sides of the triangle A B C, such that B A′ : A′C = C B′ : B′A = AC′ : C′B = m : 1, find an expression for the radius of the circumcircle of the triangle D E F formed by the intersections of A A′, B B′, C C′.

Solutions by N. B. Pendse and N. B. Mitra.

The length of A A′ can be easily found, for

$$(m+1)\ AA'^2 + mA'C^2 + B\,A'^2 = m\,b^2 + c^2$$

i.e.
$$(m+1)\ A\,A'^2 = m\,b^2 + c^2 - m\,a^2.\ \frac{1}{(m+1)^2} + a^2\,\frac{m^2}{(m+1)^2}$$

$$= m\,b^2 + c^2 - \frac{m\,a^2}{m+1}.$$

$$\therefore \quad AA' = \frac{\sqrt{m\,(m+1)\,b^2 + (m+1)\,c^2 - ma^2}}{m+1}.$$

similarly
$$B\,B' = \frac{\sqrt{m\,(m+1)\,c^2 + (m+1)a^2 - mb^2}}{m+1},$$

and
$$C\,C' = \frac{\sqrt{m\,(m+1)\,a^2 + (m+1)b - m\,c^2}}{m+1}.$$

The trilinear equation of B B′ is

$$a\,\alpha - mc\,\gamma = 0;$$

similarly CC′, AA′, are
$$b\,\beta - m\,a\,\alpha = 0,$$
$$c\,\gamma - m\,b\,\beta = 0.$$

The points B′, C′, A′ are given by

$$\frac{a\alpha}{m} = \frac{c\gamma}{1} = \frac{b\beta}{0} = \frac{2\triangle}{m+1},$$

$$\frac{b\beta}{m} = \frac{a\alpha}{1} = \frac{c\gamma}{0} = \frac{2\triangle}{m+1},$$

$$\frac{c\gamma}{m} = \frac{b\beta}{1} = \frac{a\alpha}{0} = \frac{2\triangle}{m+1},$$

Also, the points D, E, F are given by

$$\frac{a\alpha}{m} = \frac{b\beta}{m^2} = \frac{c\gamma}{1} = \frac{2\triangle}{m^2+m+1},$$

$$\frac{a\alpha}{1} = \frac{b\beta}{m} = \frac{c\gamma}{m^2} = \frac{2\triangle}{m^2+m+1},$$

$$\frac{a\alpha}{m^2} = \frac{b\beta}{1} = \frac{c\gamma}{m} = \frac{2\triangle}{m^2+m+1},$$

$$\therefore \quad \text{area BDC} = \text{CEA} = \text{AFB} = \frac{\triangle m}{m^2+m+1}.$$

$$\therefore \quad \text{area DEF} = \frac{\triangle(1-m)^2}{m^2+m+1}.$$

And $$\frac{BD}{BB'}=\frac{m+1}{m^2+m+1},$$

$$\frac{BF}{BB'}=\frac{m(m+1)}{m^2+m+1}.$$

$\therefore$ $$\frac{FD}{BB'}=\frac{1-m^2}{m^2+m+1}.$$

Similarly $$\frac{EF}{AA'}=\frac{FD}{BB'}=\frac{DE}{CC'}=\frac{1-m^2}{m^2+m+1}.$$

The circum radius of DEF is equal to

$$\frac{EF.\ FD.\ DE}{4 \text{ area } DEF}=\frac{BB'.\ AA'.\ CC'}{4\triangle}\times\frac{\left(\frac{1-m^2}{m^2+m+1}\right)^3}{(1-m)^2}$$

$$=\frac{BB'.\ AA'.\ CC'}{4\triangle}\times\frac{(1-m)(1+m)^3}{(m^2+m+1)^2}$$

$$=\frac{1-m}{4\triangle(m^2+m+1)^2}\times\sqrt{\{(a^2+m^2c^2+2mac\cos B)}$$

$$(b^2+m^2a^2+2mab\cos C)\ (c^2+m^2b^2+2mbc\cos A)\}$$

Partial Solution by G. V. Krishnaswami and Sadanad.

Question 986.

(M. K. Kewalramani) :—At any point P of a curve equal arcs PQ, PR are measured in the same direction along the curve and along the parabola of the closest contact. Prove that ultimately

$$QR=\left\{3+\frac{1}{3}\left(\frac{d\rho}{ds}\right)^2-\rho\frac{d^2\rho}{ds^2}\right\}\frac{(ds)^4}{24\rho^3}.$$

Solution by Sadanand, N. B. Mitra, and S. R. Ranganathan.

Let the parabola be $y=(ax+by)^2$. If it is of the closest contact with

$$y=\frac{s^2}{2\rho}-\frac{s^3}{6\rho^2}\frac{d\rho}{ds}-\frac{s^4}{24\rho^3}\left(1-2\left(\frac{d\rho}{ds}\right)^2+\rho\frac{d^2\rho}{ds^2}\right)+\ldots$$

$$x=s-\frac{s^3}{6\rho^2}+\frac{s^4}{8\rho^3}\frac{d\rho}{ds}+\ldots\text{[See: Edwards' } Calculus\text{, Ex. 51, P. 291]};$$

then $$a^2=\frac{1}{2\rho},\quad ab=-\frac{1}{6\rho}\frac{d\rho}{ds}.$$

Substituting for x and y, we have

$$y=\frac{s^2}{2\rho}-\frac{1}{6\rho^2}\frac{d\rho}{ds}s^3+s^4\left(-\frac{1}{6\rho^3}+\frac{5}{72\rho^3}\left(\frac{d\rho}{ds}\right)^2\right)+\ldots$$

Hence ultimately,

$$QR=(y_1-y_2)$$

$$=\left(3+\frac{1}{3}\left(\frac{d\rho}{ds}\right)^2-\rho\frac{d^2\rho}{ds^2}\right)^2\left(\frac{ds^4}{24\rho^3}\right)$$

Question 987.

(V. TIRUVENKATACHARI):—show that

$$\int_0^{\frac{1}{2}\pi} \sin\theta\ (\log\cos\theta)\ d\theta = -\frac{1}{3}\left\{1+\frac{2}{5}+\frac{2.4}{5.7}+\frac{2.4.6}{5.7.9}+\ldots\right\} = -1.$$

Solution by K. J. Sanjana and several others.

More generally let us take $\int_0^{\frac{1}{2}\pi} (\sin\theta)^{2r+1} \log\cos\theta\ d\theta \equiv \mathrm{I}$.

Putting $\cos\theta = z$, we readily get $\mathrm{I} = \int_0^1 \log z\ .\ (1-z^2)^r dz$

$$= \int_0^1 \log z\ (1 - r_1 z^2 + r_2 z^4 \ldots + (-1)^r z^{2r})\ dz$$

$$(\alpha) \qquad = -\left\{\frac{r_0}{1^2} - \frac{r_1}{3^2} + \frac{r_2}{5^2} \ldots + (-1)^r \frac{1}{(2r+1)^2}\right\}.$$

Again, $$1 = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} (\sin\theta)^{2r+1} .\ \log\ (1-\sin^2\theta)\ d\theta$$

$$= -\tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \{\sin\theta)^{2r+3} + \tfrac{1}{2}\ (\sin\theta)^{2r+5} + \tfrac{1}{3}\ (\sin\theta)^{2r+7} \ldots\}\ d\theta$$

$$= -\tfrac{1}{2}\left\{\frac{2\cdot4\ldots(2r+2)}{3\cdot5\ldots(2r+3)} + \tfrac{1}{2}\cdot\frac{2\cdot4\ldots(2r+4)}{3\cdot5\ldots(2r+5)} + \tfrac{1}{3}\cdot\frac{2\cdot4\ldots(2r+6)}{3\cdot5\ldots(2r+7)} + \ldots\right\}$$

$$(\beta) \qquad = -\tfrac{1}{2}\cdot\frac{2\cdot4\ldots(2r+2)}{3\cdot5\ldots(2r+3)}\left\{1 + \tfrac{1}{2}\frac{2r+4}{2r+5} + \tfrac{1}{3}\frac{(2r+4)\ (2r+6)}{(2r+5)\ (2r+7)} + \ldots\right\}.$$

The result in the question is obtained by putting $r=0$ in these values. By equating (α) and (β) we get the algebraical identity—

$$\frac{r_0}{1^2} - \frac{r_1}{3^2} + \frac{r_2}{5^2} - \ldots + (-1)^r \frac{1}{(2r+1)^2}$$

$$= \frac{2\cdot4\ldots(2r)\ (r+1)}{3.5\ldots(2r+1)\ (2r+3}\left\{1 + \tfrac{1}{2}\frac{2r+4}{2r+5} + \tfrac{1}{3}\frac{(2r+4)\ (2r+6)}{(2r+5)\ (2r+7)} + \ldots ad\ inf.\right\}.$$

Question 988.

(S. MALHARI RAO)—Complete the following magic square, by placing 8 other prime numbers in the vacant cells :—

7	43	71	113
73	97	23	41

Solution by Prof. N.B. Mitra, and several others.

We shall suppose that only positive primes are meant.

If we fill up the 1st two cells of the 3rd row by x & y, it is easy to see that the remaining cells should be filled up as follows :—

7	43	71	113
$154-x$	$94-y$	$48-y$	$x+2y-62$
x	y	$92+y$	$142-x-2y$
73	97	23	41

Hence $y<48$ and y, $94-y$ & $92+y$ are primes different from those given and from each other. The only value of y which satisfies these conditions is 11.

Hence the square now becomes

7	43	71	113
$154-x$	83	37	$x-40$
x	11	103	$120-x$
73	97	23	41

Hence $40<x<120$ and x, $x-40$, $120-x$, $154-x$ are all primes different from the others and from each other.

The only values of x satisfying these conditions are 53, 101 & 107.

Hence we get the following squares :—

7	43	71	113
47	83	37	67
107	11	103	13
73	97	23	41

7	43	71	113
53	83	37	61
101	11	103	19
73	97	23	41

7	43	71	113
101	83	37	13
53	11	103	67
73	97	23	41

Question 989.

(S. Malhari Rao) :—Give the solutions in positive integers of the equation

$$(x+y+z)^2 = x\,y\,z.$$

Solutions by K. J. Sanjana and N. B. Mitra.

Solving as a quadratic in z we obtain

$$z = \tfrac{1}{2}xy-(x+y) \pm \tfrac{1}{2}\sqrt{\{x^2y^2-4xy\,(x+y)\}}.$$

If $x\,y=u$, $x+y=v$, we require $u^2-4\,u\,v \equiv \square = (u-2\,k\,v)^2$, say. This gives $u = v\,k^2/(k-1) = \alpha\,v$, suppose; $\therefore\ xy = \alpha x+\alpha y$, and $\dfrac{x}{\alpha} = \dfrac{y}{y-\alpha}$, so that $x = l\alpha$, $y = l\alpha/(l-1)$.

By giving k rational values and choosing l so as to make x and y integral, any number of solutions may be obtained. Thus $k=2$ along with $l=2$, 3, 5, gives the triads

8, 8, 16; 12, 6, 18; 20, 5, 25.

For this value of k, $u^2-4\,u\,v=0$. If $k=3$, $\alpha = \frac{9}{2}$; and $l=2$, $l=10$, give the pairs of triads 9, 9, 9 or 36; 45, 5, 100 or 25.

An easy practical method would be to put $y=a\,x$, $z=b\,x$; this gives $x=(1+a+b)^2/ab$. Values of a and b should now be so chosen as to make x integral: thus for

$a=1$. 1, 1, 2, 2, 3, 4, 5, 9, ..., we get

$b=1$, 2, 4, 3, 9, 8, 5, 9, 20,...respectively;
and the triads 9, 9, 9; 8, 8, 16; 9, 9, 36; 6, 12, 18;...5, 45, 100;...may be found.

As the equation for finding x in terms of y and z is a quadratic, if one value of x for given values of y and z is known, a second value will always exist. Hence if this is x_1, we have

$$(x_1+\overline{a+b}\,x)^2 = a\,b\,x_1;$$

also
$$(x+\overline{a+b}\,x)^2 = a\,b\,x$$

$\therefore$
$$(x_1-x)\ \{x_1+x\,(1+2a+2b)\} = (x_1-x)\,a\,b\,x^2,$$

and $x_1 = x\ \{a\,b\,x-(1+2a+2b)\}$.

Thus starting with 6, 12, 18, and taking $x=6$, $x=2$, $b=3$, we get $x_1=150$, so that 12, 18, 150 is another triad of solutions. Next taking $x=12$, $a=\frac{3}{2}$, $b=\dfrac{25}{2}$, we get $x_1=2352$, and 18, 150, 2352 is another solution.

Similarly from this we shall get 150, 2352, 19321; and the process may be indefinitely continued.

Partial solutions by Sadanand, Pendse and V. Thiruvenkata Chari.

Questions for Solution.

1042. (MARTYN, M. THOMAS M.A.) :—The envelope of the curves $y \sin^2\theta = a \sin\theta - x \sin\theta \cos\theta - \frac{1}{c}(x - a\cos\theta)^2$, where θ is variable, consists of two parts. Give a dynamical interpretation of the result.

1043. (V. RAMASWAMI AIYAR, M.A.) :—On the sides BC, CA, AB of a triangle ABC pairs of points XX′, YY′, ZZ′ are taken such that XX′ = BC, YY′ = CA and ZZ′ = AB ; and perpendiculars are erected at these points to the sides. If the perpendiculars at (Y, Z′), (Z, X′), (X, Y′) intersect at P, Q, R ; and those at (Y′, Z), (Z′, X), (X′, Y) intersect at P′, Q′, R′ ; prove that the six points P, Q, R, P′, Q′, R′ all lie on a circle.

1044. (V. RAMASWAMI AIYAR, M.A.) :—Given three straight lines as well as their orthopoles : determine the circum-circle, as well as the nine points circle, of the triangle of reference. Show also how the triangle of reference could be determined.

1045. (N. P. PANDYA):—A variable parabola touches a fixed ellipse externally at P and the sub-normals at P to both the curves are equal. If a common tangent to the parabola and the auxiliary circle of the ellipse cuts the normal at P in Q, find the locus of Q.

1046. (S. KRISHNASWAMI AIYANGAR):—At every point O of a plane curve a parabola and a rectangular hyperbola of closest contact with it are drawn. If the common normal meets the conics at P and Q, then

$$9.QO^2 + 4\rho_1^2.PO = 0,$$

where ρ_1 is the radius of curvature at the corresponding point of the evolute to the curve.

1047. (S. KRISHNASWAMI AIYANGAR):—If ρ, be the radius of curvature at any point of a cardioide and ρ_o, ρ_α, $\rho_{\alpha\beta}$ at the corresponding points of its evolute, α—evolute, and the β—evolute of the α—evolute, show that

$$\rho_{\alpha\beta} = \rho_\alpha \sin\beta + \rho_o \sin\alpha \cos\beta - \frac{1}{9}\rho \cos\alpha \cos\beta.$$

1048. (V. THIRUVENKATACHARI, B.A. HONS) :—Establish the following asymptotic relations

$$\int_0^1 \frac{\sin\left(\log\frac{1}{y^p}\right)}{(1-xy)\log\left(\frac{1}{y}\right)}\,dy = \tan^{-1}\left(\frac{p}{1}\right) + x\tan^{-1}\left(\frac{p}{2}\right) + x^2\tan^{-1}\frac{p}{3} + \ldots\ldots$$

$$\int_0 \frac{\cos\left(2\log\frac{1}{y}\right)}{(1-xy)\sqrt{\log\frac{1}{y}}}\,dy=\sqrt{\pi}\left\{\frac{e^{-1}}{\sqrt{1}}+x.\frac{e^{-\frac{1}{2}}}{\sqrt{2}}+\frac{x^2e^{-\frac{1}{3}}}{\sqrt{3}}+\frac{x^3e^{-\frac{1}{4}}}{\sqrt{4}}+\ldots\ldots\right\}$$

when $|x|<k\leq 1$.

1049. (S. Ramanujan, F. R. S.) :—Show that

(1)
$$\int_0^\infty \frac{\sin nx}{x+\frac{1}{x}+\frac{2}{x}+\frac{3}{x}+\ldots\ldots}\,dx = \frac{\sqrt{\left(\frac{1}{2}\pi\right)}}{n+\frac{1}{n}+\frac{2}{n}+\frac{3}{n}+\ldots}$$

(2)
$$\int_0^\infty \frac{\sin\frac{1}{2}\pi nx\,dx}{x+\frac{1^2}{x}+\frac{2^2}{x}+\frac{3^2}{x}+\ldots} = \frac{1}{n+\frac{1^2}{n}+\frac{2^2}{n}+\frac{3^2}{n}+\ldots}$$

1050. (K. S. Srinivasachari, M.A., L.T.) :—If

$$(ax^2+bx+c)^{-1}=p_0+p_1x+p_2x^2+\ldots\ldots$$

find in terms of a, b, c, x the value of

$$p_0^2+p_1^2x^2+p_2^2x^4+\ldots\ldots$$

1051. (A. C. L. Wilkinson) :—If ABCD is a cyclic quadrilateral, the points A, B, C, D occurring in this cyclic order, the sum of the reciprocals of the perpendiculars from A, C on their Simson lines with respect to the triangles BCD, ABD respectively is equal to the sum of the reciprocals of the perpendiculars from B, D on their Simson lines with respect to the triangles ACD, ABC respectively.

1052. (A. C. L. Wilkinson) :—If the circle of curvature at any point P of an ellipse be drawn and O be any point on the tangent at P, prove that the straight line joining the points of contact of the other two tangents from O to the ellipse and its circle of curvature, the other common tangent and the tangent at P are concurrent.

1053. (A. C. L. Wilkinson) :—Two circles are such that triangles can be inscribed in the one which are circumscribed to the other. If P, Q, R be the points of contact of such a triangle with the circle about which it is circumscribed, show that the nine-point-circle of P, Q, R is fixed and that the locus of the centres, K, of the inscribed or escribed circles of the triangle PQR is the curve $IK^2=2r.NK$, where N is the common nine point centre, I the centre and r the radius of the common circumscribed circle of the triangles PQR.

Periodicals received from 16th March to 15th May 1919 (with contents.)

(1) *Acta Mathematica*, Vol. 42, No. 1:—*Zur Theorie der linearen Gleichungen*,—E. Study. The classification of sets of points,—Eric. H. Neville. *Auszug aus einem Briefe des Herrn Landau an den Herausgeber*—Edmund Landau. *Bibliographie.*

(2) *The American Mathematical Monthly*, Vol. 25, No. 10. December 1918:—The content of a second course in Calculus—E. J. Moulton. A system of Algebraic and Transcendental Equations—G. N. Bauer and H. L. Slobin. Practical solution of Linear Equations—H. T. Burgess.

(3) *The American Mathematical Monthly*, Vol. 26, Nos. 1 & 2. January and February 1919: Leonardo of Pisa and his *Liber Quadratorum*,—R. S. McClenon. On the Form of the Power series for an Algebraic Function—E. J. Wilczynski. Concerning a method for finding a Particular Integral—A. B. Coble. Book Review: Who was the first Inventor of the Calculus—F. Cajori. Discussion: Concerning a new method for Tracing Cardioids,—W. F. Rigge. Collegiate Mathematics for war service: Statistics in relation to the War—Roxana H. Vivian. On certain constructions of Descriptive Geometry—G. Loria. Difference Quotients—T. P. Ballantine. A Theorem in the Geometry of the Triangle—J. W. Clawson. On the Elementary Problem of closure on an Equilateral Hyperbola—A. Emch. Remarks on a previous Article—N. Altshiller.

(4) *The Astrophysical Journal*, Vol. 49, No. 1, January 1919: On the Dark markings of the Sky, with a catalogue of 182 such objects—E. E. Barnard. Studies based on Colors and Magnitudes in Steller Clusters—Harlow Shapeley. The Infra-Red Arc Spectrum of Iron—H. M. Randall and E. F. Barker. Discussion of some evidence on the origin of Radiation in the Tube-Resistance Furnace—Arthur S. King. The Infra-Red Arc Spectra of Cobalt, Nickel, Manganese and Chromium—H. M. Randall and E. F. Barker.

(5) *Bulletin* of the *Calcutta Mathematical Society*, Vol. 9, No. 2 1917—18:—The late Sir Gooroo Dass Banerjee—Sudhansukumar Banerji. On the figures of Equilibrium of two Rotating Masses of Fluid for the Exponential Potential, Part I—Abanibhushan Datta, Fourier's Series and its Influence on some of the Developments of Mathematical Analysis—A. C. Bose. On the Numerical calculation on

the Roots of the Equations $P_n^m(u)=0$ and $\frac{d}{du} P_n^m(u)=0$, regarded of Equations in n—Bholanath Pal. On some New Therems in the Geometry of Masses—Sasindrachandra Dhar. On the Electric Resistance of a Conducting Spheroid with given Electrodes—Sitachandra Kar.

(6) *Mathematics Teacher*, Vol. 11, No. 2, December 1918 :—Why Students Fail in Mathematics—H. A. Merrill. A Solution of Equations by Standard Curves—R. C. Colwell. War Problems in Mathematics—W. E. Breckenridge. Arithmetical Errors made by High School Pupils-T. H. Minnick. Some Relations connecting the sums of the Coaxial Minors of a Circulant—W. H. Metzler. Canada's Challenge : In Flanders Fields—T. D. McCrae. America's Answer—R. W. Lillard.

(7) *Monthly Notices of the Royal Astronomical Society*, Vol. 79, Nos. 2 & 3, December 1918 & January 1919 :—The Evolution of Binary systems—J. H. Jeans. On a certain relation between the Elements a Planets and Satellites,—R. J. Pocock. Observations of the Magnitude of Nova Aquilae, 1918, made at the Radcliffe Observatory, Oxford; and a comparison with the Radcliffe Magnitudes of Nova Persei, 1901, A. A. Rambout. The earlier spectrum of Nova Aquilae, 1918,—A. L. Cortie. The dark-line spectrum of Nova Germinorum No. 2, J. Lunt. The Stellar Magnitude Scales of the Astrographic Catalogue. Thirteenth note. The Bordeaux Magnitudes ($+17°, +16^c, +°15^c$ and $+14°$)—H. H. Turner. Z. Andromedae (231848)—T. E. Espin. Some probably Variable Stars—T. E. Espin. *Sur les relations entre les coordonnees rectilignes normales d'une meme e'toile sur deux cliches differents.*—B. Bailland. On a shower of shooting stars from the Pleiades,—A. King. The New Star of Hipparchus, and the dates of Birth and Accession of Mithridates—J. K. Fortheringham. Obeservations of Nova Aquilae, 1918—J. G. Hagen. The Spectrum of Nova Aquilae, 1911, June 15—A. L. Cortie. The Pulsations of a Gaseous Star, and the Problem of the Cepheid Variables—A. S. Eddington. On the Variable Star B. D.$+39°3476$ = Piazzi XVIII, 153—A. Stanley Williams. The Short-period Variable R. R. Ceti, C. Martin and H. C. Plummer. Magnitude Observations of five Wolf-Rayet Stars—C. Martin and H. C. Plummer. Proper motions of the stars in Zone $24°-32°$ N. declination, in relation to Spectral Types—F. W. Dyson and W. G. Thackeray. Occultations of stars by the Moon and Phenomena of Jupiter's Satellites observed in year 1918—Royal Observatory, Greenwich. Mycrometrical measures of Double Stars (13th Series) and New Double Stars—T. E. Espin. Observations of a bright projection on the limb of Jupiter—F. Sargent.



MR. S. RAMANUJAN, B. A., F. R. S.

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PROGRESS REPORT.

In continuation of the announcement in the April number of our Journal, the Committee feel very great pleasure to inform the General Body that under Art. VII (*b*) of the Constitution, Mr. S. Ramanujan B.A., F.R.S. has been appointed *an Honorary Member* of our Society. A short life-sketch of Mr. Ramanujan is given elsewhere in this number and a portrait, prepared with as much care as possible from an old group-photo, adorns this number as its frontispiece.

2. The following gentlemen have been elected members of our Society :—

1. *Mr. Mohanlal V. Vasawada, L.C.E.*—Assistant Member; Board of Revenue for Irrigation, Gwaliar State, Morar (C.I.)
2. *Mr. Harihar Nath Wancha, M.A.*—Professor of Mathematics Rangoon College, Rangoon ;
3. *Mr. Nripendranath Maitra, M.A., B.L.* (Cal.),—Deputy, Magistrate and Deputy Collector, Sitamarhi (P.C.), (Dist.—Muzaffarpur, Behar) ;
4. *Mr. Abani Kanta Ray, B. Sc.*,—M. Sc. Student, Cal. Univ. College, Krishnagar (Bengal). (at concessional rate) ;
5. *Mr. R. Sriramulu*—Lecturer, Pachaiyappas College, Madras ;
6. *Mr. K. Narasimha Rao*—Assistant Lecturer, Madras Christian College, Madras ;
7. *Mr. A. Nageshwara Aiyar, B.E.*—District Board Engineer, Ramnad.

3. The Commttee feel it their very sad duty to record the premature death of His Highness the Maharaja Bhowsinghji of Bhownagar. The late Maharaja was a highly enlightened Ruling Prince and had cheerfully got himself enrolled as a Life-Member of our Society in its very early existence. By his untimely death our Society has sustained a heavy loss.

4. The following books have been presented to our Library—

1. *Analytical Geometry* of the Straight Line and the Circle-by John Milne, George Bell & Sons, 5s/-, 1919 ;
2. *Differential Calculus* for Colleges and Secondary Schools-by Charles Davison, G. Bell & sons, 6s/-, 1919 ;
3. *An Arithmetic* for Preparatory schools-by Trevor Dennis, G. Bell & Sons. 4s/6d, 1919 ;
4. *Bombay University Calendar*, Parts I & II for 1919-20 ;
5. *Madras University Calendar*, Parts I & II for 1919, and Exam. Papers for 1918.

POONA, 10th *Aug.* 1919.

D. D. KAPADIA,
Honorary Joint Secretary.

Mr. S. Ramanujan, B.A., F.R.S.

Mr. S. Ramanujam B.A. F.R.S., a Sri Vaishnava Brahmin was born of poor parents in December 1888 at Erode in the Madras Presidency. He was put to school when he was five years old and even while in the Lower Secondary Forms he gave indication of his special bent of mind for mathematics by questioning his teachers about zero and imaginary quantities. He matriculated from the Town High School, Kumbhakonam in December 1903 and read for about a year in the F. A. Class in the Government College, Kumbhakonam. Even at that stage he was working at infinite series and different representations of numbers.

Having failed in the F. A. Examination, a natural result of his absorption in mathematics and consequent neglect of other subjects, he spent nearly 5 years moving about from place to place andworking at mathematics in his own way. In 1910, he went to Madras with two good sized note books filled with his researches in mathematics and with the help of some friends obtained a clerk's post in the Harbour Trust Office. This enabled him to remain in Madras and work at mathematics with the help of books and periodicals which were made accessible to him there. In 1912, having one day seen the remark in Mr. G. H. Hardy's tract on 'Orders of Infinity' that the precise order of $\rho(x)$ had not yet been ascertained and having himself arrived at a result relating to it, he opened correspondence with Mr. Hardy sending him some of his results on continued fractions and theory of numbers. Mr. Hardy was struck with the grandeur of the results and wrote to him an appreciative letter asking for more of his results in other branches of mathematics. On receiving these results which were mainly in Definite Integrals and Elliptic Functions. Mr. Hardy discovered in Ramanujan a great Mathematician and asked him if he would go to Cambridge. Meanwhile the Madras University recognising his Mathmatical talents had granted him a Research Studentship. Mr. E. H. Neville of Cambridge who came to Madras to deliver a course of University Lectures and who had been asked by Mr. Hardy to bring Mr. Ramanujan, with him when he returned to Cambridge, got the University of Madras to grant Mr. Ramanujan a special Scholarship of the annualvalue of £. 250 for three years, and Mr. Ramannjan sailed for England in March 1914. He was very warmly received at Cambridge by Mr. Hardy and both he and Mr. Littlewood helped him in publishing his researches in the English Periodicals. An account of his work while in England has already appeared in the February 1917 number of this Journal. At Cambridge he was given the Research Degree and the frontispiece shows him in his accademical robes. In recognition of his abilities in Mathematics, he was elected a Fellow of the Royal Society in 1917, and in 1918 he was granted a Fellowship of Trinity. The Madras University has also been pleased to give him an annual grant of £. 250 for another period of 5 years without imposing any conditions regarding residence or work. Since May 1917, he has not been keeping good health. He returned to India in March last for the sake of his health and is now residing at Kodumudi a village on the banks of the Cauvery and very near the place of his birth.

On the Cartesian Oval.

By V. Ramaswami Aiyar.

Introduction.

In tackling a problem connected with Genocchi's theorem on the rectification of the cartesian oval, I found that a method of treating the curve which dealt with the three foci symmetrically, and furnished simple expressions for the leading magnitudes connected with the oval, was a desideratum. Then I groped my way to the discovery of the fundamental parametric equations here given. By means of them the treatment becomes simple and of the kind desired ; and, while the leading properties of the curve are deduced easily and in various ways, several new results occur, some of them quite of importance. I have not, so far, found the method here noticed anticipated.

In this article I shall be able to present only a part of the work that I wish to ; I hope to present the rest subsequently.

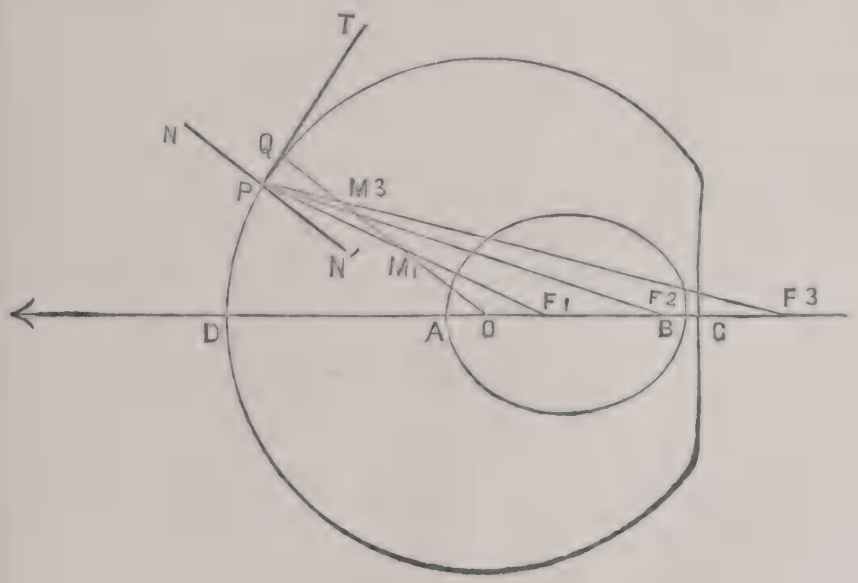
Section 1.

The Fundamental Parametric Equations.

2. A, B, C, being three given collinear points, the distances of any point P from them are connected by the well known relation

$$PA^2.BC+PB^2.CA+PC^2.AB+BC.CA.AB=0;$$

which will be referred to as the 'equation of distances.' By means of it, when any two of these distances are known, the third can be found.



3. Let now O, F_1, F_2, F_3 be four points in a line in order, we shall say, from left to right. We will call the line the axis. Let $OF_1=a$, $OF_2=b$, $OF_3=c$; so that $o<a<b<c$. Then a system of consistent values for the distances of a point P from F_1, F_2, F_3 are furnished by the equations

$$\left.\begin{aligned}\rho_1&=\sqrt{bc}+\frac{\lambda}{\sqrt{bc}}\\ \rho_2&=\sqrt{ca}+\frac{\lambda}{\sqrt{ca}}\\ \rho_3&=\sqrt{ab}+\frac{\lambda}{\sqrt{ab}}\end{aligned}\right\} \quad \ldots \quad \ldots \quad (1)$$

as may be verified at once by the equation of distances.

In these expressions there is a *parameter* λ. For every value of λ we shall have a point P, or rather a pair of points P, P′ situated symmetrically about the axis. They are the points of co-intersection of circles of centres F_1, F_2, F_3, and of radii ρ_1, ρ_2, ρ_3 corresponding to that value of λ. We shall speak of these circles as the 'circles' ρ_1, ρ_2, ρ_3. They are coaxal for every value of λ.

Varying λ, from $+\infty$ to $-\infty$, we shall have a *locus* of the points ρ, ρ'. We will call this locus 'the Cartesian Oval.' O will be called its 'triple focus'; and F_1, F_2, F_3 its 'foci' or 'single foci',

Eliminating λ between the equations (1) taken in pairs, we get the following 'bipolar' equations of the curve :

$$\left.\begin{aligned}\sqrt{b}\cdot\rho_1-\sqrt{a}\cdot\rho_2&=\sqrt{c}.\ (b-a)\\ \sqrt{c}\cdot\rho_1-\sqrt{a}\cdot\rho_3&=\sqrt{b}.\ (c-a)\\ \sqrt{c}\cdot\rho_2-\sqrt{b}\cdot\rho_3&=\sqrt{a}.\ (c-b)\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Also using all the equations in (1), we get the following 'tripolar' equation of the curve.

$$\sqrt{a}.\ (b-c).\ \rho_1+\sqrt{b}.\ (c-a).\ \rho_2+\sqrt{c}.\ (a-b).\ \rho_3=0 \quad \ldots \quad (3)$$

4. **Theorem**. *If the distances of a point P from F_1 and F_2, taken with suitable signs, satisfy the first equation in* (2), *the point P must be on the oval.*

For, let the distances so taken be δ_1, δ_2 : so that we have

$$\sqrt{b}\cdot\delta_1-\sqrt{a}\cdot\delta_2=\sqrt{c}.\ (b-a) \quad \ldots \quad \ldots \quad \ldots \quad (\alpha)$$

Putting

$$\sqrt{bc}+\frac{\lambda}{\sqrt{bc}}=\delta_1,$$

$$\sqrt{ca}+\frac{\lambda}{\sqrt{ca}}=\delta_2,$$

and solving for λ, we get the *same* value of λ from each equation. This is by virtue of the relation (α). This proves that P is the poin on the curve whose parameter is λ so determined.

It is clear that a similar statement holds with regard to the other equations in (2).

Hence, if the symbol (AB) mean 'the distance AB taken with a suitable sign,' we see that our oval is the locus of a point P moving so that any of the following equations is satisfied :

$$\left.\begin{aligned} \sqrt{b}.\ (PF_1)+\sqrt{a}.\ (PF_2)&=\sqrt{c}.\ F_1F_2\,;\\ \sqrt{c}.\ (PF_1)+\sqrt{a}.\ (PF_3)&=\sqrt{b}.\ F_1F_3\,;\\ \sqrt{c}.\ (PF_2)+\sqrt{b}.\ (PF_3)&=\sqrt{a}.\ F_2F_3\,;\end{aligned}\right\} \quad \ldots \quad \ldots \quad (4)$$

The Ordinary Form Identified.

5. Let now A, B, be two given points ; l, m, n any positive, unequal, multiples; and let a variable point P move so that

$$m\ (PA)+l\ (PB)=n.\ AB. \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Then according as n is the greatest, intermediate, or least of the quantities l, m, n, the equation becomes comparable with the first, second, or third equation in (4) ; and we conclude :

(i) that the locus of P is a Cartesian Oval having A and B for two of its foci ;

(ii) that the triple focus O is a point situated in AB produced such that $\frac{OA}{l^2}=\frac{OB}{m^2}$;

(iii) that it has for the third focus a point C situated in the line on the same side of O as A and B, such that

$$\frac{OA}{l^2}=\frac{OB}{m^2}=\frac{OC}{n^2}\,;$$

(iv) and that the point P also satisfies the equations

$$n\ (PA)+l\ (PC)=m.\ AC,$$
$$n\ (PB)+m\ (PC)=l.\ BC,$$

Having made this identification we shall leave the ordinary form and proceed to discuss the curve by means of the special equations.

Section II.

The Vertices of the Oval and its General Tracing.

6. By the vertices of the oval we shall mean the points where the curve cuts the axis. Let x be the abscissa and λ the parameter of a vertex. We shall measure the abscissa from O as origin, and positively

to the left. The foci then are the points with abscissae $-a$, $-b$, and $-c$; and $x+a$, $x+b$, $x+c$ are equal to the focal distances of the vertex numerically. So also are ρ_1, ρ_2, ρ_3 for the value of λ in question.

Hence x and λ must satisfy one equation out of each of the pairs:

$$x+a=\pm\,\rho_1\,;\ x+b=+\rho_2\,;\ x+c=\pm\rho_3.$$

We find the equations go together as follows:

(1) $x+a=\rho_1,\ x+b=\rho_2,\ x+c=\rho_3\,;$

(2) $x+a=\rho_1,\ x+b=-\rho_2,\ x+c=-\rho_3\,;$

(3) $x+a=-\rho_1,\ x+b=\rho_2,\ x+c=-\rho_3\,;$

(4) $x+a=-\rho_1,\ x+b=-\rho_2,\ x+b=\rho_3.$

We may also observe that the equations (2) are what (1) become on changing $\sqrt{a}$ into $-\sqrt{a}$; and (3) and (4) are similarly what (1) become on changing the signs of $\sqrt{b}$ and $\sqrt{c}$.

Denoting the values of x and λ given by these equations in order by

$$(x_4,\ \lambda_4)\ (x_1,\ \lambda_1)\ (x_2,\ \lambda_2)\ (x_3,\ \lambda_3)$$

we find that they are

$$\left.\begin{aligned}
x_4&=\sqrt{bc}+\sqrt{ca}+\sqrt{ab},\ \lambda_4=\sqrt{abc}\,(\sqrt{a}+\sqrt{b}+\sqrt{c}),\\
x_1&=\sqrt{bc}-\sqrt{ca}-\sqrt{ab},\ \lambda_1=\sqrt{abc}\,(\sqrt{a}-\sqrt{b}-\sqrt{c}).\\
x_2&=-\sqrt{bc}+\sqrt{ca}-\sqrt{ab},\ \lambda_2=\sqrt{abc}\,(-\sqrt{a}+\sqrt{b}-\sqrt{c})\\
x_3&=-\sqrt{bc}-\sqrt{ca}+\sqrt{ab},\ \lambda_3=\sqrt{abc}\,(-\sqrt{a}-\sqrt{b}+\sqrt{c}).
\end{aligned}\right\}\quad \ldots\ (6)$$

Now on the axis set off the points of abscissae x_4, x_1, x_2, x_3 as above given; and call them D, A, B, C. These are the vertices of the curve; and they are the points of parameters λ_4, λ_1, λ_2, λ_3.

It should be observed that the vertices and the foci lie along the axis in the order

$$\mathrm{D,\ A,\ F_1,\ F_2,\ B,\ C,\ F_3}.$$

7. For tracing the curve it is necessary to note the relative order of the values of λ which correspond to the vertices *and* of those which make ρ_1, ρ_2, ρ_3 respectively vanish, namely,

$$\lambda'=-bc,\ \lambda''=-ca,\ \lambda'''=-ab.$$

It is readily seen that they are in descending order of magnitude as follows:

$$\lambda_4,\ \lambda_3,\ \lambda''',\ \lambda'',\ \lambda_2,\ \lambda_1,\ \lambda'.$$

Also we note, from the fundamental parametric equations, that the increments (or decrements) in the values of ρ_1, ρ_2, ρ_3 corresponding to any given change in the value of λ are of the forms

$$\sqrt{a}.\delta,\ \sqrt{b}.\delta,\ \sqrt{c}.\delta.$$

(1) As λ decreases from ∞ to λ_4, the circles ρ_1, ρ_2, ρ_3 decrease in radius and come into contact at the vertex D. The values of ρ_1, ρ_2, ρ_3 corresponding to $\lambda=\lambda_4$ are F_1D, F_2D, F_3D. Hence prior to λ reaching the value λ_4 the radii of the circles ρ_1, ρ_2, ρ_3, have been of the forms $F_1D+\sqrt{a}.\delta$, $F_2D+\sqrt{b}.\delta$, $F_3D+\sqrt{c}.\delta$; and, plotting these for a typical δ, we see that the circle ρ_1 is entirely within the circle ρ_2, and the latter within the circle ρ_3. Our tracing points ρ,ρ' have thus been imaginary until the value λ_4 was reached, when, they both appear at the position D.

(2) As λ decreases from λ_4 to λ_3, the radii of the circles ρ_1, ρ_2, ρ_3, are to start with, of the forms

$$F_1D-\sqrt{a}.\delta,\ F_2D-\sqrt{b}.\delta,\ F_3D-\sqrt{c}.\delta.$$

Plotting these for a typical δ, it is at once seen that the circles ρ_1, ρ_2, ρ_3 interesect one another. Thus as λ decreases from λ_4, the coaxal circles ρ_1, ρ_2, ρ_3, form an intersecting system; and this must continue until the next contact occurs, that is, until the value λ_3 is reached. Then the circles touch one another at the vertex C, and the values of ρ_1, ρ_2, ρ_3, there are F_1C, F_2C, F_3C. Thus, as λ decreases from λ_4 to λ_3 our tracing points P,P′ are real and trace out an oval extending from D to C on both sides of the axis.

(3) As λ decreases from λ_3 to λ_2, the radii of the circles are, to start with, of the forms $F_1C-\sqrt{a}.\delta$. $F_2C-\sqrt{b}.\delta$, $F_3C-\sqrt{c}.\delta$. Plotting these for a small δ, it is seen that the circle ρ_2 becomes entirely contained in the circle ρ_1, while the circle ρ_3 goes entirely outside the circle ρ_1. This kind of *non-intersection* goes on until the next contact is reached, that is until $\lambda=\lambda_2$. In the interval, the radius ρ_3 diminished to zero when λ passed through the value $''\lambda'=ab$, and then became negative numerically increasing. The like happened with the radius ρ_2 as λ passed through the value $\lambda''=-ca$. When λ becomes equal to λ_2, the circles P_1, P_2, P_3 come into contact at B, the value of the radii then reached being F_1B, $-F_2B$, and $-F_3B$ respectively. The tracing points P, P′ have been imaginary in the interval.

(4) As λ decreases from λ_2 to λ_1, the radii of the circles are, to start with, of the forms

$$F_1 B - \sqrt{a}.\delta, \quad -F_2 B - \sqrt{b}.\delta, \quad -F_3 B - \sqrt{c}.\delta.$$

Plotting these typically, it is at once seen that the circles ρ_1, ρ_2, ρ_3 start in a state of real intersection and the tracing points P, P′ are real. This goes on until the next contact is reached, that is till $\lambda =$ when the radii of the circles become $F_1A, -F_2A_2$ and $-F_3A$. Thus as λ decrease from λ_2 to λ_1 our tracing points are real and trace out an oval extending from B to A on both sides of the axis.

(5) In like manner it is seen that as λ decreases from λ_1 to -8, our tracing points are imaginary. The circle ρ_1 is entirely contained in the circle ρ_2 and the latter in the circle ρ_3; and this goes on however large λ becomes negatively.

8. Summarising, we see that the real part of the curve consists of two ovals—together called the Cartesian Oval — one extending from D to C and traced as λ ranges from λ_4 to λ_3 and the other from B to A and traced as λ ranges for λ_2 to λ_1.

The latter oval is entirely contained in the former; for ρ_1 has throughout the range λ_4 to λ_1 been positive and decreasing. The two constituent ovals are thus distinguishable as the *outer* and the *inner* oval.

It is to be particularly noted that the values of ρ_1, ρ_2, ρ_3 are positive for points on the outer oval; while *for points on the inner oval, ρ_1 is positive, but ρ_2 and ρ_3 are negative.* The equations (2) are therefore true in an *arithmetical* sense for the outer oval; and they would be true in the same sense for the inner oval upon changing the signs of ρ_2 and ρ_3.

We further observe that the foci F_1 and F_2 are contained within both the ovals, while F_3 is outside both. Hence F_3 is termed the *external* focus. Also, the triple focus O will be within, on, or outside the inner oval according as the abscissa of the vertex A is positive, zero, or negative; that is according as

$$\sqrt{bc} \gtrless \sqrt{ca} + \sqrt{ab}. \qquad \ldots \qquad \ldots \quad (7)$$

Section III.

The fourth Parametric Equation.

9. Let P be a point on the oval of parameter λ. The distances of P from F_1, F_2, F_3 being known, its distance from any other point on

the axis can be found by means of the equation of distances. Let ρ denote the distance to the triple focus O. We find

$$\rho^2 = bc + ca + ab + 2\lambda_2 \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

10. Eliminating λ between the values of ρ^2 and of ρ_1, ρ_2, ρ_3 in succession, we get the bipolar equations

$$\left.\begin{aligned} \rho^2 &= 2\sqrt{bc}.\,\rho_1 + ab + ac - bc \\ \rho^2 &= 2\sqrt{ca}.\,\rho_2 + bc + ba - ca \\ \rho^2 &= 2\sqrt{ab}.\,\rho_3 + ca + cb - ab \end{aligned}\right\} \quad \ldots \quad \ldots \quad (9)$$

11. Let $\Theta_1, \Theta_2, \Theta_3$ be the angles that F_1P, F_2P, F_3P make with the axis, so that (ρ_1, Θ_1), (ρ_2, Θ_2), (ρ_3, Θ_3) are the polar co-ordinates of P with respect to F_1, F_2, F_3 as origins, the initial line in each case being the axis pointing *to the left*. We have

$$\left.\begin{aligned} \rho^2 &= \rho_1^2 - 2a\rho_1 \cos\Theta_1 + a^2 \\ \rho^2 &= \rho_2^2 - 2b\rho_2 \cos\Theta_2 + b^2 \\ \rho^2 &= \rho_3^2 - 2c\rho_3 \cos\Theta_3 + c^2 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (10)$$

12. Using these values in (9), we get the following polar equations of the oval :

$$\left.\begin{aligned} \rho_1^2 - 2\rho_1(\sqrt{bc} + a\cos\Theta_1) + (a-b)(a-c) &= 0 \\ \rho^2 - 2\rho_2(\sqrt{ca} + b\cos\Theta_2) + (b-a)(b-c) &= 0 \\ \rho_3^2 - 2\rho_3(\sqrt{ab} + c\cos\Theta_3) + (c-a)(c-b) &= 0. \end{aligned}\right\} \quad \ldots \quad (11)$$

The polar equations make it evident that the curve is the inverse of itself with respect to the circles of centres F_1, F_2, F_3 and squares of radii equal to $(a-b)(a-c)$, $(b-a)(b-c)$, $(c-a)(c-b)$. The first and third are real circles, while the second is imaginary. We shall speak of them as the circles F_1, F_2, F_3. Any two of the foci are inverses with respect to the circle whose centre is the third. The circles F_1 and F_2 invert each constituent oval into the other; while the circle F_3 inverts each into itself.

The circle F_3 cuts each constituent oval into two parts one of which lies without, and the other within, the circle F_3. They may be distinguished accordingly as the *outer* and *inner* parts of that oval. the terminal points of the parts are the points of contact of the tangents from F_3.

We shall generally take our typical point P of the curve to be on the outer part of the outer oval. Theorems proved for the case will be found to hold when P is elsewhere with suitable modifications suggested by continuity.

13. In our curve let us allow a to increase and become equal to b. Then the first and second equation in (11) become

$$r=2(\sqrt{bc}+b\cos\theta), \quad b<c \qquad \ldots \quad (12)$$

an elliptic limacon.

Next, let c decrease and become equal to b. Then the second and third equations in (11) become

$$r=2(\sqrt{ab}+b\cos\theta,)\ b>a \qquad \ldots \quad (13)$$

a hyperbolic limacon.

Thirdly, let a, b, c all become equal. Then each equation becomes

$$r=2a\,(1+\cos\theta), \qquad \ldots \quad (14)$$

a cardioid. Thus the limacons and the cardioid are extreme cases of the cartesian oval.

Lastly, let O advance infinitely near to, and become coincident with, F_1. The outer and inner ovals coalesce and the curve becomes the circle F_1 reckoned twice over. This also is an extreme form of the oval.

14. Before passing to consider the cartesian equations of the curve we may take notice of a special case of the equations (9). Let a, b, c be such that $ab+ac-bc=0$. Then the first equation in (9) becomes

$$\rho^2=2\sqrt{bc}\ \rho_1. \qquad \ldots \qquad \ldots \qquad \ldots \quad (15)$$

This equation, then, represents a cartesian oval. It is readily seen that it inverts with respect to O into the bipolar form $rr'=k^2$, that is a Cassinian Oval. The condition $ab+ac-bc=0$ is equivalent to $a^2=(a-b)(a-c)$; or the circle of inversion F_1 passes through the triple focus O. Hence we have the theorem :

If in a Cartesian Oval the circle of inversion F_1 passes through the triple focus O, the curve inverts with respect to O into an oval of Cassini. (16)

15. The equations (9) are equally suitable for deducing the cartesian equations of the curve. If x, y be the co-ordinates of P, abscissas being counted positive to the left as before, we have

$$\rho^2=x^2+y^2,\ \rho_1^2=(x+a)^2+y^2, \text{ \&c.}$$

Hence we get

$$\left.\begin{aligned}(x^2+y^2-ab-ac+bc)^2&=4bc\,\{(x+a)^2+y^2\}\\(x^2+y^2-bc-ba+ca)^2&=4ca\,\{(x+b)^2+y^2\}\\(x^2+y^2-ca-cb+ab)^2&=4ab\,\{(x+c)^2+y^2\}.\end{aligned}\right\} \qquad \ldots \quad (17)$$

Each of these reduces to

$$(x^2+y^2-bc-ca-ab)^2-4abc\ (2x+a+b+c)=0, \quad \ldots \quad (18)$$

which may be considered the standard form of the cartesian equation of the oval.

16. When an equation is of the form $x^2=y^2+z^2$, the analytical geometer knows it at once as the envelope of the system of curves

$$x=y\cos\theta+z\sin\theta \quad \ldots \quad \ldots \quad (19)$$

where θ is a parameter.

Accordingly, from equations (17) we see that the curve is the envelope of each of the systems of circles

$$\left.\begin{aligned} x^2+y^2-ab-ac+bc&=2\sqrt{bc}\ \{(x+a)\cos\theta+y\sin\theta\}\ ;\\ x^2+y^2-bc-ba+ca&=2\sqrt{ca}\ \{(x+b)\cos\theta+y\sin\theta\}\ ;\\ x^2+y^2-ca-cb+ab&=2\sqrt{ab}\ \{(x+c)\cos\theta+y\sin\theta\}\ ; \end{aligned}\right\} \ \ldots \ (20)$$

Written in their standard forms, these circles are

$$(x-\sqrt{bc}\cos\theta)^2+(y-\sqrt{bc}\sin\theta)^2=a(b+c+2\sqrt{bc}\cos\theta)\ ;\ \&c. \quad (21)$$

Let G_1, G_2, G_3 denote the circles of centre O and radii $\sqrt{bc}$, $\sqrt{ca}$, $\sqrt{ab}$ respectively. They are of importance in connection with the oval. We shall call them the 'generating circles' of the oval. From equations (21), it can be readily seen that

The Cartesian Oval is the envelope of a variable circle whose centre moves on any of the generating circles G_1, G_2, G_3, and which cuts the corresponding circle of inversion F_1, F_2 F_3, orthogonally.

These will be spoken of as the *non-axial*, or *secondary*, systems of bitangential circles of the oval.

From equations (18) we see that the cartesian oval is the envelope of the system of circles

$$x^2+y^2-bc-ca-ab=u(2x+a+b+c)+\frac{abc}{u}\ ; \quad \ldots \quad (22)$$

that is, of the system

$$(x-u)^2+y^2=\frac{(u+a)(u+b)u+c)}{u} \quad \ldots \quad (23)$$

which forms the equation of the *axial*, or *primary*, system of bitangential circles of the oval. Having thus introduced the analytical treatment of the curve, we proceed in the next section to develop a different line of study, also based on the fourth parametric equation, using Calculus.

SECTION IV.

The Triple Focal Perpendicular on the Tangent.

It is desirable to have a clear idea at the outset as to how the normal at a point P lies with reference to the joins of P to O, F_1, F_2, F_3.

Let P describe the outer oval from D to C. The distances of P to O, F_1, F_2, F_3 are

$$\rho=\sqrt{bc+ca+ab+2\lambda},\ \rho_1=\sqrt{bc}+\lambda/\sqrt{bc},\ \text{etc},$$

and they are all positive and decreasing. If PT be the tangent at P drawn in the direction of motion it follows that PO, PF_1, PF_2, PF_3 make acute angles with PT. *Hence the normal at* P *being drawn, the lines* PO, PF_1, PF_2, PF_3 *all lie on one side of it, namely the side of* PT.

Next let P describe the inner oval from B to A. Then the distances ρ and ρ_1 are positive and diminishing; while ρ_2 and ρ_3 are, *in magnitude*, increasing. Hence PO and PF_1 make acute angles with the tangent PT drawn in the direction of movement; while PF_2 and PF_3 make obtuse angles with the same. Hence the normal at P being drawn as before, we see PO and PF_1 will lie on one side of it, namely that of PT; while PF_2 and PF_3 will lie on the other. Therefore, *at any point* P *on the inner oval the normal lies within the angle* F_1PF_2.

The above will help us to draw the figure correctly for what follows.

19. Let P be a point on the curve, of parameter λ. Join it to O, F_1, F_2, F_3. Draw OQ perpendicular to the tangent at P, cutting PF_1, PF_2, PF_3 at M_1 M_2, M_3, and the tangent at P in Q. Then we get a figure with a few—I would say *five*—leading properties, from which numerous others follow. We shall deal with the first two of them in the present section.

20. *The first property is that*

$$PM_1=\sqrt{bc},\ PM_2=\sqrt{ca},\ PM_3=\sqrt{ab}. \qquad \ldots \quad (24)$$

Dem. Take the fundamental equations

$$\rho^2=bc+ca+ab+2\lambda,$$

$$\rho_1=\sqrt{bc}+\frac{\lambda}{\sqrt{bc}},$$

$$\rho_2=\sqrt{ca}+\frac{\lambda}{\sqrt{ca}},$$

$$\rho_3=\sqrt{ab}+\frac{\lambda}{\sqrt{ab}},$$

Differentiating with respect to the arc, measured from C towards D in the outer oval, and from A towards B in the inner oval, so that λ and s increase together, we get

$$\rho\frac{d\rho}{ds}=\frac{d\lambda}{ds},$$

$$\frac{d\rho_1}{ds}=\frac{1}{\sqrt{bc}}\cdot\frac{d\lambda}{ds}, \text{ etc.}$$

The first of these means $\rho \cos \mathrm{OPQ}=\frac{d\lambda}{ds}$;

that is $\frac{d\lambda}{ds}= t$, where t denotes PQ. (25)

Hence the second equation tells us

$$\cos \mathrm{F_1PQ}=\frac{t}{\sqrt{bc}};$$

that is $$\frac{t}{\mathrm{PM_1}}=\frac{t}{\sqrt{bc}}.$$

Hence $\mathrm{PM_1}=\sqrt{(bc)}$; and similarly $\mathrm{PM_2}=\sqrt{(ca)}$, and $\mathrm{PM_3}=\sqrt{(ab.)}$.

Cor. 1. As the whole lengths $\mathrm{PF_1}$, $\mathrm{PF_2}$, $\mathrm{PF_3}$ are $\sqrt{(bc)}+\lambda/\sqrt{(bc)}$, etc.; and the parts $\mathrm{PM_1}$, etc., are equal to $\sqrt{(bc)}$, etc., therefore the remaining parts $\mathrm{M_1F_1}$ $\mathrm{M_2F_2}$ $\mathrm{M_3F_3}$ are equal respectively to

$$\frac{\lambda}{\sqrt{(bc)}},\ \frac{\lambda}{\sqrt{(ca)}};\ \frac{\lambda}{\sqrt{(ab)}}. \quad \ldots \quad (26)$$

Cor. 2. Hence rectangle $\mathrm{PM_1}.\ \mathrm{M_1F_1}$ = rectangle $\mathrm{PM_2}.\ \mathrm{M_2F_2}$ = rectangle $\mathrm{PM_3}$, $\mathrm{M_3F_3}=\lambda$. (27)

We are thus afforded a simple geometrical meaning of the parameter λ.

Cor. 3. The sines of the angles that the normal makes with the focal vectors $\mathrm{PF_1}$, $\mathrm{PF_2}$, $\mathrm{PF_3}$ are as $\sqrt{a}:\sqrt{b}:\sqrt{c}$.

For, these angles are equal to the angles $\mathrm{PM_1Q}$, $\mathrm{PM_2Q}$, $\mathrm{PM_3Q}$ in our figure; whose sines are $\frac{t}{\sqrt{(bc)}}$, $\frac{t}{\sqrt{(ca)}}$, $\frac{t}{\sqrt{(ab)}}$; and are therefore as $\sqrt{a}:\sqrt{b}:\sqrt{c}$.

21. **Theorem**: *If the normal at P cut the axis in G, then* $OG=abc/\lambda$. (28)

Dem: The normal is parallel to OM_1. Therefore by similar triangles

$$GO : OF_1 = PM_1 : M_1F_1 ;$$

that is,
$$OG : a = \sqrt{bc} : \frac{\lambda}{\sqrt{bc}} ;$$

$$\therefore \qquad OG = \frac{abc}{\lambda}.$$

Cor. 1. The centres of curvature at the vertices D, A, B, C of the oval are the points of abscissae ξ_4, ξ_1, ξ_2, ξ_3 where

$$\left.\begin{aligned} \xi_4 &= \frac{\sqrt{abc}}{\sqrt{a}+\sqrt{b}+\sqrt{c}} \\ \xi_1 &= \frac{\sqrt{abc}}{\sqrt{a}-\sqrt{b}-\sqrt{c}} \\ \xi_2 &= \frac{\sqrt{abc}}{\sqrt{b}-\sqrt{c}-\sqrt{a}} \\ \xi_3 &= \frac{\sqrt{abc}}{\sqrt{c}-\sqrt{a}-\sqrt{b}} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (29)$$

These follow from the theorem by substituting for λ the values of the parameters of the vertices from equations (6)

Cor. 2. The radii of curvature at the vertices D, A, B, C are σ_4, σ_1, σ_2, σ_3, where

$$\left.\begin{aligned} \sigma_4 &= \frac{(\sqrt{b}+\sqrt{c})\,(\sqrt{c}+\sqrt{a})\,(\sqrt{a}+\sqrt{b})}{\sqrt{a}+\sqrt{b}+\sqrt{c}} \\ \sigma_1 &= \frac{(\sqrt{b}+\sqrt{c})\,(\sqrt{c}-\sqrt{a})\,(\sqrt{b}-\sqrt{a})}{\sqrt{b}+\sqrt{c}-\sqrt{a}} \\ \sigma_2 &= \frac{(\sqrt{c}+\sqrt{a})\,(\sqrt{b}-\sqrt{a})\,(\sqrt{b}-\sqrt{c})}{\sqrt{c}+\sqrt{a}-\sqrt{b}} \\ \sigma_3 &= \frac{(\sqrt{a}+\sqrt{b})\,(\sqrt{c}-\sqrt{a})\,(\sqrt{c}-\sqrt{b})}{\sqrt{a}+\sqrt{b}-\sqrt{c}}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (30)$$

These values are obtained by subtracting the abscissae of the centres of curvatures at the vertices, as given in the last Corollary, from the abscissae of the vertices, given in equations (6). Accordingly, when positive, they denote concavity to the right; and when negative, concavity to the left.

Cor. 3. The curve is concave to the right at the vertices D and A, and concave to the left at the vertex B.

Cor. 4. At the vertex C the curve is concave to the right, or to the left, according as

$$\sqrt{a}+\sqrt{b} \gtrless \sqrt{c}. \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

When at C the curve is concave to the right, we shall say the curve is an *indented* oval; if to the left, an *unindented* oval.

Cor. 5. The vertex C is a point of undulation, if

$$\sqrt{a}+\sqrt{b}=\sqrt{c}. \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

Cor. 6. The radius of curvature at D is greater than the radius of curvature at A, and the latter greater than the radius of curvature at B, in absolute magnitude.

Cor. 7. In the unindented oval, the radius of curvature at C is greater that at D in absolute magnitude.

Cor. 8. On every indented oval there is a pair of real points of parameter zero.

For, the outer oval is traced as λ varies from

$$\lambda_4=\sqrt{(abc)}(\sqrt{a}+\sqrt{b}+\sqrt{c}) \text{ to } \lambda_3=\sqrt{(abc)}(\sqrt{c}-\sqrt{a}-\sqrt{b})$$

and zero lies between these values.

Cor. 9. In every indented oval, there is an obvious double tangent. Its points of contact are the points of zero parameter.

For, the normal at either point cuts the axis at an infinite distance. Hence

$$\frac{abc}{\lambda}=\infty \quad \therefore \lambda=0.$$

Cor. 10. The distance of the points of contact of the double tangent from F_1, F_2, F_3 and O are

$$\sqrt{(bc)},\ \sqrt{(ca)},\ \sqrt{(ab)} \text{ and } \sqrt{(bc+ca+ab)}. \quad \ldots \quad (33)$$

For, making $\lambda=0$ in the fundamental parametric equations, the result follows.

Geometrical Investigation of the Secondary Systems of Bitangential Circles.

22. Let P, p_1 be a pair of inverse points on the curve, which are inverses in respect of the circle F_1. Let the normals at P, p_1 meet at g_1.

By the properties of inverse curves we have $g_1P = g_1p_1$. The circle of centre g_1 and radius g_1P or g_1p_1—which obviously touches the curve at P, p_1—will cut the circle F_1 orthogonally, as it passes through a pair of inverse points with respect to it, namely P and p_1.

To find the locus of g_1 as P varies: From O draw perpendiculars on the tangents to the oval at P, p_1 cutting the radius vector F_1p_1P at M_1 and m_1. By the property in question, we have $PM_1 = p_1m_1$, each being equal to $\sqrt{(bc)}$. Further they are in the same sense as will be obvious from the figure drawn with due regard to the observations in § 18. Consequently Pp_1 is equal to M_1m_1. Hence the triangles Pg_1p_1 and M_1Om_1, whose corresponding sides are parallel, are equal in all respects. Thus g_1P is equal and parallel to OM_1. Therefore Og_1 is equal and parallel to M_1P. Hence $Og_1 = \sqrt{(bc)}$ and the locus of g_1 is the 'generating' circle G_1. Therefore, if a variable circle move with centre on G_1, always cutting the circle F_1 orthogonally, its envelope is the Cartesian Oval. And a like proof holds for the *other* secondary systems of bitangential circles.

Cor. I. Hence, P being any point on the oval, if parallels be drawn through O to tne focal vectors PF_1, PF_2, PF_3 cutting the normal PG at g_1, g_2, g_3; then $g_1g_2g_3$ are the centres of the bitangential circles of the systems (G_1), (G_2), (G_3) which touch the curve at P.

Cor. 2, Consequently, the radii of the circles of the secondary systems of bitangential circles which touch the oval at P are equal to OM_1, OM_2 and OM_3.

Cor. 3. In the *indented* oval, the radii of the bitangential circles touching the curve at a point of contact of the double tangent are equal to OF_1, OF_2, OF_3; that is, are equal to a, b, c. ... (34)

Cor. 4. If g_1, g_2, g_3, G be the centres of the bitangential circles touching the oval at P, the set (P, g_1, g_2, g_3, G) is anharmonically constant.

For, it subtends at O a pencil, which, by parallels, is equal to that subtended at P, by a fixed set of points on the axis, namely

$$(O, F_1, F_2, F_3, \infty).$$

23. *The second property* of the triple focal perpendicular on the tangent consists of the following metrical relations.

Denoting OM_1, OM_2, OM_3 by n_1, n_2, n_3 we have

$$n_1 = \sqrt{\frac{a\rho_2\rho_3}{\rho_1}}, \quad n_2 = \sqrt{\frac{b\rho_3\rho_1}{\rho_2}}, \quad n_3 = \sqrt{\frac{c\rho_2\rho_1}{\rho_3}}. \qquad \ldots \quad (35)$$

Proof—Applying the equation of distances to connect OF_1, OM_1, OP and reducing, we get

$$\rho_1 . \, OM_1^2 = \frac{(\lambda + ab)(\lambda + ac)}{\sqrt{bc}} = a\rho_2\rho_3$$

Hence $n_1 = \sqrt{\frac{a\rho_2\rho_3}{\rho_1}}$, etc.

Cor. 1. From the last article it follows that *the values obtained are those of the radii of the secondary systems of* bitangential circles which touch the curve at the point λ.

Cor. 2. From the values of n_1, n_2, n_3 we get

$$\left.\begin{aligned} n_2 n_3 &= \sqrt{bc} \,.\, \rho_1 = \lambda + bc \\ n_3 n_1 &= \sqrt{ca} \,.\, \rho_2 = \lambda + ca \\ n_1 n_2 &= \sqrt{ab} \,.\, \rho_3 = \lambda + ab \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (36)$$

Cor. 3. Hence, in our figure,

$$OM_2 \,.\, OM_3 = PM_1, \, PF_1 ;$$

with two similar relations. (37)

Cor. 4. Also, by subtraction, we obtain

$$OM_1 . \, M_2M_3 = n_1(n_3 - n_2) = a(c - b) ;$$

and two similar relations. (38)

24. **Theorem** : If n denote the normal PG, then,

$$n = \frac{l_1}{\lambda} \sqrt{abc\rho_1\rho_3\rho_2}, \qquad \ldots \quad \ldots \quad (39)$$

For, by parallels,

$$PG : M_1O = PF_1 : M_1F_1 ;$$

that is $$n : \sqrt{\frac{a\rho_2\rho_3}{\rho_1}} = \rho_1 : \frac{\lambda}{\sqrt{bc}}$$

$$\therefore \qquad n = \frac{1}{\lambda} \sqrt{abc\rho_1 \rho_2 \rho_3}.$$

Cor. 1. We have

$$n_1 n_2 n_3 = n\lambda. \qquad \ldots \qquad \ldots \qquad \ldots \quad (40)$$

Cor. 2. $$n = \frac{1}{\lambda}\sqrt{(\lambda+bc)(\lambda+ca)(\lambda+ab)} \qquad \ldots \qquad \ldots \quad (41)$$

Cor. 3. If the abscissa of G be u, then,

$$n = \sqrt{\frac{(u+a)(u+b)(u+c)}{u}} \qquad \ldots \qquad \ldots \quad (42)$$

Cor. 4. Thus the radius of the circle, with centre at any point G on the axis and having double contact with the curve, is

$$\sqrt{\frac{GF_1 . GF_2 . GF_3}{GO}} \qquad \ldots \qquad \ldots \quad (43)$$

The circle is real provided G does not lie in the segments OF_1, F_2F_3.

25. By means of the last formula we can appreciate the effect produced on the oval when F_1, F_2, F_3 remain fixed, but the triple focus O moves to a considerable distance to the left. Let G be a typical point, first, in $F_1 F_2$; and secondly, in $F_1 F_3$ produced. The formula shows that when O recedes and GO becomes very large, the radius of the double contact circle, centre G, becomes, in each case, very small. Therefore, as regards the inner oval we infer that it tends to become very thin and look like a thin ellipse of foci F_1, F_2. As regards the outer oval, we realize the matter thus. As O moves, starting from a position near F_1, our oval is at first unindented. The indented form begins to appear as soon as O has passed a position O′ such that

$$\sqrt{O'F} + \sqrt{O'F_2} = \sqrt{O'F_3}.$$

As O recedes, the indentation, gentle at first, becomes very sharp, and very long, approaching the big cardioidal figure, with cusp at F_3, that the curve rapidly becomes. That this is the form can be gathered from the fact that when O recedes to a considerable distance the foci F_1, F_2, F_3 are coincident relatively to the distance of O from them. The double contact circles in question, with centres in $F_1 F_3$ produced are the circles inscribed in the creek of this cardioidal figure, whose form is like a thin parabola of focus F_3; and the circles inscribed in it become indefinitely small.

26. *To find the differential coefficients of* n_1, n_2, n_3 with respect to λ.

From the values $n_1 = \sqrt{\frac{a\ \rho_2 \rho_3}{\rho_1}}$, etc;

and $$\frac{d\rho_1}{d\lambda}=\frac{1}{\sqrt{bc}},\quad \frac{d\rho_2}{d\lambda}=\frac{1}{\sqrt{ca}},\quad \frac{d\rho_3}{d\lambda}=\frac{1}{\sqrt{ab}},$$

we easily get

$$\left.\begin{aligned}\frac{dn_1}{d\lambda}&=\tfrac{1}{2}\cdot\frac{n_2+n_3-n_1}{n_2n_3}\\ \frac{dn_2}{d\lambda}&=\tfrac{1}{2}\cdot\frac{n_3+n_1-n_2}{n_3n_1}\\ \frac{dn_3}{d\lambda}&=\tfrac{1}{2}\cdot\frac{n_1+n_2-n_3}{n_1n_2}\end{aligned}\right\}\quad \ldots\qquad \ldots\qquad \ldots\quad (44)$$

We thus meet with a set of simultaneous differential equations, easily solved, to which the geometry of the cartesian oval affords an illustration.

27. It is interesting to see that when n_1, n_2, n_3 and n are given, we can find a, b, c and λ.

For, λ is at once given by

$$n_1n_2n_3=n\lambda.$$

When λ has been found, the values of bc, ca, ab are given by $n_2n_3=\lambda+bc$, etc.; whence a, b, c are known.

Hence, given the four bitangential circles of a cartesian oval touching the curve at a point P, the circle belonging to the axial system being distinctly specified, we can determine the oval.

Section V.

The Third Property of the Triple focal Perpendicular.

28. The property to be now considered is one of special importance. OQ being, as before, the triple focal perpendicular on the tangent at P, cutting the focal vectors PF_1, PF_2, PF_3 in M_1, M_2, M_3; the property in question is $OQ=\frac{1}{2}(OM_1+OM_2+OM_3)$... (45)

Denoting OQ by p, our Proposition is $p=\frac{1}{2}(n_1+n_2+n_3)$. ... (46)

Proof: In OQ produced take M_1' the reflexion of M_1 in the tangent at P.

Then, rect. $OM_1.OM'_1=OQ^2-M_1Q^2$;

$$\begin{aligned}&=OP^2-M_1P^2;\\ &=(ab+ac+bc+2\lambda)-bc,\\ &=(ab+\lambda)+(ac+\lambda);\\ &=n_1(n_2+n_3).\end{aligned}$$

But $OM_1=n_1$

$\therefore$ $OM_1'=n_2+n_3$

$\therefore$ $OQ=\frac{1}{2}(OM_1+OM_1')=\frac{1}{2}(n_1+n_2+n_3)$.

That is, $p=\frac{1}{2}(n_1+n_2+n_3)$.

29. We shall now introduce three new quantities which are important in the analysis of the oval. These are the lengths M_1Q, M_2Q and M_3Q. We shall denote them by q_1, q_2, q_3.

We evidently have

$$p = n_1 + q_1 = n_2 + q_2 = n_3 + q_3 \quad \ldots \quad \ldots \quad (47)$$

From these and the result of last article, we get

$$p = q_1 + q_2 + q_3 \quad \ldots \quad \ldots \quad \ldots \quad (48)$$

which is *another form* of the property in question. It shows that, in our figure,

$$QO = QM_1 + QM_2 + QM_3 \quad \ldots \quad \ldots \quad (49)$$

From (47) and (48), we get

$$n_1 = q_2 + q_3; \quad n_2 = q_3 + q_1; \quad n_3 = q_1 + q_2;$$

whence

$$\left.\begin{aligned} q_1 &= \tfrac{1}{2}(n_2 + n_3 - n_1) \\ q_2 &= \tfrac{1}{2}(n_3 + n_1 - n_2) \\ q_3 &= \tfrac{1}{2}(n_1 + n_2 - n_3). \end{aligned}\right\} \quad \ldots \quad \ldots \quad (50)$$

30. We have already denoted the length PQ by t.
In terms of t we obviously have

$$\left.\begin{aligned} q_1 &= \sqrt{bc - t^2} \\ q_2 &= \sqrt{ca - t^2} \\ q_3 &= \sqrt{ab - t^2}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (51)$$

Hence, substituting in (48), we get

$$p = \sqrt{bc - t^2} + \sqrt{ca - t^2} + \sqrt{ab - t^2} \quad \ldots \quad \ldots \quad (52)$$

which is the p, t equation of the Cartesian Oval.

The p, t equation is important in any curve as, *inter alia*, it gives a grip over the curvatures of the curve. And when, as here, we have p given *explicitly* as a function of t, the convenience is the greater.

31. Let us investigate the general p, t equation $p = \phi(t)$. ... (i)

Let ω be the angle which the perpendicular makes with a fixed line, this angle being so reckoned that we have

$$t = \frac{dp}{d\omega}.$$

From (i), differentiating with respect to ω, we get

$$t = \phi'(t) \cdot \frac{dt}{d\omega};$$

Hence,

$$\frac{dt}{d\omega} = \frac{d^2p}{d\omega^2} = \frac{t}{\phi'(t)}. \quad \ldots \quad \ldots \quad (ii)$$

The radius of curvature being denoted by σ, we have

$$\sigma = p + \frac{d^2p}{d\omega^2}.$$

That is $$\sigma = \phi(t) + \frac{t}{\phi'(t)} \quad \ldots \quad \ldots \quad \text{(iii)}$$

Let σ_1 denote the radius of curvature of the evolute of the curve. Then

$$\sigma_1 = \frac{d\sigma}{d\omega}.$$

Now the operation

$$\frac{d}{d\omega} = \frac{dt}{d\omega}.\frac{d}{dt} = \frac{t}{\phi'(t)}.\frac{d}{dt}.$$

Applying these equivalent operations to σ, we get

$$\sigma_1 = \left\{ \frac{t}{\phi'(t)}.\frac{d}{dt} \right\} \left\{ \phi(t) + \frac{t}{\phi'(t)} \right\}. \ldots\ldots \quad \ldots \quad \text{(iv)}$$

And the radius of curvature of the n^{th} evolute is given by

$$\sigma_n = \left\{ \frac{t}{\phi'(t)}.\frac{d}{dt} \right\}^n \left\{ \phi(t) + \frac{t}{\phi'(t)} \right\}. \quad \ldots \quad \text{(v)}$$

32. Let us apply the formula obtained above, namely,

$$\sigma = \phi(t) + \frac{t}{\phi'(t)},$$

to the case of our oval.

We have $\phi(t) = \sqrt{bc - t^2} + \sqrt{ca - t^2} + \sqrt{ab - t^2}$; and we have the letters q_1, q_2, q_3 to denote the radicals involved.

Differentiating, $\phi'(t) = \frac{-t}{q_1} + \frac{-t}{q_2} + \frac{-t}{q_3}$.

$$\therefore \quad \frac{\phi'(t)}{t} = -\frac{q_1q_2 + q_2q_3 + q_3q_1}{q_1q_2q_3}.$$

Hence $$\sigma = \phi(t) + \frac{t}{\phi'(t)} = (q_1 + q_2 + q_3) - \frac{q_1q_2q_3}{\Sigma(q_1q_2)}.$$

$$\therefore \quad \sigma = \frac{(q_1 + q_2)(q_2 + q_3)(q_3 + q_1)}{q_1q_2 + q_2q_3 + q_3q_1}. \quad \ldots \quad \ldots \quad (53)$$

For example, at the vertex D, $t = 0$ and we have

$$q_1 = \sqrt{bc},\ q_2 = \sqrt{ca},\ q_3 = \sqrt{ab};$$

and we get for the radius of curvature the value

$$\frac{(\sqrt{a} + \sqrt{b})(\sqrt{b} + \sqrt{c})(\sqrt{c} + \sqrt{a})}{\sqrt{a} + \sqrt{b} + \sqrt{c}}$$

which verifies the result already obtained.

33. Working out the general formula in § 31 for the radius of curvature of the evolute, we get

$$\sigma_1 = \frac{3tq_1q_2q_3(q_1+q_2)(q_2+q_3)(q_3+q_1)}{(q_1q_2+q_2q_3+q_3q_1)^3} \quad \ldots \quad \ldots \quad (54)$$

The evolute has cusps (1) at the centres of curvature at the vertices, for which $t=0$; and (2) at the centres of curvature at the points of contact of the tangents drawn to the oval from the external focus, at which points $q_3=0$. These are the only real cusps. The evolute is thus visibly, an 8 cusped curve. Four of these cusps relate to the inner oval, and four to the outer oval. In the unindented oval the evolute corresponding to each constituent oval has a look like that of the evolute of an ellipse. And the reader interested can readily find out the modification of form produced when the curve is an indented oval.

34. There is another method of investigating the radius of curvature at any point. At the very outset, in § 20, we met with the small inconspicuous equation

$$t = \frac{d\lambda}{ds}.$$

Now, $$t = \frac{dp}{d\omega}.$$

$\therefore$ $$\frac{dp}{d\omega} = \frac{d\lambda}{ds}.$$

Hence, we derive $$\frac{d\omega}{ds} = \frac{dp}{d\lambda}.$$

That is, $$\frac{1}{\sigma} = \frac{dp}{d\lambda}.$$

But $$p = \tfrac{1}{2}(n_1+n_2+n_3)$$

$\therefore$ $$\frac{1}{\sigma} = \tfrac{1}{2}\left\{\frac{dn_1}{d\lambda}+\frac{dn_2}{d\lambda}+\frac{dn_3}{d\lambda}\right\}.$$

And, we found in § 26 that

$$\frac{dn_1}{d\lambda} = \tfrac{1}{2}\,\frac{n_2+n_3-n_1}{n}\,;\ \text{etc.}$$

Hence we get the formula.

$$\frac{4}{\sigma} = \Sigma\,\frac{n_2+n_3-n_1}{n_2n_3}. \quad \ldots \quad \ldots \quad \ldots \quad (55)$$

This is readily transformed into that of § 32.

For $$q_1 = \tfrac{1}{2}(n_2+n_3-n_1),\ n_1 = q_2+q_3,\ \text{etc.}$$

Hence
$$\frac{4}{\sigma}=\Sigma\ \frac{2q_1}{(q_1+q_3)(q_1+q_2)}$$
$$=\Sigma\ \frac{2q_1(q_2+q_3)}{(q_1+q_2)(q_2+q_3)(q_3+q_1)}$$
$$=\frac{4\Sigma q_1q_2}{\Pi(q_1+q_2)}$$
$$\therefore\qquad \sigma=\frac{\Pi(q_2+q_3)}{\Sigma(q_1q_2)}.$$

35. The formula obtained for the radius of curvature admits of an important transformation which yields several simple geometrical constructions for the centre of curvature.

From the triangle OPQ, we have
$$OP^2=PQ^2+QO^2.$$
$$\therefore\quad bc+ca+ab+2\lambda=t^2+(q_1+q_2+q_3)^2$$
$$=t^2+(bc-t^2)+(ca-t^2)+(ab-t^2)+2\Sigma q_1q_2.$$
Hence we obtain the important result
$$\lambda=-t^2+\Sigma q_1q_2, \quad \ldots \quad \ldots \quad (56)$$
which expresses λ in terms of t.

This gives
$$\Sigma q_1q_2=\lambda+t^2.$$
Hence, from the formula
$$\sigma=\frac{(q_1+q_2)(q_2+q_3)(q_3+q_1)}{q_1q_2+q_2q_3+q_3q_1}$$
we get
$$\sigma=\frac{n_1n_2n_3}{\lambda+t^2}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (57)$$

36. Now we saw already that
$$n_1n_2n_3=n\ \lambda,$$
Hence we have
$$\frac{\sigma}{n}=\frac{\lambda}{\lambda+t^2}. \quad \ldots \quad \ldots \quad \ldots \quad (58)$$
This gives the following construction :

Let P be a point on the oval. Join PF_1 and cut off $PM_1=\sqrt{bc}$. Join OM_1 and produce it and draw PQ perpendicular to it. Then PQ is the tangent at P. Draw PG parallel to OQ cutting the axis in G. Then PG is the normal.

Now along PQ take Q_o so that
$$PQ.\ QQ_o=PM_1\ .\ M_1F_1,$$
Also along PQ take P_o so that $PP_o=QQ_o$. Join Q_oG; and, through P_o draw P_oS parallel to Q_oG, cutting PG in S. Then S is the centre of curvature at P.

For, by parallels

$$PS : PG = PP_o : PQ_o ,$$
$$= QQ_o : PQ + QQ_o, \text{ since } PP_o = QQ_o$$
$$= PQ - QQ_o : PQ^2 + PQ \,.\, QQ_o.$$

Now $PQ.\ QQ_o = PM_1 \,.\, M_1F_1$, by construction $= \lambda$, by its geometrical meaning.

$$\therefore \qquad PS : PG = \lambda : t^2 + \lambda$$

i.e.,
$$PS : n = \lambda : t^3 + \lambda.$$

$$\therefore \qquad PS = \sigma, \qquad \text{by (58).}$$

37. Since $n_2 n_3 = \sqrt{bc} \,.\, \rho_1$, the relation (57) gives

$$\frac{\sigma}{n_1} = \frac{\sqrt{bc} \cdot \rho_1}{\lambda + t^2} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (59)$$

which leads to the following construction :

Draw the normal as before and along it, take $Pg_1 = OM_1$; from Q draw QX perpendicular on PM_1 ; along PM_1 take $XX_1 = M_1F_1$; join X_1 to g_1 ; through F_1 draw F_1S parallel to X_1g_1 cutting the normal at S. Then S is the centre of curvature required.

For, by parallels

$$PS : Pg_1 = PF_1 : PX_1$$
$$= PF_1 : PX + XX_1 ;$$
$$= PF_1 .\ PM_1 : PX .\ PM_1 + PM_1 .\ XX_1.$$

But
$$PF_1 .\ PM_1 = \rho_1 .\ \sqrt{bc}$$
$$PX .\ PM_1 = PQ^2 = t^2$$
$$PM_1 .\ XX_1 = PM_1 .\ M_1F_1 = \lambda.$$

$$\therefore \qquad PS : Pg_1 = \rho_1\sqrt{bc} : t^2 + \lambda$$

ie
$$\frac{PS}{n_1} = \frac{P_1\sqrt{bc}}{\lambda + t^2.}$$

Hence by (59) $PS = \sigma$, which proves the construction.

We have two *other* similar constructions.

"Some difficulties met with in reading Mathematics without a Teacher : A Complaint against Text-books."*

By. Major W. A. Garstin, Pes'awar.

I was very pleased when the opportunity was offered to me of joining this Association at its start and I readily took advantage of the occasion because, although I cannot say that I am a mathematician, I do lay claim to being one of those others interested in mathematics to encourage and help whom is one of the primary objects of this Association. While, then, I welcomed the opportunity of becoming a member of the Association it never occurred to me that I should ever have the fortitude to read a paper before its members. And I do not doubt that I should never have done so had not an encouraging word been given to me by a friend,—himself a distinguished mathematician. We were discussing the methods and aims of mathematical teaching, and I was telling him of some of the difficulties which I had met with in reading the recognized text-books—difficulties which I was emboldened to maintain were due as much to defective methods of presentation as to my own dullness. The result of that brief discussion is the paper I am now to read to you. I cannot hope to bring to your notice anything very original ; indeed, it would seem presumptuous for anyone like myself, with no experience whatever as a teacher to address an audience such as this, on a subject connected with the vexed and difficult question of mathematical teaching. But I feel I shall not have wasted your time if I am able to bring home to those of you whose profession it is to teach, some of the inherent difficulties met with by one who has come to mathematics with the sole desire to learn and understand.

The title which I have chosen for the subject matter of this paper, "Some difficulties met with in reading Mathematics without the aid of a Teacher: A complaint against Text-books," suggests at once he comment that the difficulties must have been both various and many. That certainly is true; but the difficulties about which I am going to address you this afternoon I have narrowed down to a few of a more or less general character, such as are inherent in a subject as abstract as mathematics. They have nothing to do with the application of mathematics, and I am not going to trouble you about the hard points of this or that particular theorem. Very naturally I have encountered many difficulties of this latter kind where the assistance of a teacher would have been invaluable to me ; but my object in the present paper is altogether a different one. It is nothing more or less than to ask

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you to consider some root matters of prime importance, in the presentation of which, so it seems to me, our English text-books fail lamentably to give that instruction which, in my opinion, it is legitimate to expect them to provide. They are difficulties relating to the fundamental ideas of mathematics as represented (or supposed to be represented) in our modern elementary text-books. Now it seems to me to be of the utmost importance that the mathematical student should acquire at, as early a stage in his mathematical reading as possible, a firm and clear grasp of the fundamental ideas of his subject, and should have a thorough understanding of its aim and object and the nature of the methods by which it works, so that he may gain a due appreciation of what mathematics as a science really is. This need one would have thought was so obvious that one could hardly imagine any text-book professing to teach mathematics not entering upon at least some discussion of these matters; yet it has been my experience that it is just in regard to them that the elementary text-books are profoundly silent; either because, I suppose, the ideas are considered not to be suitable for the beginner or because they are not necessary for the purpose of mathematics, as that hard-grained subject is known in the schools. To neither of these views do I subscribe; and in my opinion it is because this side of the subject is so neglected that not only is mathematics looked upon by the ordinary student as a mere mechanical juggling with symbols, but students with marked mathematical ability shew themselves to possess but the most confused and hazy ideas of its fundamental concepts. I do not mean, I may hasten to explain, anything of a metaphysical nature, such as how it is we cannot conceive that two and two make five (though some of us have been known, I daresay, to arrive at that result in our everyday task)—that part of the *theory* of mathematics belongs to the philosophy of the subject, of which Mr. Bertrand Russel is so able and well-known an exponent. But there are ideas fundamental to mathematics which are, I think, quite inadequately treated in our elementary text-books, whereas too much time, and space to consider, cannot be spent in making such notions clear I venture and in taking care that by means of numerous examples and exercises the student really does understand them.

An excellent instance of the kind of difficulty I am dealing with is given by Mr. Hardy in the preface to the first edition of his "*Pure Mathematics*". He is writing about the notion of a limit, a notion "that", he writes, "has always presented grave difficulties to mathematical students even of great ability". It has been his good fortune, he explains, to have had a share in the teaching of a good many of the

ablest candidates for the Mathematical Tripos and, to quote his own words: "it is very rarely indeed that I have encountered a pupil who could face the simplest problem involving the ideas of infinity, limit or continuity with a vestige of the confidence with which he would deal with questions of a different character and of far greater intrinsic difficulty." "I have indeed", he goes on to say, "in an examination asked a dozen candidates, including several future Senior Wranglers to sum the series $(1+x+x^2+\ldots\ldots)$ and not received a single answer that was not practically worthless——and this from men quite capable of solving difficult problems connected with the curvature and torsion of twisted curves". "I cannot believe"[2], he says, "that this is due solely to the nature of the subject," and, then, after admitting that, no doubt, there are difficulties in those ideas but that they are not so great as many other difficulties inherent in mathematics which every young mathematician completely overcomes, Mr. Hardy arrives at the conclusion that "the fault is not that of the subject or of the student, but of the text-book and the teacher". These words by an eminent authority exactly represent the kind of difficulty which I have encountered in reading by myself—difficulty in getting at the bed-rock of the matter and not at attaining mere technical skill. For, I take my stand on "mathematics for mathematics' sake", and not for the sake of passing an examination or necessarity applying it. And the illustrations upon which I will now enter are therefore those which have chiefly to do with pure mathematics.

The first point to which I would draw your attention is the lack in the current elementary text-books of any adequate explanation of what is the aim and object of (pure) mathematics and how its various branches have arisen; no attempt is made to view the subject as a whole so that the great underlying principles which govern the subject may be seen and the student thereby led to realize that all its various activities, algebra, trigonometry, geometry, analyticalgeometry and their multifarious developments, are indeed branches of a parent stem, are, what Browning would call, 'the broken arcs of a perfect whole.' What is it then that the Mathematician is striving to obtain? He starts with mere numbers and a relation like $2+3=5$; but he soon begins to generalize and replaces these particular figures by the general letters a, b, and x; that is, he replaces "some" by "any", and is thus led on to a world of general theorems and deductions. But the content of his letters begin to extend by the conception of positive and negative quantity, and the representation of these two different concepts gives rise to an entirely new branch of mathematics: namely, analytical

geometry. Then again the treatment of equations of the second degree only led to solutions which for many ages were regarded simply as 'impossible', but to interpret which became imperatively neceessary if the science was not to be impeded at every fresh advance by ever-increasing restrictions and many interesting results were not to remain meaningless. Thus new symbols were invented and old ones made to clothe ideas vastly extended beyond that of mere number or quantity, but yet so defined as to obey the fundamental laws of algebra. And finally came the epoch-making recognition of the two different aspects of quantity as discrete and as continuous, with the resultant analysis reared on the basis of continuity. In all this, what the mathematician is really seeking is 'Generality,' a progress from the idea of 'some' things to 'any' things, an advance from the particular to the general. And with this idea of Generality go the ideas of the Variable and of Form to form, what Dr. Whitehead has aptly called, the Mathematical Trinity. This trinity forms the parent stem from which all branches spring and have their being. Now where in any current text-book has any full and clear exposition been given of these fundamental notions at all comparable to that given by Dr. Whitehead in his "*Introduction to Mathematics*"—a book costing one shilling and easily understood by any boy? Certainly not in any text-book I have ever met. Indeed, what I believe are commonly used as text-books make no attempt at all at any such explanation; and yet the ideas are fundamental to mathematics and must be thoroughly grasped if any proper understanding is to be gained of the object and aim of mathematics. My experience has been that the various divisions of the subject are treated as so many water-tight compartments, so many isolated chapters having no connecting thread between them, with the result that the subject of, say, imaginary qnantities looks like a mere excursion into the fantastic.

The mention of imaginaries furnishes me with a good illustration to point. The introduction of the student to complex numbers which our elementary text-books provide is to my mind extraordinarily meagre and unsatisfactory. While one book may give the briefest possible explanation of the symbols used, another is content with a bald definition of a complex number and its representation graphically, a third may be a little fuller but still utterly fail to convey a proper understanding of the part played by the use of these symbols in mathematical analysis in order that the student may grasp the basic idea underlying their conception; how their introduction has given complete *generality* and *symmetry* (the notion of form again) to the

theory of algebraic equations. I wonder whether one of Mr. Hardy's clever students would have had as much difficulty in giving a plain and convincing explanation of what the symbol $(a+bi)$ means, how it arose, and the reason for its choice, as he had in giving an intelligible explanation of the meaning of the 'sum to infinity' of a geometric series? The important fundamental ideas are the symbolical nature of these so-called imaginary expressions, the manner in which a concrete representation is given to this symbolism by employing the usual system of rectangular axes in a plane, the complex quantity $(a+bi)$ being represented by the point M in the plane XOY whose co-ordiantes are $x=a$ and $y=b$, and the fact that to every proposition established for complex quantities there is a corresponding theorem of geometry, though these are not the greatest advantages arising from their representation. Real numbers correspond to the x—axis, which for this reason is called the *axis of reals*. If the student is to get clear ideas of the meaning and use of these symbolical expressions then it seems to me that a great deal more explanation and illustration is needed than is usually given in the ordinary text-book. After all the ideas are not easy, and unless the nature of the symbols used is made clear, the integral part they play in mathematical analysis explained, and the mode in which they arose traced, the reader of the text-book, although he may be able to solve many academic exercises on the application of DeMoivre's theorem, will have but the haziest idea of what is really important, namely, the splendid mathematical conceptions which made all these remarkable results possible. In explaining the most elementary ideas of all connected with limits, Mr. Hardy devotes a chapter of fifty pages without advancing beyond the ordinary geometric series; and yet I am sure that the student who has digested those pages will have gained much more lasting (mathematical) profit than one who has solved all the examples on geometrical progression in Hall & Knight's "*Higher Algebra.*" Again take the notion of a *function*. Compare, for instance, Mr. Hardy's treatment of classes of functions in chapter II of his book with what is considered sufficient in a text-book, say, of the class of C. Smith's *Algebra*. As Mr. Hardy himself points out in the preface of his book, his treatment is "more systematic and illustrated with much greater detail than is usual in English books." He is at pains to bring out, and he succeeds admirably in doing so, the really important point in functional relationship, $y=f(x)$, and that is that any absolutely arbitrary law may be assumed for finding the value of y from that of x. The word *function* in its most general sense means nothing more or less than this; to every value of x corresponds a value of y. This is the important part of the matter; let it be driven home, as Mr.

Hardy drives it home, by a wealth of illustrations and exercises. And so in Edouard Goursat's (Professor of Mathematics in the University of Paris) preface to his *Mathematical Analysis.*, I read "Since mathematical analysis is essentially the science of the continuum, it would seem that every course in analysis should begin, logically, with the study of irrational numbers. The theory of incommensurable numbers is treated in so many excellent well-known works that I have thought it useless to enter upon such a discussion." Well, as the authors of the American translation of the work remark, such "excellent well-known books" are *not* common in English.

I would again remind you that the difficulties to which I refer in this paper are not those connected with the application of mathematics but arise out of what I consider to be the inadequate representation of its fundamental ideas. My next example, however, is of a more specific kind in that it deals with a particular theorem, but it illustrates well the common failure of the ordinary text-book to explain the precise nature of the conception on which the mathematical theory is based. To the student, when first introduced to it, the mathematical theory of probability appears as some thing singularly ineffectual, either because it is so against his common experience, as when for instance he does happen to play in a game of chance, the success or failure of his particular venture seems to upset entirely the mathematical theory, or because he is asked to consider such examples as the throwing of ideally perfect dice or the tossing of ideally perfect coins—things to be classed with such mathematical fictions as frictionless pulleys and weightless elephants! Now, of course, these perfect conditions are only chosen to fix the mathematical ideas; and the conditions are necessarily conceived as ideal because the mathematical theory of probability deals with the ideal since it aims at results which are to be exact. It must be made clear to the student that 'probability' as a mathematical quantity is not, as Prof. Chrystal well expresses it, an attribute of any particular event happening on any particular occasion, but can only be predicated of an event happening or conceived to happen, "on the average" or "in the long run." The important idea here then is that of happening a great many times; and I agree with Prof. Chrystal that the word *frequency* would be a much better word to use than *probability* to convey the idea of a very large number of occasions. These considerations are not, I think, sufficiently insisted upon in the proofs of the elementary theorems of the subject. I take as my example the proof, as I usually find it given in the text-books, that the probability of an event being known, it is required to find the probability of its happening exactly r

times in n trials. The usual text-book treatment is that if any particular r sets are selected out of the total number n the chance that the event will happen in every one of these r trials and fail in all the rest is $p^r q^{n-r}$, by the law of independent events; from which it follows that the terms in the expansion by the binomial theorem of $(p+q)^n$ represent respectively the required probabilities. Now, compare the above treatment with the following which I have taken from an elementary text-book on the theory of statistics:—"If we deal with one event only, we expect in N trials, Nq failures and Np successes. Suppose we combine with the results of this first event the results of a second. The two events are quite independent, and therefore, according to the rule of independence, of the Nq failures of the first event $Nq.\ q$ will be associated, *on an average*, with failures of the second event, and $Nq.\ p$ with successes of the second event. Similarly of the Np successful first events $Np.\ q$ will be associated, on an average, with failures of the second event and $Np.\ p$ with successes. In trials of two events we would therefore expect approximately Nq^2 cases of no successes, $2Npq$ cases of one success and one failure, and Np^2 cases of two successes. The results of a third event may be combined with those of the first two in precisely the same way, to expect, on the average, Nq^3 cases of no successes, $3\ Npq^2$ cases of one success, $3\ Np^2q$ cases of two successes, and Np^3 cases of three successes. Similarly for a fourth event. Hence it is evident that all the results are included under a very simple rule: the frequencies of 0, 1, 2, ...successes are given

for one event	by the	binomial	expansion	$N\ (p+q)$
for two events	„	„	„	$N\ (p+q)^2$
for three events	„	„	„	$N\ (p+q)^3$
for four events	„	„	„	$N\ (p+q)^4$

and so on. Quite generally in fact: the frequencies of 0, 1, 2... success in N trials are given by the successive terms in the binomial expansion of $N(p+q)^n$."

The above method seems to me to supply just the necessary connexion between theory and fact to let the student see how the mathematical theory is the ideal.

I should like to give you more examples of the same nature in which the main object of the text-books appears to be to get the student through a number of theorems as quickly as possible to the sole end that he may gain facility in mere technique. The treatment of series would furnish me with an excellent instance in point—the binomial

expansion, the exponential series, the logarithmic series, and the various trigonometrical series. They are treated without any attempt to explain —and such explanation I submit should,come very early—how they all lead up to and form part of the general problem of the development of a function as a power series and that to be able so to develop a function is a great advantage. For example, the most effective way of defining $\sin x$ and $\cos x$ is not by the usual geometrical method bnt as limits respectively of the series.

$$\sin = x - x^3/3\,! + x^5/5\,! - \ldots$$
$$\cos x = 1 - x^2/2\,! + x^4/4\,! - \ldots$$

Indeed unless a student proceeds as far as the Calculus, and there are many students for whom mathematics forms a part of their curriculum, who never get as far as that, he never is told that there are such theorems as Taylor's and Maclaurin's; he never learns that all those various series are but particular cases of a general theorem which for the width and the general character of its results is a marvellous achievement. But I must not detain you longer by multiplying illustrations as, before closing, I wish to say a few words about a subject on which a great mass of literature has accumulated, and I hope that the instances I have cited will have sufficed to make my meaning clear to you.

I refer to the Differential Calculus; a big subject, but there is only one point in it to which I wish here to draw your attention. That is, the danger which the student who really tries to think things out might be led into of inferring, that the calculus is a method of approximation only; a very close approximation but still an approximation, a mere asymptotic approach to a value which is never actually reached. Thus in Lamb's excellent text-book, a numerical example of the manner in which the ratio $\delta y/\delta x$ approximates to its limiting value is given in the case of $y = \log_{10} x$ for the neighbourhood $x = 1$, $D_x y = \cdot 43429$; a table is worked out showing how for different values of δx in the neighbourhood of 1 and the corresponding values of δy, the ratio $\delta y : \delta x$ approximates closer and closer to the value of $D_x y$. Well the student might reasonably reply: Yes, I quite see that, but how does one pass from the state of getting nearer and nearer to the value to the state of actually reaching it? What he has read about limits preparatory to the calculus and what he reads when he comes to the calculus itself is so surrounded with such expressions as "in the neighbourhood of", "as we approximate to", "as we approach" and so on, that I think

it is not surprising if he asks: But when are we going to get there? His difficulty, of course, is the old one of the paradox of Achilles and the tortoise.—Achilles who never overtook the toroise although he got "as close as we please" (as the stock text-book phrase is) to doing so because while he was making up on the tortoise the latter travelled a certain fraction of the distance which Achilles had to cover to reach the spot where the tortoise was originally, and hence the tortoise still remained just that finite fraction ahead of him. The paradox is due to confusing two distinct kinds of motion, one discrete in which the passage is from one value to another by definite and separate stops or jumps, and the other continuous in which the change of position passes through all intermediate values. Zeno's famous paradox indeed is an excellent one to bring home to the student the difference between continuous and discontinuous variation. Now I think that the reason, or at least one reason, for the confusion of thought which might lead a student to imagine that the calculus is only an approximate method arises from making the idea of the limit subordinate to the calculus. It is, of course, nothing of the sort. As far as the doctrine of limits is concerned *qua* such, the calculus is a mere incident. I mean that in the notion of the limit,it is of capital importance for the student to understand thoroughly that the limit of a function is *not* its value. The limits may possibly be equal to the values of the function; but, as Mr. Hardy points out, whether it is so or not "has nothing whatever to do with the notion of the limit." Let the student grasp this fact and he will then realize what a very incidental affair the calculus really is—applying only to those common or garden functions which have a value (or values) equal to the limit, that is, those only which are continuous. Let the notion of a limit be drilled into the student before he passes to a study of the calculus and he will see that, as regards the calculus, the crux of the whole matter lies in the fact that the limit of $f(x+h)$, as h approaches zero, is $f(x)$, and that unless this is so, the differential coefficient cannot exist. At present, the limit is taught as something existing for the calculus with the consequence that there is a danger of the student never getting really to understand the notion of a limit and incidentally of gaining a false perspective of the true place of the calculus in the hierarchy of mathematical ideas.

I have been led to invite your attention to this point regarding the differential calculus because I see from the agenda of to-day's meeting a proposal that this Association should undertake to write a standard book on the Calculus to meet the requirements of the Punjab University students. This, if you will permit me to say so, savours

rather of a book to meet the requirements of the Punjab University examinations. May I express the pious hope that the examiners will combine with the authors and will rise to the occasion to produce a work which will merit the title of 'standard' in the best sense? that the authors will take a broad 'humane' view of their subject and will give us a treatise worthy the student of any University who has a real desire for knowledge, and not a work designed merely to give a facility in working mechanical exercises, which, however ingenious they may be, are of no educative value whatsoever. I deny that to teach a student that the differential coefficient of x^n is nx^{n-1} is to teach him the differential calculus. I quite see the other side of the question: thousands of persons can use the slide rule or a vernier without being acquainted with the theory of their construction. But I think the object of this Association is to foster a spirit less utilitarian than that. Approach the subject from the standpoint of "mathematics for mathematics' sake" and is it too much to hope that mathematics can be made to take its place among the humanities? Or, should we not at the most expect more than that such a 'value' should be the 'limit' to which we can approach as close as we please but never reach? Let it then be our ideal. It is as I have read mathematics by myself with this ideal before me that I have encountered those difficulties about which it has been my privilege to address you this afternoon.

Since writing the above my attention has been drawn to a new set of text-books on algebra (including trigonometry) by T. Percy Nunn in Longman's Modern Mathematical Series. I have had time only to give them a hasty perusal, but from what I have seen, they seem to me to be a thorough break-away from the older type of text-book against which my remarks have been directed. They appear to be a departure on the very lines I have been advocating, and I would earnestly commend them to your notice.

Astronomical Notes.

1. *The Problem of Cepheid Variation.*—In the Notes for June, mention was made, of the investigations by Prof. Eddington on the Dynamics of the Pulsation theory of the cepheid variables. Prof. Jean is a great opponent of this theory. His investigations on the subject have led him to a different conclusion. His discussion is based on a study of the light curves and spectral variations of these stars. He suggests that the physical cause of variation of the cepheids and long-period variables is the same, being of the nature of explosions or tidal eruptions on a rotating body, and that the explosions are of the nature of 'forced' and not 'free' oscillations.

2. *The parallax of the Pleiades.*—This group of brilliant stars has attracted attention from the earliest times. Several attempts have been made to determine the distance of this important star cluster. Prof. Kapteyn has developed an indirect method of finding the mean parallax of a cluster by counting the number of stars it contains under different magnitudes, assuming that the luminosity curve is known. Dr. Schonten of Holland has recently applied this method to ascertain the distances of some well-known clusters. The results in the case of thirteen of them are given in the Observatory No. 537. The parallax he obtains for the Pleiades group is 0″036 with a probable error of ± 0″010, which gives the distance of the cluster to be 88 light years. This is of the same order of magnitude as the parallaxes found by Kapteyn and Plummer by other methods.

3. In Astr. Nach 4994, an investigation has been made of the orbit of the Comet 1832 II, applying Einstein's Relativity Theory of Gravitation. The elements of this orbit are slightly different from those obtained from Newton's Laws. The period determined is 760.9 years, the Einstein effect being to shorten it by 10.9 years.

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SOLUTIONS.

Questions 991 and 992.

991. (S. NARAYANAN) :—ABC is a triangle whose incentre is I and orthocentre H. If the in-centre of the exmedial triangle of ABC be K and KH bisected at L prove that the nine-points centre of ABC is the middle point of LI.

992. (S. NARAYANAN) :—The internal bisectors of the angles A, B, C of a triangle meet the sides of the medial triangle DEF at P, Q, R. Prove that the conic which touches these sides at P, Q, R touches also the principal Feuerbach tangent of DEF.

Solutions by N. Sankara Iyer, L. N. Subramanyam, S. R. Ranganatham and M. K. Kewalramani.

If K is (α, β, γ) referred to $\triangle$ ABC, we get

$$\frac{2\triangle}{a}-\alpha=\frac{2\triangle}{b}-\beta=\frac{2\triangle}{c}-\gamma=2r.$$

We have thus to prove that

$$2(O_1+H_1)=2I_1+2L_1=2I_1+K_1+H_1,$$

where O_1, H_1, &c., denote the α—co-ordinates of the points ;

i.e., we have to prove that

$$2O_1+H_1=2I_1+K_1,$$

i.e.,
$$2R\cos A+2R\cos B\cos C=2r+\frac{2\triangle}{\alpha}-2r=\frac{2\triangle}{\alpha},$$

i. e.,
$$\cos A+\cos B\cos C=\sin B\sin C$$

which is true.

For Question 992, we take D E F as the triangle of reference. It is clear that the internal bisector of A meets E F at the isotomic conjugate of the meeting point of E F and the internal bisector of D. The equation of this bisector is $\gamma=\beta$ or $\frac{z}{c}=\frac{y}{b}$ in areals. The equation of DP is therefore $\frac{1}{cz}=\frac{1}{by}$ in areals, or $b^2\beta=c^2\gamma$.

The required conic is therefore $\Sigma\sqrt{(a^2\alpha)}=0$.

The Feuerbach tangent is $\sum\frac{a\alpha}{b-c}=0$.

This will touch the ellipse, if

$$\sum \frac{a^2(b-c)}{a} = 0, \text{ i.e., } \Sigma a(b-c) = 0,$$

which is true.

Question 994.

(K. B. MADHAVA) :—Conduct the following game of Bridge with Hearts as trumps, in which A has to lead and A and B together have to make seven tricks.

	Hearts Q. Clubs 7, 6,	Diamonds A, 3, 2 Spades K, 8
Hearts 7, 6. Diamonds. J, 9, 8 Clubs. Q, J. 9 Spades. None	B Y Z A	Hearts 9, 8 Diamonds 10, 5 Clubs none Spades Q, 7, 6
	Hearts A, K, 5 Clubs A, 10, 8, 4.	Diamonds K. Spades none

Solution by R. Littlehailes, R. D. Karve, N. Sankara Aiyar, P. A. Subramani Aiyar E. Appalachari, and T. R. Narayana Aiyar.

(1)	A leads Heart A :	Y follows Heart 6 :	B, Heart Q :	Z Heart 8
(2)	A, Diamond K :	Y, Diamond 8 :	B, Diamond A ;	Z, Diamond 5.
(3)	B, Diamond 3 :	Z, Diamond 10 :	A, Trump K ;	Y, Diamond 9.
(4)	A, Heart 5 :	Y, Heart 7 :	B, Club 6 ;	Z Heart 9.
(5)	Z, Spade 7 :	A, Club 4 :	Y, Diamond J ;	B, Spade 8.
(6)	B, Spade K :	Z, Spade 5 :	A Club 8 ;	Y Club 9.
(7)	B, Diamond 2 ;	Z, Spade 6 :	A, Club 10 ;	Y Club J.
(8)	B Club 7 ;	Z, Spade Q :	A, Club A :	Y Club Q.

N.B.—Trick (4) is won by Z and all the others by A and B. The success of the game consists in playing into Z's hands at a time when he can play only spades thus giving B two tricks.

In tricks 5 and 6 it is open to Y to reserve Diamond J and throw down Club 9 and Club J ; in which case B leads Club 7 in trick (7) and A secures both the tricks 7 and 8.

QUESTIONS FOR SOLUTION.

1054. (MARTYN M. THOMAS) :—If s_r denote the sum of the squares of the fractions $\frac{1}{1}, \frac{1}{3}, \frac{1}{3}, \ldots\ldots \frac{1}{r}$, and p_r denote the sum of their products taken two at a time, show that

$$nC_1\, s_1 - nC_2(s_2+p_2) + nC_3(s_3+p_3) - nC_4(s_4+p_4) + \ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots (-1)^{r-1} nC_r(s_r+p_r) = \frac{1}{n^2}.$$

1055. (ENQUIRER) :—Show that

$$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}} = \frac{1}{2^2} + \frac{5}{2^7} + \frac{7\cdot 9}{2!}\frac{1}{2^{12}} + \frac{9\cdot 11\cdot 13\cdot}{3!}\cdot\frac{1}{2^{17}} \ldots\ldots\ldots$$

1056. (M. K. KEWALRAMANI) :—If

$\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$, show that

$$\sum \frac{\cos^2(p\alpha+q\beta+r\gamma)}{(2p)!\,(2q)!\,(2r)!} = \frac{\cos 2n\alpha + \cos 2n\beta + \cos 2n\gamma}{(2n)!}\cdot 2^{2n-2},$$

with a corresponding expression for sines; summation being extended to all positive integral values of p, q, r including zero, subject to the condition $p+q+r=n$.

1057. (M. K. KEWALRAMANI) :—Show that

$$\sum \frac{(a+b+c)^{2p}(b+c+d)^{2q}(c+d+a)^{2r}(d+a+b)^{2s}}{(2p)!\,(2q)!\,(2r)!\,(2s)!}$$
$$= \frac{2^{n-4}}{(2n)!}\left\{2\cdot 3\cdot {}^n\sigma^{2n} + 2\Sigma^2 t(\sigma+a)^{2n} + \Sigma(\sigma-a-b)^{2n}\right\}$$

where $2\sigma = a+b+c+d$ and the summation extend to all positive integral values of p, q, r, s including zero but subject to the condition $p+q+r+s=n$.

1058. (N. P. PANDYA) :— A fixed ellipse is intersected by a variable parabola in four points A, B, C, D. If the axis of the parabola be always perpendicular to the axis major of the ellipse, find the locus of the centre of gravity of the quadrilateral ABCD.

1059. (K. J. SANJANA, M. A.) :—Prove that the sums of the products of the quantities $(2^2-1)^{-1}$, $(4^2-1)^{-1}$, $(6^2-1)^{-1}$, ... *ad inf.*, taken three at a time and four at a time, are respectively

$$\frac{10-\pi^2}{32} \text{ and } \frac{1680-180\pi^2+\pi^4}{6144};$$

and show how to find the sums of the products taken five, six, ...at a time.

1060. (K. J. Sanjana, M. A.) :—Show how to find the sum of the products r at a time of the quantities

$$(3^2-1)^{-1},\ (5^2-1)^{-1},\ (7^2-1)^{-1},\ldots\ldots ad\ inf.;$$

and find the values for $r=3$ and $r=4$.

1061. (Lakshmishankar, N. Bhatt) :—The Euler line of the triangle AB_1C_1 meets the sides AB_1, AC_1 in B_2, C_2; the Euler line of AB_2C_2 meets these sides in B_3, C_3; and this process is continued indefinitely. If N_r is the nine-point centre of the rth triangle AB_rC_r thus formed, and AN_r meets B_rC_r in D_r, prove that the straight line $N_x\,N_{x+2y}$ and $D_x\,D_{x+2y}$ are parallel, x and y being any positive integers.

1062. (C. Krishnamachari) :—Prove that

$$1+\binom{n-1}{1}\frac{1}{3}+\binom{n-1}{2}\frac{1}{5}+\ldots+\frac{1}{2n-1}$$

$$=2^{n-1}\left\{\frac{1}{2n-1}+\frac{n-1}{(2n-1)\,(2n-3)}+\right.$$

$$\left.\frac{(n-1)\,(n-2)}{(2n-1)\,(2n-3)\,(2n-5)}+\ldots+\frac{\lfloor n-1}{(2n-1)\,(2n-3)\ldots 1}\right\}$$

1063. (C. Krishnamachari) :—Show that

$$(a)\quad \frac{1}{2^{n+1}}\left\{\frac{1}{n}+\frac{2n-1}{n(n-1)}+\frac{(2n-1)\,(2n-3)}{n(n-1)(n-2)}+\ldots\ldots\right.$$

$$\left.+\frac{(2n-1)\,(2n-3)\ldots 3}{\lfloor n}\ \frac{\pi}{4}\right\}$$

$$=1-(n+1)\tfrac{1}{3}+\frac{(n+1)(n+2)}{2\,!}\ \tfrac{1}{5}-\ldots\ldots$$

$$(b)\quad 1-\binom{n-\frac{1}{3}}{1}\frac{1}{3}+\binom{n-\frac{1}{2}}{2}\frac{1}{5}-\ldots\ldots$$

$$=2^{n+\frac{1}{2}}\left[\frac{1}{2n+1}-\tfrac{1}{2}\binom{n-\frac{1}{2}}{1}\frac{1}{2n+3}+\frac{1}{2^2}\binom{n-\frac{1}{2}}{2}\frac{1}{2n+5}-\ldots\right]$$

1064. (S. Krishnaswami Iyengar) :—If in Question 986 we have a rectangular hyperbola instead of a parabola, prove that Q K

$$=\frac{(ds)^5}{120\rho^4}\left\{\rho^2\frac{d^3\rho}{ds^3}-\frac{1}{3}\rho\frac{d^2\rho}{ds^2}-6\rho\frac{d\rho}{ds}\frac{d^2\rho}{ds^2}+4\left(\frac{d\rho}{ds}\right)^3+36\frac{d\rho}{ds}\right\}.$$

1065. (S. Krishnaswamiengar):—If ρ be the radius of curvature of the curve $r^m=a^m \sin m\theta$ at the point whose distance measured along the curve from a fixed point is s, prove that

$$(m-1)(m+1)^2\rho\frac{d^2\rho}{ds^2}-m(m+1)^2\left(\frac{d\rho}{ds}\right)^2-m(m-1)^2=0.$$

Show how to solve this equation and hence to find the equation of the given curve in terms of ρ and s.

Similar formulae are got, by giving particular values to m, for a parabola, a rectangular hyperbola, a lemniscate and a cardiode.

1066. (P. A. Subramanya Iyer) :—Solve completely

$$a\,xy\left(\frac{dy}{dx}\right)^2+(x^2-ay^2-b)\,\frac{dy}{dx}-xy=0.$$

1067. (V. Tiruvenkatachari) :—Show that

$$\int_0^{\frac{1}{2}\pi}\frac{\log\sin x\log\cos x}{\sqrt{(\sin x\cos x)}}\,dx=\tfrac{1}{8}\,\frac{(\Gamma\frac{1}{4})^2}{\sqrt{\pi}}\left\{(\log 2)^2+\pi\log 2-\frac{\pi^2}{4}\right\}.$$

1968. (A C. L. Wilkinson) :—E is any straight line parallel to the Euler line of a triangle ABC, E_1. E_2. E_3 are the Euler lines of the triangles formed by AB, AC, E ; BC, BA, E ; CA, CB, E ; prove that

(1) E_1. E_2. E_3, are parallel respectively to BC, CA, AB ;

(2) $E_1E_2E_3$ form a triangle equal in all respects to ABC ;

(3) the loci of the vertices of the triangles formed by $E_1E_2E_3$ as E varies are three straight lines parallel to the axis of the parabola which touches the sides of the triangle ABC and its Euler line ;

(4) if E passes through C, then E_1, E_2 intersect at the point where the Euler line of the triangle ABC meets AB ;

(5) the locus of the centre of homology of the triangles ABC, E_1 E_2 E_3 is a straight line parallel to the axis of the parabola defined in (3).

1069. (A. C. L. Wilkinson).—Normals PA, PB, PC, PD are drawn to a hyperbola ; PA meets the hyperbola again in A′, PB in B′, PC in C′, PD in D′. I(A′B′C′D′ are concyclic, prove that the locus of P is an ellipse of which the equi-conjugate diameters coincide with the asymptotes of the hyperbola.

1070. (S. Ramanujam, F.R.S.) :—Show that

(i) $$\sqrt{\sqrt[5]{\tfrac{1}{2}}+\sqrt[5]{\tfrac{4}{5}}}=\sqrt[5]{1}+\sqrt[5]{2}+\sqrt[5]{8}$$
$$=\sqrt[5]{\frac{16}{125}}+\sqrt[5]{\frac{8}{125}}+\sqrt[5]{\frac{2}{125}}-\sqrt[5]{\frac{1}{125}}$$

(ii) $$\sqrt[8]{\sqrt[5]{\frac{32}{5}}-\sqrt[5]{\frac{27}{5}}}=\sqrt[5]{\frac{1}{25}}+\sqrt[5]{\frac{3}{25}}-\sqrt[5]{\frac{9}{25}},$$

(iii) $$\sqrt[4]{\frac{3+2\sqrt[4]{5}}{3-2\sqrt[4]{5}}}=\frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}.$$

Mathematics (in brief) from Current Periodicals.

The Mathematical Gazette, Vol. IX, No. 139, March 1919.

The Graphical Treatment of Differential Equations, by S. Brodetsky, Ph. D. (to be reported in a later number). *Mathematics and the Pivotal Industries*, by W. P. Milne. (The object of the paper is to discuss the the importance of the Mathematical Association lending a helping hand in drawing up suitable mathematical syllabuses for the various 'pivotal' industries such as, Agriculture, Commerce, Engineering, Mining &c., and to see whether the Mathematical Association itself, and indirectly the Secondary Schools, would not benefit by occasionally viewing mathematics from the different view-points of industry rather than by focussing the whole attention on the academic aspect of the subject). *The Teaching of Geometry to First Year Pupils* by B. A. Howard. (The stand point of the paper is that our chief aim should be the development of the reasoning faculty; insisting always that this can only be done by sweeping away the abuses of Euclidean tradition and giving a more human and less philosophic orientation to our teaching. A schedule of first year work is outlined and discussed in the light of this statement of aim) *Cubic Graphs of the Form* $y=ax^3+bx^2+cx+d$, by A. Lodge. (The object of the paper is to call attention to some simple means of interpolating points in a cubic graph and of indicating gradients without actual calculation).

The Astrophysical Journal, Vol. XLIX, No. 2, March 1919.

Theory of Imperfect Gratings, by C. M. Sparrow. (The paper attempts to present the whole subject from the stand point of the vector-method and contains in addition a theory of the defective Fabry and Parot interferometor with consequences relating to the method of coincidences as a means of measuring wave-lengths). *Studies based on the Colours and Magnitudes in Stellar Clusters Tenth Paper: A critical Magnitude in the sequence of stellor Luminositics* by Harlow Shapley. (In the case of cluster-type variables, the absolute photographic magnitude —0·2 defines a limit of stellar luminosity at which the liability to Cepheid variation suddenly stops. This limit is probably coincident with a turning point in the development of the internal structure of a star. The question whether analogous dynamical changes occur at the same epoch of luminosity is also discussed). *Aerial Photometry*, by M. Luckish. (Certain photometric measurements and allied observations relating to Landscapes, Clouds, Haze, Sky and Water are presented, introduced by a few paragraphs dealing with the general characteristics of natural lighting),

The American Mathematical Monthly, Vol. XXVI, No. 3, March 1919.

On the Envelope of the Wallace Lines of an inscribed Quadrangle, by D. F. Barrow. (If we take a quadrangle inscribed in a circle and omit each vertex in turn, we obtain four triangles, and the feet of the perpendiculars from any point on the circle upon the four Simpson lines with regard to these triangles lie on a line called the Wallace Line of the point with regard to the quadrangle. The envelope of the Wallace Lines is an involute of a four cusped hypocycloid whose cusps lie on a circle of radius three times that of the circumcircle. Other theorems are deduced and generalizations indicated),

Annals of Mathematics—Second Series, Vol. 20, No. 3, March 1919.

On Quaternions and their Generalization and the History, of the Eight Square Theorem, by L. E. Dickson. (The object is to present the history of the generalizations to 4 and 8 squares of the formula, $(a^2+b^2)(\alpha^2+\beta^2)=r^2+s^2$, where $r=a\alpha-b\beta$, and $s=a\beta+b\alpha$ and an elementary exposition of Hurwitz's proof that such a formula holds only for 2, 4 or 8 squares. The writer shows that for these three cases, the formula admits of a simple interpretation concerning the norm of numbers which are ordinary complex numbers, quaternions or numbers of Cayley's algebra with 8 units. No knowledge of quaternions or the latter algebra is presupposed.) *Non-Symmetric Kernels of positive Type*, by Dr. Caroline E. Seely. (The paper considers properties of non-symmetric kernels, analogous to those of continuous symmetric functions of positive type, treated in Goursat, *Cours d'Analyse*, Vol. 3, p. 449). *Elementary Properties of the Stieltjes Integral*, by H. E. Bray. (Stieltjes Integral is defined and its existence proved. This is followed by other theorems leading up to a theorem on the change of order of integration of an iterated integral and another on integration by parts.) *A Kinematical Property of Ruled Surfaces*, by G. K. Whittemore. (The author proves that every ruled surface S (not developable) may be generated by a radius fixed in a sphere whose centre moves with unit velocity along the line of striction and which turns about the tangent to this curve from the binormal towards the principal normal with angular velocity equal to the reciprocal of the parameter of distribution, deduces two corollaries and the necessary and sufficient condition that the parameter of distribution may be equal to the radius of torsion of the line of striction at the corresponding points.) *Systems of Linear Inequalities* by Lloyd L. Dines. (The existence and character of the solution of a system of inequalities

$$\{\, a_{r1}\, x_1+a_{r2}\, x_2+\ldots\ldots+a_{rn}\, x_n>0,\ r=1,\ 2,\ 3,\ldots n\,\}$$

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PROGRESS REPORT.

Mr. Balakram, M.A., I.C.S.—one of the enthusiastic well-wishers of the Society—has been kind enough to present a cheque for Rs. 1,000 to our Society, with a desire that as far as practicable, the money should be utilized in completing our sets of the more important periodicals. The Committee have accepted this kind offer of Mr. Balakram with many thanks, and steps will be taken to utilize the amount in the best possible manner.

(2) The Committee feel great pleasure to announce that *Mr. T. V. Venkatarama Aiyar*, Corner House, Mylapore, Madras—has been elected a Life Member of the Society.

(3) The following gentlemen have been elected members of our Society—

1. *Mr. Indar Singh Puri, M.A.*—Deputy Examiner, A.G.'s Office, Allahabad;
2. *Lala Mehr Chand Suri, M.A.*—Professor of Mathematics, Forman Christian College, Lahore;
3. *Pandit Parama Nand, M.A.*—Professor of Mathematics, Prince of Wales College, Jammu (Kashmere State);
4. *Lala Ram Dass Navda, M.A.*—Professor of Mathematics Sanatan Dharam College, Lahore;
5. *Mr. C. Ranganathan, B.A., L.T.*—Mathematics Assistant, London Mission High School, Gooty (at concessional rate);

6. *Mr. B. S. Madhava Rao, B.Sc.* – M.Sc. Student, Calcutta Univ., 6 Badur Bagan Lane, Calcutta (at concessional rate) ;

7. *Mr. G. V. Krishnaswami*—Asst. Prof. of Mathematics, St. Joseph's College, Trichinopoly ;

8. *Mr. T. K. Devlalkar, M.A., B.Sc.*—Lecturer in Science, Karnatak College, Dharwar ;

9. *Mr. Satyendra Nath Sen B.A., M.Sc.*—Professor of Physics, Canning College, Lucknow.

(4) The following books have been received –

1. Bombay University Calendar, Part I, 1920-21 ;
2. Madras University Calendar, Vol. I & II for 1920 ;
3. Theory of Determinants, Vol. III (period 1861 to 1880) by Sir Thomas Muir, Macmillan & Co., London 1920, 35/- ;
4. Statics & Dynamics (first part)—by R. C. Fawdry, G. Bell & Sons, London, 1919, 5s/-

D. D. KAPADIA,

Hon. Joint Secretary.

Multiplication of Infinite Integrals

By K. B. Madhava, M. A.,

(*Continued from page* 180, *J.I.M.S., Vol. XI*).

V. The Analogues of Cauchy's and Merten's Theorems

By the analogue of Cauchy's theorem we mean the result that, if

$$\int_0^\infty u_1(x)\,dx$$

is convergent, and if

$$\int_0^\infty v_1(x)\,dx$$

be also convergent, then will

$$\int_0^\infty w_1(x)\,dx$$

be also convergent. (5·1)

In the case of "Merten's theorem", if

$$\int_0^\infty u_1'(x)\,dx \quad \text{and} \quad \int_0^\infty v(x)dx$$

are convergent, we have to prove that

$$\int_0^\infty w(x)\,dx$$

is convergent; or, if it happens,

$$\int_0^\infty w_1(x)\,dx$$

is convergent. (5·2

Of these Merten's theorem * is the more general, for if

$$\int_0^\infty w_1(x)\,dx$$

is convergent, it is known that

$$\int_0^\infty w(x)\,dx$$

is convergent.

* Cf Bromwich : *Infinite Series*, p. 429.

We will therefore prove only Merten's theorem.

Since,

$$\int_0^\infty v\,(x)\,dx$$

is convergent, it is possible to choose a value x_1 of x, such that for all values of $x > x_1$, we shall have

$$\Theta\,(x) = \int_x^\infty v\,(x)\,dx \qquad \ldots \qquad \ldots \qquad \ldots \quad (5{\cdot}3)$$

an ε function; that is to say $\Theta\,(x) \to 0$ with $1/x$.

We have obviously $\quad V = V(x) + \Theta(x). \ \ldots \qquad \ldots \qquad \ldots \quad (5{\cdot}31)$

Now,

$$W\,(x) = \int_0^x u(y)\ V(x-y)\,dy \qquad \text{from (5·31)}$$

$$= \int_0^x u(y)[V - \Theta(x-y)]dy \quad \text{from (5·31)}$$

$$= VU(x) - \int_0^x u\,(y)\Theta(x-y)dy$$

$$= VU(x) - H(x), \text{ say.} \qquad \ldots \qquad \ldots \qquad \ldots \quad (5{\cdot}4)$$

Consequently,

$$|\ H(x)\ | \leq \int_0^x |\ u(y)\ | \times |\ \Theta(x-y)\ |\ dy.$$

Now since $\Theta\,(x) \to 0$ with $1/x$, the numbers, $\Theta_1\,(x)$ [which means the modulus of $\Theta(x)$] can have at best a finite upper limit, H, say; and in addition, be such that for all values of x greater than some number, say x_2, $\Theta_1(x) < \varepsilon$.

Hence $\quad |\ H\,(x)\ | < \left[\int_0^{x-x_2} dy\ u_1(y)\varepsilon + \int_{x-x_2}^x u_1\,(y)\ H\ dy\right.$

$$\leq \varepsilon.\ U + H[U(x) - U(x-x_2)]$$

$$\therefore \lim_{x\to\infty} |\ H(x)\ | \leq \varepsilon.\ \overline{U} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (5{\cdot}5)$$

which tends to zero because $U(x)$ is convergent and when x is indefinitely great $U(x) = U(x-x_3)$.

Consequently, proceeding in (5·4) to the limit as $x \to \infty$,

$$\lim_{x \to \infty} \int w(x)\, dx = \lim_{x \to \infty} W(x)$$

$$= \lim_{x \to \infty} V.U(x) - H(x) = U.V. \quad \ldots \quad \ldots \quad \ldots \quad (5{\cdot}6)$$

That is to say,

$$\int_0^\infty w\,(x)\, dx$$

converges, under these circumstances, to the value U·V.

Hence the analogue of Merten's theorem is proved. It may happen that in some cases W is also absolutely convergent.

By proceeding in a similar manner, but replacing the signs of equality in (5·4) etc., by inequalities on either side, we have "the analogue of Hardy-Merten's theorem" for the infinite integrals (Th. 4° of § II), viz.

If $\int_0^\infty u_1\,(x)\, dx \to U$, and $\int_0^\infty v\,(x) dx$ oscillates finitely between the limits V_1 and V_2 and $\lim_{x \to \infty} v\,(x) \to 0$,

then

$$\int_0^\infty d\,x \int_0^x u\,(y)\, v\,(x-y)\, dy$$

also oscillates finitely between the limits $U\,V_1$ and $U\,V_2$; and may take any value between these limits.

VI. The Analogue of Hardy's first Theorem.

In this theorem, as well as in the theorems following, we do not assume the absolute convergence of the integrals U and V; they should of course be convergent. We shall then write down the integral

$$\int_0^\infty w(x)\, dx,$$

and see what additional conditions could be set upon $u\,(x)$ and $v\,(x)$, in order that the integral just mentioned may be convergent.

Consider,

$$W(x) = \int_0^x u(y)\, V(x-y)\, dy \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}5)$$

$$= \int_0^{\xi} u(y)\, V(x-y)\, dy + \int_{\xi}^{x} u(y)\, V(x-y)\, dy, \text{ where } \xi \text{ is some internal point}$$

$$= I_1 + I_2, \text{ say.} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6{\cdot}1)$$

Now $|I_2| = \int_{\xi}^{x} |u(y)| \cdot |V(x-y)|\, dy$

Since $V(x-y)$ has a definite finite limit, as $y \to \infty$, it has a finite upper limit, say H_2, in the interval $\xi \leq y \leq x$. Also, let υ be the greatest of the values of $u_1(y)$ in the same interval. Then

$$|I_2| = \int_{\xi}^{x} \upsilon\, H_2\, dy.$$

$$= H_2\, \upsilon\, (x-\xi). \quad \ldots \quad \ldots \quad \ldots \quad (6{\cdot}2)$$

Let us now choose $\xi = \lambda_1 x$, where λ_1 is a proper fraction, so that x, ξ, $x-\xi$ all tend to infinity together.
Consequently,

$$\lim_{x \to \infty} |I_2| \leq H_2 (1-\lambda_1) \lim_{x \to \infty} x\, \upsilon$$

$$\leq K_2 \lim_{x \to \infty} x\, \upsilon, \text{ say.} \quad \ldots \quad \ldots \quad (6{\cdot}21)$$

Now consider I_1, where

$$I_1 = \int_0^{\xi} u(y)\, V(x-y)\, dy.$$

We can easily see that we have

$$I_1 = \int_0^{\xi} u(y)\, dy \times \int_0^{x-\xi} v(y)\, dy + \int_0^{\xi} U(y)\, v(x-y)\, dy$$

$$= 1 + I_3, \text{ say.} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6{\cdot}3)$$

[For, differentiating with respect to ξ, we have

$$u(\xi)\,V(x-\xi)=u(\xi)\int_0^{x-\xi} v(y)\,dy - v(x-\xi)\int_0^{\xi} u(y)\,dy + v(x-\xi)\,U(\xi),$$

which is true.]

Now $|I_3| \leq \int_0^{\xi} |U(y)| \, |v(x-y)|\,dy.$

Since $U(y)$ has a definite limit U as $y \to \infty$, it has a finite upper limit, say H_1, in the interval $0 \leq y \leq \xi$. Also, let υ be the greatest of the values of $v_1(x-y)$ in the same interval; we have already chosen ξ, so that it tend to ∞ with x.

Hence

$$\lim_{x\to\infty} |I_3| \leq H_1 \lim_{x\to\infty} \xi\upsilon$$

$$\leq K_1 \lim_{x\to\infty} x\,V. \tag{6.41}$$

Moreover since, in the limit that $x\to\infty$, ξ and $x-\xi$ tend to the same limit,

$$\lim_{x\to\infty} I \to \int_0^{\infty} u(y)\,dx \times \int_0^{\infty} v(x)\,dx$$

$$\to U\,V. \tag{6.5}$$

Now proceeding with (6·1.) we find,

$$\lim_{x\to\infty} W(x) = \lim_{x\to\infty} I+I_2+I_3.$$

$$\left.\begin{aligned} &= UV+H_2 \lim_{x\to\infty} \nu(x-\xi). \\ &\quad +H_1 \lim_{x\to\infty} \upsilon\xi \end{aligned}\right\} \tag{6.6}$$

$$\left.\begin{aligned} &= UV+K_2 \lim_{x\to\infty} x\nu \\ &\quad +K_1 \lim_{x\to\infty} \upsilon x \end{aligned}\right\} \tag{6.61}$$

$$= U\,V,$$

where K_1 and K_2 are some constants.

The above will be true if $\lim_{x\to\infty} x\nu = 0$ and $\lim_{x\to\infty} \upsilon x = \infty$,

that is to say, if $\lim_{x\to\infty} x\,u(x)=0$

and $\lim_{x\to\infty} x v(x)=0$ (6·7)

Hence Hardy's First theorem may be stated as follows :—

If $\int_0^\infty u(x)\,dx$ and $\int_0^\infty v(x)\,dx$ are convergent and have the values U and V; and in addition (6·7) are also satisfied, the integral $\int_0^\infty dx \int_0^x u(y)\,v(x-y)\,dy$ converges to the value U. V.

Ex.

$$\int_0^\infty dx \int_0^x \frac{(x-y)^{\alpha-1}}{(1+x^2)\,(1+x-y)}\,dy$$

$$= \frac{\pi^2}{2}\ \mathrm{cosec}\ \alpha\pi. \qquad 0<\alpha<1.$$

Here $\lim_{x\to\infty} x.\frac{x^{\alpha-1}}{1+x} \to 0$

and $\lim_{x\to\infty} x\frac{1}{1+x^2} \to 0.$

VII. Analogue of Hardy's second Theorem.

Hardy's second theorem is a generalisation of the first and is easily obtained from (6·6).

In § VI, it was made clear that the internal point ξ, which is at our command has to be so chosen that x, ξ, and $x-\xi$ have all to tend to infinity together; and in that article we took the simplest case $\xi=\lambda x$. Now there exists* a whole scheme of functions, say $\psi(x)$, *viz.*

$\log x$, $l_2(x)$ [*i.e.*, $lg_1 lg\ x$]; or $l^\alpha(x)$, $l_2^\beta(x)\,l_3^\gamma(x)\ldots_s$;

of which $\log x$ is typical, which are such that they tend to infinity with x; and if we set ξ of the same order of greatness as say,

* Cf. Hardy's Tract *Orders of Infinity*.

$\frac{x}{\psi(x)}$ and remember that $\lim_{x\to\infty} \frac{x}{\log x} \to\infty$,

we secure the three conditions that we want.

Now resuming (6·6), we have,

$$\lim_{x\to\infty} W(x)-UV < H \lim_{x\to\infty} [\nu.(x-\varepsilon)+\varepsilon.\upsilon] \quad \ldots \quad \ldots \quad (6.6)$$

where H is the greater of the two, H_1 and H_2.

Now $\underline{\nu}$ ψ and $\underline{\upsilon}$ can obviously be chosen such that the limit of the expression in squared brackets is zero.

$$\text{If now } x\,\psi(x)u(x)\to 0 \quad \text{with } \frac{1}{x} \quad \ldots \quad \ldots \quad (7\,1)$$
$$\text{and } x\ v(x)/\psi(x)\to 0$$

We have

$$\lim_{x\to\infty} \left[W(x)-UV \right]$$
$$< H.\,\varepsilon \cdot \lim \left\{ \frac{1}{\psi(x)} - \frac{\xi}{x\psi(x)} + \frac{\xi\psi(x)}{x} \right\} \to 0.$$

It is obvious that we can choose an x, say x_4, so large that corresponding to an arbitrarly assigned small positive number ε, we shall have both

$$\nu < \frac{\varepsilon}{x\psi(x)} \ \&\ \upsilon < \frac{\varepsilon\,\psi(x)}{x} \text{ for all values of } x, \text{ greater than } x_4$$

[*Seeing that ν has been obtained § VI, as the greatest of the moduli of $u(y)$ in the interval (ξ, x), it is possible te obtain the inequality

$$\nu < \frac{\varepsilon}{\xi\psi(\xi)}$$

also. In that case the argument needs only slight alteration, thus

$$\left[\lim_{x\to\infty} W(x)-UV \right]$$
$$< H.\,\varepsilon \lim \left\{ \frac{x}{\xi\psi(\xi)} - \frac{1}{\psi(\xi)} + \frac{\xi\psi(x)}{x} \right\}$$
$$< H.\,\varepsilon \lim \left\{ 1 + \frac{\psi(x)}{\psi\left(\frac{x}{\psi(x)}\right)} \right\}$$
$$< 2\ H.\,\varepsilon \text{ which tends to zero.}$$

Hence "*the analogue of the second* theorem of Hardy,":

If U and V are convergent and in addition (7·1) are also satisfied the integral

$$\int_0^\infty dx \int_0^x v(x)u(x-y)dy,$$

converges to the value UV. (7·3)

The simplest case of course is to take $\psi(x) = \log x$.

VIII. The Analogue of Hardy's third Theorem.

The theorem is as follows :—

If $\int_0^\infty u(x)\,dx$ and $\int_0^\infty v(x)\,dx$ are convergent respectively to the values U and V, and in addition $|xu(x)| < C_1$ and $|xv(x)| < C_2$ where C_1 and C_2 are constants, then $\int_0^\infty w(x)\,dx$ converges to the value U V.

The proof of this depends upon a lemma analogous to Tauber—Pringsheim's theorem * (usually called the converse of Abel's theorem), *viz.* that a series Σa_n may be inferred to be convergent if $\frac{\Sigma A_n}{n}$ exists and

$$\lim_{n\to\infty} \frac{1}{n}(a_1 + 2a_2 + \ldots na_n) = 0.$$

Let us set $F(x) = \int_0^x f(y)\,dy$,

and integrate by parts

$$\frac{1}{x}\int_0^x y f(y)\,dy = F(x) - \frac{1}{x}\int_0^x F(y)\,dy$$

$$= F(x) - \frac{1}{x}\int_0^x dy \int_0^y f(z)dz \qquad (8.11)$$

provided the function behaves all right at the lower limit.

* [Bromwich : *p.* 251., Ex, 28, where other references are given].

This relation enables us to infer that

$$\lim_{x\to\infty} F(x) = \int_0^\infty f(y)\,dy,$$

exists and is equal to a number F if

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x yf(y)dy \longrightarrow 0 \quad \ldots \quad \ldots \quad (5{\cdot}19)$$

and

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x dy \int_0^y f(z)dz \longrightarrow F. \quad \ldots \quad \ldots \quad (8{\cdot}13)$$

In fact, the last two are necessary and sufficient conditions for the convergence of the integral $\int_0^\infty f(x)dx$ to F.

We shall now proceed with our main theorem : we have to show that the integral $\int_0^\infty w(x)dx$ is convergent. Hence we have got to show that

$$\frac{1}{x}\int_0^x y\, w(y)dy \to 0 \text{ with } \frac{1}{x}.$$

It is convenient to write

$$\overline{W}(x) = \int_0^x y\, w(y)dy \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8{\cdot}2)$$

and analogous notation for $\overline{V}$ and $\overline{U}$.

Also we can easily verify that

$$\overline{W}(x) = \int_0^x y\, u(y)\overline{V}(x-y)dy + \int_0^x yv(y)\overline{U}(x-y)dy$$

$$= I_2 + I_1, \text{ say.}$$

Now $|I_1| \leq \int_0^x |yv(y)| \cdot |\overline{U}(x-y)|\,dy.$

and since $U(x)$ is convergent, it follows that we can choose a number x_5 of x such that for all values of x greater than x_5, we can have $U(x-y) = U + \varepsilon$, when y lies in the interval considered.

Also, since V is a convergent integral

$\int_0^x y\, v\,(y)$ is of the (small o) order of x, by (8·11).

Hence

$$| I_1 | \leq \int_0^x [y\, v\,(y)]\, [U + \varepsilon_1]\, dy$$

$$\leq U\, o\,(x) + \varepsilon_1 \int_0^x | y\, v\,(y) |\, dy$$

so that

$$\lim_{x \to o} \left| \frac{1}{x} I_1 \right| \to o, \text{ if } \varepsilon_1 \int_0^x | yv(y) |\, dy \text{ can be made to tend to zero.}$$

This latter will certainly happen if

$| yv(y) | <$ a constant C_2, when $y \to \infty$. (8·3)

Similarly we can show that

$$\lim_{x \to \infty} \left| \frac{1}{x} I_3 \right| \to 0.$$

if $| yu(y) | <$ a constant C_1, when $y \to \infty$. (8·31)

Hence under these two conditions

$$\lim_{x \to \infty} \frac{1}{x} \overline{W}(x) = 0 \quad ... \quad ... \quad ... \quad (8·11)$$

Similarly it may be shown that (8·12) is satisfied.

Hence the integral $\int_0^\infty w\ (x)\ dx$ is convergent and has the value U V.

Finally, we may permit ourselves to make the following observation. Hardy has replaced (in the paper cited already) our (8·13) with the condition

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x dy\int_0^y f(z)\,dz \to 0$$

for the convergence of (8·1). This no doubt is sufficient, but as we have shown in our proof this is not necessary. It seems to us therefore that we can replace his (*small o*) inequality with a (capital O) inequality ; and unless his result is a casual slip, it seems to us that the latter case is the one of greater interest. We need only take a simple illustration with $f(x) = e^{-x}$, to bear out the force of this remark ; for, with this example,

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x dy\int_0^y e^{-z}\,dz$$

$$= \lim_{x\to\infty} \frac{1}{x}\int_0^x (1-e^{-y})\,dy$$

$$= \lim_{x\to\infty} \frac{1}{x}\,[x+e^{-x}-1]$$

which is 1 and not zero.

CERTAIN DEFINITE INTEGRALS AND SERIES

Connected with Bernoulli's Numbers.

[By C. Krishnamachari, M.A., Assistant Professor,

College of Engineering, Bangalore.]

[In solving Q. 913 of Prof. Sanjana, I arrived at certain definite integrals which I set as Q. 951. A study of Whittaker, page 126, suggested transformations of these integrals. Mr. Bhimasena Rao pointed out that the solution of Q. 609 by Mr. Madhava assumes that

$$\sum_1^\infty \frac{r^{4n+1}}{e^{2\pi r}-1} = \int_0^\infty \frac{x^{4n+1}}{e^{2\pi x}-1}\,dx$$

and suggested that I may solve Questions 387 and 609 by the theory of residues. A successful attempt at this problem led to some further results. These form the subject of the present paper.]

Part I.

§ 1. (1) Lemma— $\lim_{y\to 0} y^r (\log y)^s = 0$

where r and s are any positive integers. This can be proved by the substitution

$$y = e^{-x}.$$

(2) We easily obtain by integration by parts and using the above lemma, or by means of the transformation suggested in the above lemma, that

$$\int_0^1 (\log y)^{2n}\, y^m\, dy = \frac{(2n)!}{(m+1)^{2n+1}}$$

$$\int_0^1 (\log y)^{2n-1}\, y^m\, dy = -\frac{(2n-1)!}{(m+1)^{2n}}.$$

§ 2. Expanding $\frac{1}{1+y^2}$ and integrating term by term, we get

$$\int_0^1 \frac{(\log y)^{2n}}{1+y^2}\,dy = (2n)!\left\{\frac{1}{1^{2n+1}} - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \ldots\right\}$$

$$= \frac{E_n}{2}\left(\frac{\pi}{2}\right)^{2n+1}$$

whence by some obvious transformations, we obtain the following integrals for E_n:

$$E_n = \left(\frac{2}{\pi}\right)^{2n+1} \int_0^{\frac{1}{2}\pi} \left(\log \tan \frac{x}{2}\right)^{2n} dx \qquad [y = \tan \frac{x}{2}$$

$$= \left(\frac{2}{\pi}\right)^{2n+1} \int_0^{\infty} \frac{z^{2n}}{\cosh z}\, dz. \qquad [y = e^{-z}.$$

$$= 4^{n+1} \int_0^{\infty} \frac{z^{2n}}{e^{\pi z} + e^{-\pi z}} dz.$$

Since we have

$$\sec z = \Sigma\, E_n \frac{z^{2n}}{(2n)\,!}$$

we obtain by substituting for E_n the above integral,

$$\sec z = 2 \int_0^{\infty} \frac{e^{2zt} + e^{-2zt}}{e^{\pi t} + e^{-\pi t}}\, dt.$$

§ 3. We similarly obtain

$$\int_0^1 \frac{(\log y)^{2n-1}}{1+y}\, dy = -\frac{(2^{2n-1}-1)}{2n} B_n\, \pi^{2n},$$

whence as before

$$B_n = -\frac{2n}{\pi^{2n}(2^{2n-1}-1)} \int_0^{\frac{1}{4}\pi} \frac{(\log \tan x)^{2n-1}}{\cos x\,(\cos x + \sin x)} dx \qquad [y = \tan x.$$

$$= +\frac{2n}{2^{2n-1}-1} \int_0^{\infty} \frac{z^{2n-1}}{e^{\pi z}+1} dz. \qquad [y = e^{-\pi z}$$

Since

$$\operatorname{cosec} z = \frac{1}{z} + 2 \sum_1^{\infty} \frac{2^{2n-1}-1}{(2n)\,!} B_n\, z^{2n-1}, \text{ (Hobson, page 363),}$$

we obtain

$$\operatorname{cosec} x = \frac{1}{x} + \int_0^{\infty} \frac{e^{xz} - e^{-xz}}{e^{\pi z}+1} dz.$$

§ 4. We also have

$$\int_0^1 \frac{(\log y)^{2n-1}}{1-y^2}dy = -\frac{\pi^{2n}(2^{2n}-1)}{4n}B_n.$$

$$\therefore \quad B_n = -\frac{4n}{(2^{2n}-1)\,\pi^{2n}}\int_0^{\frac{\pi}{4}} \frac{(\log \tan x)^{2n-1}}{\cos 2x}dx \quad [y=\tan x.$$

$$= \frac{2n}{\pi^{2n}(2^{2n}-1)}\int_0^{\infty} \frac{z^{2n-1}}{\sinh z}dz. \quad [\text{Whittaker page 126, } y=e^{-z}.$$

$$= \frac{2n}{2^{2n}-1}\int_0^{\infty} \frac{z^{2n-1}}{\sinh \pi z}dz.$$

From the expansion

$$\frac{1}{e^z+1} = \frac{1}{2} - \sum_1^{\infty}(-1)^{n-1}\frac{Bn}{\underline{|2n}}\,(2^{2n-1})\,x^{2n-1} \quad \text{(Bromwich, page 235)},$$

we obtain

$$\frac{1}{e^x+1} = \frac{1}{2} - \int_0^{\infty} \frac{\sin xz}{\sinh \pi z}\,dz.$$

Also since

$$\tan x = \sum_1^{\infty} \frac{2^{2n}\,(2^{2n-1})}{\underline{|2n}}\,B_n\,x^{2n-1}, \text{ (Hobson, page 363)}$$

we at once obtain

$$\tan x = 2\int_0^{\infty} \frac{e^{2xz}-e^{-2xz}}{e^{\pi z}-e^{-\pi z}}\,dz.$$

§5. From

$$\int_0^1 \frac{(\log y)^{2n-1}}{1-y}\,dy = -\frac{2^{2n-1}\,\pi^{2n}}{2n}\,B_n,$$

we derive

$$B_n = -\frac{2n}{\pi^{2n}\,z^{2n-1}}\int_0^{\frac{\pi}{4}} \frac{(\log \tan x)^{2n-1}}{\cos x\,(\cos x - \sin x)}\,dx.$$

$$= 4n\int_0^\infty \frac{t^{2n-1}}{e^{2\pi t}-1}\,dt. \qquad [y = e^{-2\pi t}$$

$$= \frac{2n\,(2n-1)}{\pi}\int_0^\infty t^{2n-2}\log\left(\frac{1}{1-e^{-2\pi t}}\right)dt,$$

by an easy integration by parts and noting that

$$\frac{1}{e^{2\pi t}-1} = \frac{1}{2\pi}\frac{d}{dt}\log\frac{1}{1-e^{-2\pi t}}.$$

From the expansion

$$\frac{1}{e^z-1} = -\frac{1}{2}+\frac{1}{z}+\sum_1^\infty (-1)^{n-1}\,B_n\,\frac{z^{2n}}{(2n)!}$$

we obtain the well-known result

$$\frac{1}{e^x-1} = -\frac{1}{2}+\frac{1}{x}+2\int_0^\infty \frac{\sin xt}{e^{2\pi t}-1}\,dt.$$

And since it is known that, (see Bromwich page 233,)

$$\frac{1}{e^x-1} = \frac{1}{x}-\frac{1}{2}+\sum\frac{2y}{y^2+4n^2\pi^2},$$

we get

$$\sum_1^\infty \frac{y}{y^2+4n^2\pi^2} = \int_0^\infty \frac{\sin xt}{e^{2\pi t}-1}\,dt.$$

§ 6. A simple consideration shows that the formulae can be transformed into one another. Thus

$$\frac{B_n}{4n} = \int_0^\infty \frac{t^{2n-1}}{e^{2\pi t}-1}dt$$

$$= \frac{1}{2}\int_0^\infty t^{2n-1}\left(\frac{1}{e^{\pi t}-1}-\frac{1}{e^{\pi t}+1}\right)dt.$$

$$= 2^{2n-1}\left\{\int_0^\infty t^{2n-1}\left(\frac{1}{e^{2\pi t}-1}-\frac{1}{e^{2\pi t}+1}\right)dt.\right.$$

$$= 2^{2n-1}\frac{B_n}{4n} - 2^{2n-1}\int_0^\infty \frac{t^{2n-1}}{e^{2\pi t}+1}\,dt.$$

$$\therefore \int_0^\infty \frac{t^{2n-1}}{e^{2\pi t}+1}\,dt = \left(1-\frac{1}{2^{2n-1}}\right)\frac{B_n}{4n}.$$

But $\coth \pi x = \dfrac{1+e^{-2\pi x}}{1-e^{-2\pi x}}$.

$$= 1+2\,e^{-2\pi x}+2\,e^{-4\pi x}+2\,e^{-6\pi x}+\ldots\ldots$$

And $$\int_0^\infty x^{2n-1}\left(\frac{1}{\tanh \pi x}-1\right)dx = 2\int_0^\infty x^{2n-1}\left(e^{-2\pi x}+e^{-4\pi x}\ldots\right)dx$$

$$= \frac{B_n}{2n}.$$

$$\therefore \quad B_n = 2n\int_0^\infty \frac{x^{2n-1}}{\sinh \pi x\,(\cosh \pi x+\sinh \pi x)}\,dx.$$

[*Note.*—In passing, the following integrals which can be easily transformed into integrals having 0 and ∞ for limits may be noted:—

$$\frac{1}{1^{2n+1}}+\frac{1}{2^{2n+1}}+\frac{1}{3^{2n+1}}+\ldots = \frac{1}{(2n)!}\int_0^1 \frac{(\log y)^{2n}}{1-y}$$

$$= \frac{1}{(2n)!}\int_0^\infty \frac{x^{2n}}{e^x-1}\,dx.$$

$$\frac{1}{1^{2n+1}}-\frac{1}{2^{2n+1}}+\frac{1}{3^{2n+1}}-\ldots = \frac{1}{(2n)!}\int_0^\infty \frac{x^{2n}}{e^x+1}\,dx.$$

$$\frac{1}{1^{2n+1}}+\frac{1}{2^{2n+1}}+\ldots+\frac{1}{k^{2n+1}} = \frac{1}{(2n)!}\int_0^1 (\log y)^{2n}\frac{1-y^k}{1-y}\,dy.$$

$$\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\ldots\ldots+\frac{1}{k^{2n}} = -\frac{1}{(2n-1)!}\int_0^1 (\log y)^{2n-1}\frac{1-y^k}{1-y}\,dy.$$

Part II.

§ 8. We can make a very interesting application of the theory of residues to establish some well-known results. Thus

$$\frac{z}{e^z-1}=1-\frac{z}{2}+\sum_1^\infty (-1)^{n-1}\frac{B_n}{(2n)!}z^{2n}.$$

$\therefore$ $(-1)^{n-1}\dfrac{B_n}{(2n)!}$ is the residue of the function

$$f(z)=\frac{1}{z^{2n}(e^z-1)}$$ at the origin.

The other poles of $f(z)$ are $z=\pm 2\pi ri$ $(r=1, 2, \ldots\ldots)$.

The residue at $\pm 2\pi ri$ is

$$\frac{1}{(2\pi ri)^{2n}}\cdot\frac{1}{e^{2\pi ri}}=(-1)^n\frac{1}{(2\pi r)^{2n}}.$$

And $z f(z)\to 0$ as $z\to\infty$ uniformly. Hence the sum of the residues of the function at all its poles is zero. Hence we obtain

$$(-1)^{n-1}\frac{Bn}{\underline{2n}}+(-1)^n\,2\frac{1}{(2\pi r)^{2n}}=0$$

i.e. $$B_n=\frac{2\,(2n)!}{(2\pi)^{2n}}\sum_1^\infty\frac{1}{r^{2n}}.$$ (a well-known result).

Similarly consider the function

$$\sec z=\sum_0^\infty E_n\frac{z^{2n}}{(2n)!}.$$

The poles are $z=\pm(2n+1)\dfrac{\pi}{2}$, and observing that $\dfrac{E_n}{(2n)!}$ is the residue of $f(z)=\dfrac{1}{z^{2n+1}}\cdot\dfrac{1}{\cos z}$ at the origin, we obtain as before

$$E_n=\frac{2\,(2n)!}{\left(\frac{\pi}{2}\right)^{2n+1}}\sum_1^\infty\frac{(-1)^{r-1}}{(2r-1)^{2n+1}}.$$ (a familiar result).

Note.—§ 8 is suggested in MacRobert's *Theory of Functions.*

Similarly, we can find an expression for $\phi_n(z)$, the Bernoullian polynomial function of degree n, *viz.*

$$\phi_n(x) = x^n - \frac{n}{2} x^{n-1} + B_1 \binom{n}{2} x^{n-2} - B_4 \binom{n}{4} xn-4+ \ldots\ldots$$

We have

$$z \frac{e^{xz}-1}{e^z-1} = \sum_1^\infty \phi_n(x) \frac{z^n}{n!}.$$

We see at once that $\phi_n(x)/(n)!$ is the residue of the function

$$f(z) = \frac{1}{z^n} \cdot \frac{e^{xz}-1}{e^z-1}, \text{ at the origin.}$$

Suppose x positive and less than unity. Then $z f(z) \to 0$ uniformly as $z \to \infty$ and hence the sum of the residues of $f(z)$ at all its poles on the plane is zero.

The residue at $(2\pi ir)$ is

$$\frac{e^{2\pi irx}-1}{(2\pi ir)^n e^{2\pi ir}}$$

and the residue at $-2\pi ir$ is

$$\frac{e^{-2\pi irx}-1}{(-2\pi ir)^n e^{-2\pi ir}}.$$

If n is even, say $2m$, the sum of the two residues

$$= (-1)^m \frac{2 \cos 2\pi rx - 2}{(2\pi r)^{2m}}.$$

Hence the sum of the residues at all the poles

$$= (-1)^m 2 \Sigma \frac{\cos 2\pi rx - 1}{(2\pi r)^{2m}}$$

$$= (-1)^m 2 \sum_1^\infty \frac{\cos 2\pi rx}{(2\pi r)^{2m}} + (-1)^{m+1} 2 \Sigma \frac{1}{(2\pi r)^{2m}}.$$

If n is odd, say $2m+1$, the sum of the residues at $\pm 2\pi ir$

$$= (-1)^n \frac{2 \sin 2\pi rx}{(2\pi r)^{2n+1}}$$

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Hence we obtain

$$\phi_{2n}(x) = (-1)^{n+1} 2\ (2n)! \sum_{1}^{\infty} \frac{\cos 2\pi r x}{(2\pi r)^{2n}} + (-1)^n B_n$$

$$\phi_{2n+1}(x) = (-1)^{n+1} 2\ (2n+1)! \sum_{1}^{\infty} \frac{\sin 2\pi r x}{(2\pi r)^{2n+1}}.$$

[A trigonometric method of proving these results is suggested in Ex. 16, page 256, Bromwich.]

Similarly we know that

$$\frac{e^x z}{e^z + 1} = \sum_{0}^{\infty} \psi_n(x) \frac{z^n}{n!}$$ (See Bromwich, page 240).

Proceeding as before, we obtain two analogous formulæ, *viz.*

$$\psi_{2n}(x) = (-1)^n\ 2(2n)!\ \sum_{0}^{\infty} \frac{\sin (2r+1)\ \pi x}{[(2r+1)\ \pi]^{2n+1}}$$

$$\psi_{2n+1}(x) = (-1)^{n+1}\ 2\ (2n+1)! \sum_{0}^{\infty} \frac{\cos (2r+1)\ \pi x}{[(2r+1)\ \pi]^{2n+2}}.$$

§ 9. We shall next pass on to the problems of Mr. Ramanujan (Q. 387) and Mr. Bhimasena Rao (Q. 609) referred to in the Introduction.

Consider the function $f(z) = \dfrac{z^{4n+1}}{(e^{2\pi z} - 1)(e^{-2\pi i z} - 1)}$

Its poles are $\pm r$, $\pm ri$. Consider $\int f(z)\, dz$ taken round the indented quadrant of a circle with the origin as center and semi-circles round the poles. It is zero since the function has no poles inside the contour. Along a semicircle round α, write $z = \alpha + \delta e^{i\theta}$.

$$\int f(z)\, dz = \int_{\pi}^{0} f(\alpha + \delta e^{i\theta})\ i\delta e^{i\theta} d\theta.$$

Making $\delta \to 0$ and noting that $f(\alpha + \delta e^{i\theta})\ \delta e^{i\theta} \to \alpha'$, uniformly as $\delta \to 0$ (Whittaker, p. 117, § 6.23), we see that

$$\int f(z)\, dz = -\pi i \alpha', \quad [\alpha' \text{ being the residue at } \alpha].$$

For the integral along the semi circle round 'ir' the upper and lower limits are $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ and the integral similarly equals $-\pi i a'$ (a' being the residue at ir).

Along the quadrants, the integrals are zero since $z\,f(z)\to 0$ uniformly as $z\to\infty$ and as $z\to 0$. Making the radii of the semi circles tend to zero, and the radii of the quadrants zero and infinity, we obtain

$$\int_{O'A} + \int_{AB} + \int_{BO''} f(z)\,dz - \pi i \varepsilon a' + \int_{o''o'} f(z)\,dz = 0.$$

The residues at 'r' and 'ri' are equal and

$$= -\frac{1}{2\pi i}\,\frac{r^{4n+1}}{e^{2\pi r}-1}.$$

Along OA, z is real and $= x$.

$$\int_{OA'} = \int_0^\infty \frac{x^{4n+1}}{\left(e^{2\pi x}-1\right)\left(e^{-2\pi i x}-1\right)}\,dx.$$

Similarly $$\int_{EO'} = \int_\infty^0 \frac{(ix)^{4n+1}}{\left(e^{2\pi i x}-1\right)\left(e^{2\pi x}-1\right)}\,i\,dx$$

$$= \int_0^\infty \frac{x^{4n+1}}{\left(e^{2\pi x}-1\right)\left(e^{2\pi i x}-1\right)}\,dx.$$

Hence the above equation, from the theory of residues gives

$$\int_0^\infty \frac{x^{4n+1}}{e^{2\pi x}-1}\,dx\left[\frac{1}{e^{2\pi i x}-1}+\frac{1}{e^{-2\pi i x}-1}\right] = -\Sigma\frac{r^{4n+1}}{e^{2\pi r}-1}.$$

i. e. $$\Sigma\frac{r^{4n+1}}{e^{2\pi r}-1} = \int_0^\infty \frac{x^{4n+1}}{e^{2\pi x}-1}\,dx \qquad \text{(after simplication)}$$

$$= \frac{B_{n+1}}{4\,(2n+1)}.$$

In the case of Mr. Ramanujan's Q. 387, which is evidently a particular case of this, there is an additional term $-\frac{1}{8\pi}$ on the right. This is because, for the function $f(z)$ the origin is a pole and the residue at this pole is zero if n is any positive integer, and $-\frac{1}{4\pi^2 i}$ if n is zero.

In this particular case, we have at the origin

$$\int_{o''o'} f(z)\,dz = \int_{\frac{1}{2}\pi}^{0} f(\delta e^{i\theta})\; i\delta e^{i\theta} d\theta.$$

$$= -\frac{\pi}{2} i a,$$

where a' is the residue at the origin.

Hence the additional term on the right; and we obtain thus Mr. Ramanujan's problem

$$\sum \frac{r}{e^{2\pi r} - 1} = \frac{1}{24} - \frac{1}{8\pi}.$$

To obtain the second of Mr. Bhimasena Rao's results, examine the function

$$\int \frac{z^{4n+1}}{(e^{\pi z}+1)\,(e^{-\pi i z}+1)}\,dz.$$

taken round the same contour with semi circles round the poles $(2r+1)$, $(2r+1)i$,

Residue at $(2r+1)$ = residue at $(2r+1)i$

$$= \frac{(2r+1)^{4n+1}}{\pi i \{ e^{(2r+1)\pi} + 1 \}}.$$

Hence we obtain that the sum of the integrals round all the semi circles is equal to

$$= -2. \frac{(2r+1)^{4n+1}}{\{ e^{(2r+1)\pi} + 1 \}}.$$

$$\int_{OA} + \int_{AB} + \int_{BO} = 2 \sum \frac{(2r+1)^{4n+1}}{e^{(2r+1)\pi} + 1}$$

Now $\int_{AB} \to 0$ since $z f(z) \to 0$ as $z \to \infty$,

$$\int_{OA} + \int_{BO} = \int_{0}^{\infty} \frac{x^{4n+1}}{e^{\pi x} + 1} \left[\frac{1}{e^{-\pi i x} + 1} + \frac{1}{e^{\pi i x} + 1} \right] dx$$

$$= \int_0^\infty \frac{x^{4n+1}}{e^{\pi x}+1}\, dx$$

$$= B_{2n+1}\frac{(2^{4n+1}-1)}{2(2n+1)} \text{ (see § 3 above).}$$

Hence the second result.

§ 9*a*. A very interesting result, which I think is highly probable is obtained by examining $\int f(z)dz$ round the same contour, when

$$f(z) = \frac{z^n}{(e^{2\pi z}-1)(e^{-2\pi iz}-1)}$$

Residue at 'r' + residue at 'ri'

$$= -\frac{1}{2\pi i}\frac{r^n}{e^{2\pi r}-1}\left[1+(-1)^{\frac{n-1}{2}}\right].$$

Exactly as before

$$\int_{OA} = \int_0^\infty \frac{x^n}{(e^{2\pi x}-1)(e^{-2\pi ix}-1)}\, dx$$

$$\int_{BO} = i^{n-1}\int_0^\infty \frac{x^n}{(e^{2\pi ix}-1)(e^{2\pi x}-1)}\, dx.$$

Hence we obtain

$$-\frac{(1+i^{n-1})}{2}\sum\frac{r^n}{e^{2\pi r}-1} = \int_0^\infty \frac{x^n}{(e^{2\pi x}-1)(e^{-2\pi ix}-1)}\, dx$$

$$+ i^{n-1}\int_0^\infty \frac{x^n}{(e^{2\pi ix}-1)(e^{2\pi x}-1)}\, dx.$$

Putting $n=4m+1$, we obtain the above result.

Putting $n=4m$, we get

$$-\frac{(1-i)}{2}\sum\frac{r^{4m}}{e^{2\pi r}-1}=\int_0^\infty \frac{ix^{4m}}{(e^{2\pi n}-1)(e^{-2\pi ix}-1)}dx$$

$$-i\int_0^\infty \frac{x^{4m}}{(e^{2\pi ix}-1)(e^{2\pi x}-1)}dx$$

$$=\int_0^\infty \frac{x^{4m}}{e^{2\pi ix}-1}dx\left[-\frac{1}{2}+\frac{i}{2}\cot\ \pi x-i\left(-\frac{1}{2}-\frac{i}{2}\cot\ \pi x\right)\right].$$

$$=\int_0^\infty \frac{x^{4m}}{e^{2\pi x}-1}\,dx.\ (i-1)\ \tfrac{1}{2}\ (1+\cot\ \pi x).$$

Whence we obtain

$$\sum\frac{r^{4m}}{e^{2\pi r}-1}=\int_0^\infty \frac{x^{4m}}{e^{2\pi x}-1}\ (1+\cot\ \pi x)\ dx.$$

Putting $n=4m-1$, since $1+i^{n-1}=1+i^{4m-2}=0$, we get

$$\int_0^\infty \frac{x^{4m-1}}{(e^{2\pi x}-1)(e^{-2\pi ix}-1)}\ dx=\int_0^\infty \frac{x^{4m-1}}{(e^{2\pi xi}-1)(e^{2\pi x}-1)}dx$$

i.e.

$$\int_0^\infty \frac{x^{4m-1}}{e^{2\pi x}-1}\ dx\ \left[-\frac{1}{2}+\frac{i}{2}\cot\ \pi x\right]$$

$$=\int_0^\infty \frac{x^{4m-1}}{e^{2\pi x}-1}\ dx\ \left[-\frac{1}{2}-\frac{i}{2}\cot\ \pi x\right];$$

Whence we obtain the result

$$\int_0^\infty \frac{x^{4m-1}}{e^{2\pi x}-1}\ \cot\ \pi x=0.$$

Putting $n=4m+2$, we obtain exactly as before

$$\sum \frac{r^{4m+2}}{e^{2\pi r}-1} = \int_0^\infty \frac{x^{4m+2}}{e^{2\pi n}-1}(1+\cot \pi x)dx.$$

Putting $n=4m-2$, we obtain the same result with $4m-2$ substituted for $4m+2$.

It thus appears that $\int_0^\infty \frac{x^{4m-1}}{e^{2\pi x}-1} \cot \pi x \, dx = 0$

and we are led to the general conclusion that

$$\sum_1^\infty \frac{r^n}{e^{2\pi r}-1} = \int_0^\infty \frac{x^n}{e^{2\pi x}-1}. \quad \{1+f(x)\}\, dx,$$

where

$f(x)=0$ if $n=4m+1$, and $=\cot \pi x$. if $n=4m$ or $4m \pm 2$,

except when $n=1$, which is Mr. Ramanujan's problem.

§ 10. The above method will now be utilized for obtaining some further results.

(*a*) Examine $\int \frac{z^{4n+1}}{\left(e^{\pi z}-e^{-\pi z}\right)\left(e^{\pi i z}-e^{-\pi i z}\right)} dz$ taken round an infinite circle with the origin as centre.

Residue at each of the points r, ri, $-r$, $-ri$ is

$$(-1)^r \frac{r^{4n+1}}{e^{\pi r}-e^{-\pi r}} \quad \frac{1}{2\pi i}.$$

Residue at the origin is as before zero, if n is any positive integer and $+\frac{1}{4\pi^2 i}$, if n is zero.

Hence we obtain

$$\sum_1^\infty (-1)^r \frac{r^{4n+1}}{e^{\pi r}-e^{-\pi r}} = 0;$$

$$\sum_1^\infty (-1)^r \frac{r}{e^{\pi r}-e^{-\pi r}} = -\frac{1}{8\pi}.$$

(*b*) Consider $\int \frac{f(z)}{\sin \pi z} dz$ taken over a circle of radius $R \to \infty$.

The poles are $\pm r$, 0, and the poles of $f(z)$.

Residue at $r = (-1)^r \frac{f(r)}{\pi}$.

Residue at $-r = (-1)^r \frac{f(-r)}{\pi}$.

Residue at $0 = \frac{f(0)}{\pi}$.

$$\therefore \quad \sum_1^\infty (-1)^r \frac{f(r)+f(-r)}{\pi} + \frac{f(0)}{\pi} = \frac{1}{2\pi i} \int \frac{f(z)}{\sin \pi x} dz - \mathrm{R} \frac{f(z)}{\sin \pi z}$$

where the last term on the right means that we should take the sum of the residues of the function with respect to the poles of $f(z)$.

If in particular $f(z)$ is such that $z f(z) \to 0$ as $z \to \infty$, we get

$$\frac{f(0)}{\pi} + \sum_1^\infty (-1)^r \frac{f(r)+f(-r)}{\pi} = -\mathrm{R} \frac{f(z)}{\sin \pi z}.$$

Ex. In particular let $f(z)$ be an even function. We get

$$f(0) + 2 \sum_1^\infty f(r) = -\pi \mathrm{R} \frac{f(z)}{\sin \pi z},$$

Put $f(z) = \frac{x}{z^2 - x^2}$. We get

$$-\frac{1}{x} + 2 \sum_1^\infty (-1)^r \frac{x}{r^2 - x^2} = -\mathrm{R} \frac{\pi f(z)}{\sin \pi z} = -\frac{\pi}{\sin \pi x}$$

i.e.

$$\pi \operatorname{cosec} \pi x = \frac{1}{x} + 2 \sum_1^\infty (-1)^{r-1} \frac{x}{r^2 - x^2}.$$

(Ex. 19, page 190, Bromwich)

(*c*) Consider similarly $\int \frac{f(z)}{\cos \pi z} dz$.

We obtain if $z f(z) \to 0$ as $z \to \infty$.

$$\sum_0^\infty (-1)^{r+1} \left[f\left(\frac{2r+1}{2}\right) - f\left(-\frac{2r+1}{2}\right) \right] = -\pi \mathrm{R} \frac{f(z)}{\cos \pi z}.$$

If $f(z)$ is an odd function, this gives

$$\sum_{0}^{\infty} (-1)^{r+1} \; 2 f\left(\frac{2r+1}{2}\right) = -\pi \, \mathrm{R} \frac{f(z)}{\cos \pi z}$$

Ex. Let $f(z) = \frac{z}{z^2 - x^2}$. We get

$$\sum_{0}^{\infty} (-1)^{r+1} \frac{2r+1}{\left(\frac{2r+1}{2}\right)^2 - x^2} = \pi \sec \pi x \quad \text{(Page 190, Bromwich).}$$

(d) Consider $\int \frac{f(z)}{\tan \pi z} dz$. We get

$$\sum_{-\infty}^{\infty} f(r) + \pi \, \mathrm{R} \frac{f(z)}{\tan \pi z} = 0.$$

Ex. 1. Let $f(z) = \frac{a}{(x-z)(a-z)}$. We get

$$\sum_{-\infty}^{\infty} \frac{a-x}{(x-n)(a-n)} = \pi (\cot \pi x - \cot \pi a).$$

Ex 2. Let $f(z) = \frac{x}{z(x-z)}$.

We see here that the origin is a pole for $f(z)$ as well as for $\cot \pi z$.

Hence we should examine the residue at the origin of $\frac{f(z)}{\tan \pi z}$ and substitute this for $f(0)$.

$$\frac{f(z)}{\tan \pi z} = \frac{x}{z(x-z)} \frac{\cos \pi z}{\sin \pi z}$$

$$= \frac{1}{z} \left(1 + \frac{z}{x} + \ldots\right) \left(1 - \frac{\pi^2 z^2}{\underline{|2}} + \ldots\right) \frac{1}{\pi z} \left(1 + \frac{\pi^2 z^2}{\underline{|3}} - \ldots\right)$$

$$= \frac{1}{\pi x z} + \ldots\ldots \text{terms containing other powers.}$$

$\therefore$ Residue at the origin $= \frac{1}{\pi x}$.

Residue at $z = x$ is $-\cot \pi x$.

$\therefore$ $$-\sum_{\infty}^{\infty}{}' \left(\frac{1}{x-n} + \frac{1}{n}\right) + \frac{1}{x} = \pi \cot \pi x.$$

Ex. 3. $f(z)=\dfrac{1}{(x-z)^2}$. We get

$$-\sum_{-\infty}^{\infty}\frac{1}{(x-n)^2}=-\pi\,\mathrm{R}\frac{\cot\pi z}{(x-z)^2}$$

We have to find the residue of $\dfrac{\cot\pi z}{(x-z)^2}$ at $z=x$. Now let

$$\phi(x)=\cot\pi x.$$

Write $\cot\pi z=\cot[\pi x-\pi(x-z)]$

$$=\phi(z+y),\text{ where } y=-x+z.$$

$$=\phi(x)+y\,\phi'(x)+\frac{y^2}{\underline{2}}\phi''(x)+\frac{y^3}{\underline{3}}\phi'''(x)+\ldots\ldots$$

Again $\phi(x)=\cot\pi x$

$\phi'(x)=-\pi\operatorname{cosec}^2\pi x$

$\phi''(x)=2\pi^2\operatorname{cosec}^2\pi x\cot\pi x$

$\phi'''(x)=+2\pi^3[3\operatorname{cosec}^4\pi x-2\operatorname{cosec}^2\pi x]$,

Hence we obtain

$$\frac{\cot\pi z}{(x-z)^2}=\frac{\phi(x-y)}{y^2}.$$

$\therefore$ Residue at $z=x$ is $-\phi'(x)$.

$\therefore$ $$\sum_{-\infty}^{\infty}\frac{1}{(x-n)^2}=\pi^2\operatorname{cosec}^2\pi x.$$

For $\dfrac{\cot\pi z}{(x-z)^3}$ residue at $z=x$ is $-\dfrac{1}{2!}\phi''(x)$.

$\therefore$ $$\sum_{-\infty}^{\infty}\frac{1}{(x-n)^3}=+\frac{\pi}{\underline{2}}\phi''(x)$$

$$=+\pi^3\operatorname{cosec}^2\pi x\cot\pi x.$$

For $\dfrac{\cot\pi z}{(x-z)^4}$, residue at $z=x$ is $\dfrac{1}{3!}\phi'''(x)$.

$\therefore$ $$\sum_{-\infty}^{\infty}\frac{1}{(x-n)^4}=\pi^4\left(\operatorname{cosec}^4\pi x-\frac{2}{3}\operatorname{cosec}^2\pi x\right).$$

Here put $x=\dfrac{1}{2}-\dfrac{1}{2n}$. We get Q. 1000, first part. The Second part is similarly obtained.

Put $x=\dfrac{1}{2n}$, we get Q. 1001, first part. The other parts are similarly obtained.

More generally, we obtain

$$\sum_{-\infty}^{\infty} \frac{1}{(x-n)^r} = \frac{\pi}{\lfloor r-1} \cdot (-1)^r \cdot \frac{d^{r-1}}{dx^{r-1}} \cot \pi x.$$

[See Bromwich, Ex. 17, Page 190].

Ex. 4 Let $f(z) = \frac{x}{z^2-x^2}$. $f(z)$ is an even function.

We get

$$-\frac{1}{x} + \sum_{1}^{\infty} \frac{2x}{n^2-x^2} = -\pi \, \mathrm{R} \, \frac{x}{z^2-x^2} \cot \pi z.$$

$$= -\frac{\pi}{\tan \pi x}.$$

$$\therefore \quad \pi \cot \pi x = \frac{1}{x} + \sum_{1}^{\infty} \frac{2x}{x^2-n^2}. \quad \text{(a well known result).}$$

§ 11. Next consider the residues of functions of the type $f(x) \frac{\sin ax}{\sin \pi x}$, etc. We require the following well-known theorem in the theory of residues ; viz. If C is a circle of radius $R \to \infty$, and $f(z)$ is a function such that $z f(z) \to k$ as $z \to \infty$, we have

$$\int f(z)\, dz = \int z f(z) \frac{dz}{z}.$$

$$= k \int \frac{dz}{z} = 2\pi i \, k.$$

(See : Forsyth, *Theory of Functions*).

§ 12. Consider the function $f(x) \frac{\sinh ax}{\sinh \pi x}$.

The poles are $\pm ri$, 0. Residue at the origin is 0 since $\sinh 0 = 0$.

The sum of the residues at 'ri' and ∞ 'ri' is

$$\frac{f(ri)\, i \sin ra}{\pi (-1)^r} - \frac{f(-ri)\, i \sin ra}{(-1)^r \pi}$$

$$= (-1)^r \frac{1}{\pi} i \sin ra \, \{ f(ri) - f(-ri) \}.$$

$$\therefore \quad \sum_{1}^{\infty} (-1)^r \, i \, [f(ri) - f(-ri)] \sin ra + \mathrm{R} \, f(z) \frac{\sinh az}{\sinh \pi z} = 0.$$

Ex. 1. Let $f(z)=\dfrac{1}{x-z}$.

$$z f(z)\to -1 \text{ as } z\to\infty.$$

Hence we obtain

$$\frac{\sinh ax}{\sinh x\pi}=\Sigma\ (-1)^{r-1}\frac{\sin ar}{\pi}\frac{2r}{x^2+r^2}.$$

[Ex. 23, Page 257 Bromwich].

Ex. 2. If we similarly examine $f(x)\dfrac{\cosh az}{\sinh \pi z}$, we obtain

$$\mathrm{R}\, f(z)\frac{\cosh az}{\cosh \pi z}+f(o)+\Sigma(-1)^r\frac{f(ri)+f(-ri)}{\pi}\cos ar=0.$$

Put $f(z)=\dfrac{1}{x-z}$. We get

$$\frac{\cosh ax}{\sinh \pi x}=\frac{1}{x}+\Sigma\ (-1)^r\ \frac{2x\cos ar}{\pi(x^2+r^2)}.$$

[Ex. 23, page 257 Bromwich].

§ 13. Similarly by taking $f(x)\dfrac{\sin ax}{\sin \pi x}$ and $f(x)$, $\dfrac{\cos ax}{\sin \pi x}$ we get

$$\sum_1^\infty (-1)^r\ \{f(r)-f(-r)\}\sin ar=-\pi\,\mathrm{R}\frac{\sin az}{\sin \pi z}f(z)$$

$$f(0)+\sum_1^\infty (-1)^r\{f(r)+f(-r)\}\cos ar=-\pi\,\mathrm{R}\frac{\cos az}{\sin \pi z}f(z).$$

If $f(x)=\dfrac{1}{x-z}$, we get

$$\sum_1^\infty (-1)^r\frac{\sin ar}{x^2-r^2}=\frac{\pi}{2}\frac{\sin ax}{\sin \pi x}.$$

$$\sum_1^\infty (-1)^r\frac{\cos ar}{x^2-r^2}=-\frac{1}{2x^2}+\frac{\pi}{2x}\frac{\cos ax}{\sin \pi x}.$$

———

ASTRONOMICAL NOTES.

1. *Comets.* A comet (1919 *f*) was recorded on two plates taken on December 10, 1919 at Hamburg by Dr. Baade. It is probably identical with Holmes's Comet whose return to perhelion should have happened about November 30.

Another comet (1919 *g*) was discovered by Mr. J. F. Skjellerup at the Cape of Good Hope on December 18. It was also observed at the Royal Observatory by Mr. Woodgate. The position of the comet March 11 will be R.A. 20H. 36M. Decl. 8°. 7 N. It should be visible early in the morning during this month.

A new object of about the 10th magnitude (provisionally designated 1920 *a*) was discovered by Senor Comas Sola at Barcelona. It was first reported to be a comet but now appears to be a minor planet.

2. *Perturbation of Neptune.* In Harward Circular 215 Prof. W.H. Pickering draws the attention of Astronomers to the fact that Neptune is now gradually deviating from its computed position in the manner that it should do, if disturbed by an unknown outer planet. The deviation so far observed amounts to a little over 2″ and if this is due to the perturbing action of an outer planet it is expected to increase to about 15″ in the course of a few decades. The unknown planet is believed to be small, about terrestrial size and to be travelling in a highly elliptical orbit, the present distance from the Sun being about 68 times that of the earth. Prof. Pickering states that it should be located at present in R. A. 6H. 35M. and Dec. 23° N., Mag. about 15 and as it is surrounded by numerous brighter stars in the Milky Way, identification will be difficult.

3. *Eclipses.* In the year 1920, there will be four eclipses, the first two of which will occur during the month of May.

i. A total Eclipse of the Moon, May 3, 1920; the beginning of the eclipse will be generally visible in India.

The circumstances of the eclipse are as under:—

Moon enters Penumbra May 3	4—19 A.M.	Indian Standard Time.
Moon enters Umbra	5—31 „	
Total eclipse begins	6—45 „	
Total eclipse ends	7—57 „	
Moon leaves Umbra	9—11 „	
Moon leaves Penumbra	10—23 „	

Magnitude of the eclipse will be 1·224 (the moon's diameter being taken as unity).

ij. A partial Eclipse of the Sun, May 18, 1920, invisible in India: the eclipse will be generally visible in Australia and the southern part of the Indian Ocean.

NIZAMIAH OBSERVATORY,
HYDERABAD.

T. P. Bhaskara Sastri.

SOLUTIONS.

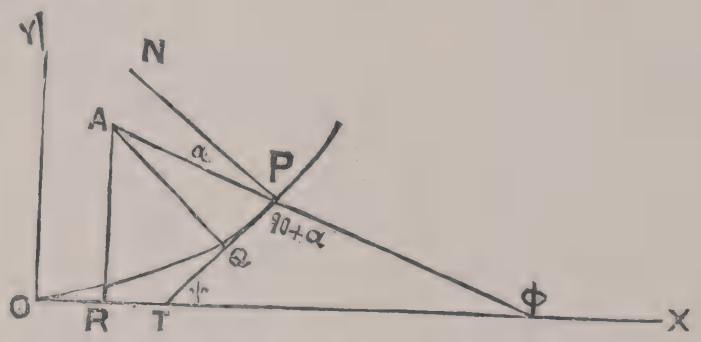
Question 819.

(K. Appukuttan Eradi, M.A.) :—If ρ, ρ' be the radii of curvature at corresponding points of a curve and its α—evolute, where α is a function of the arc measured from a fixed point on the curve, show that

$$\rho'\left(\frac{1}{\rho}+\frac{d\alpha}{ds}\right)+\sin\alpha=\frac{d}{ds}\left\{\frac{\cos\alpha}{\frac{1}{\rho}+\frac{d\alpha}{ds}}\right\}.$$

Solution by Martyn M. Thomas, K. B. Madhava and K. R. Rama Iyer.

Choose the tangent and normal at the fixed point O, as fixed axes of coordinates ; let A be a point on the α—evolute corresponding to P on the given curve.



Let the coordinates of A be (X, Y) referred to the fixed axes, and (u, v) referred to the tangent and normal at P. Let P be (x, y).

From A drop perpendiculars AR, AQ on OX and the tangent PT.

Now, X = OR = Projection of OA on X axis

= algebraical sum of the projections of OP, PQ, QA on X—axis.

$$= x-u\cos\psi-v\sin\psi.$$

Similarly, $Y = y-u\sin\psi+v\cos\psi$.

$$\therefore\quad \frac{dX}{ds} = \cos\psi-\frac{du}{ds}\cos\psi+u\sin\psi\frac{1}{\rho}-\frac{dv}{ds}\sin\psi-v\cos\psi\frac{1}{\rho}$$

$$=\cos\psi\left(1-\frac{du}{ds}-\frac{v}{\rho}\right)+\sin\psi\left(\frac{u}{\rho}-\frac{dv}{ds}\right) \quad \ldots \quad \ldots \quad (1)$$

$$\frac{dY}{ds}=\sin\psi\left(1-\frac{du}{ds}-\frac{v}{\rho}\right)-\cos\psi\left(\frac{u}{\rho}-\frac{dv}{ds}\right) \quad \ldots \quad \ldots \quad (2)$$

$$\therefore\quad \left(\frac{dS}{ds}\right)^2=\left(\frac{dX}{ds}\right)^2+\left(\frac{dY}{ds}\right)^2=\left(1-\frac{du}{ds}-\frac{v}{\rho}\right)^2+\left(\frac{u}{\rho}-\frac{dv}{ds}\right)^2 \quad \ldots \quad (3)$$

Also $\tan(90+\alpha+\psi)=\frac{dY}{dX}=\frac{\left(1-\frac{du}{ds}-\frac{v}{\rho}\right)-\cot\psi\left(\frac{u}{\rho}-\frac{dv}{ds}\right)}{\cot\psi\left(1-\frac{du}{ds}-\frac{v}{\rho}\right)+\left(\frac{u}{\rho}-\frac{dv}{ds}\right)}$

$$\therefore \quad \cot\alpha=\frac{\frac{u}{\rho}-\frac{dv}{ds}}{1-\frac{du}{ds}-\frac{v}{\rho}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

$\therefore$ Substituting in (3) from (4),

$$\left(\frac{dS}{ds}\right)^2=\left(\frac{u}{\rho}-\frac{dv}{ds}\right)^2\left(\tan^2\alpha+1\right)$$

$$\therefore \quad \frac{dS}{ds}=\pm\sec\alpha\left(\frac{u}{\rho}-\frac{dv}{ds}\right) \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Since $u=v\tan\alpha$, $\frac{du}{ds}=\tan\alpha.\frac{dv}{ds}+v\sec^2\alpha.\frac{d\alpha}{ds}$.

$\therefore$ Substituting for u and $\frac{du}{ds}$ in (4), we have,

$$1=v\sec^2\alpha\left(\frac{1}{\rho}+\frac{d\alpha}{ds}\right)$$

$$\therefore \quad v=\frac{\cos^2\alpha}{\frac{1}{\rho}+\frac{d\alpha}{ds}} \;\&\; u=\frac{\sin\alpha\cos\alpha}{\frac{1}{\rho}+\frac{d\alpha}{ds}}.$$

Taking the lower sign in (5),

$$\frac{dS}{ds}=\sec\alpha\left(\frac{dv}{ds}-\frac{u}{\rho}\right)$$

$$\therefore \quad \frac{dS}{d\phi}.\frac{d\phi}{ds}=\sec\alpha.\frac{d}{ds}\left\{\frac{\cos^2\alpha}{\frac{1}{\rho}+\frac{d\alpha}{ds}}\right\}-\frac{1}{\rho}.\frac{\sin\alpha}{\frac{1}{\rho}+\frac{d\alpha}{ds}}$$

where $\phi=90+\alpha+\psi$.

$$\therefore \quad \rho'.\frac{d\alpha+d\psi}{ds}=\cos\alpha.\frac{d}{ds}\left\{\frac{1}{\frac{1}{\rho}+\frac{d\alpha}{ds}}\right\}-\frac{2\sin\alpha}{\frac{1}{\rho}+\frac{d\alpha}{ds}}.\frac{d\alpha}{ds}-\frac{1}{\rho}.\frac{\sin\alpha}{\frac{1}{\rho}+\frac{d\alpha}{ds}}$$

$$\therefore \quad \rho'\left(\frac{d\alpha}{ds}+\frac{1}{\rho}\right)=\cos\alpha.\frac{d}{ds}\left\{\frac{1}{\frac{1}{\rho}+\frac{d\alpha}{ds}}\right\}-\frac{\sin\alpha}{\frac{1}{\rho}+\frac{d\alpha}{ds}}.\frac{d\alpha}{ds}-\sin\alpha$$

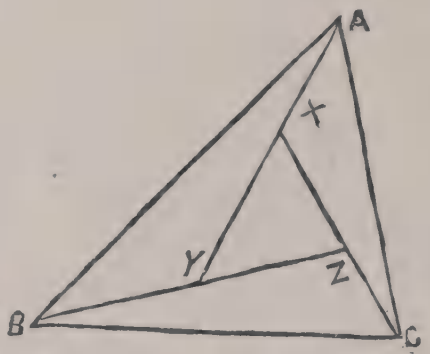
$$\therefore \quad \rho\left(\frac{d\alpha}{ds}+\frac{1}{\rho}\right)+\sin\alpha=\frac{d}{ds}\left\{\frac{\cos\alpha}{\frac{1}{\rho}+\frac{d\alpha}{ds}}\right\}.$$

Question 821.

(M. K. KEWALRAMANI.):—If through A, B, C lines AXY, BYZ, CZX are drawn so as to make the same angle θ with AB, BC, CA respectively and form the triangle XYZ, prove that $\rho = 2\sigma \sin(\omega - \theta)$ where ω is the Brocard angle of the triangle ABC, σ the radius of the first Lemoine Circle of ABC, and ρ is the radius of the Cosine Circle of XYZ.

Solution by N. Sankara Aiyar.

$\angle YXZ = \angle XAC + \angle XCA = A$. Hence the triangles ABC & XYZ are similar.



Now, $\frac{CX}{CA} = \frac{\sin(A-\theta)}{\sin A}$ *i.e.* $CX = \frac{b \sin(A-\theta)}{\sin A}$ & $CZ = \frac{a \sin\theta}{\sin C}$

$$\therefore \quad XZ = 2R\left\{\frac{\sin B \sin(A-\theta)}{\sin A} - \frac{\sin A \sin\theta}{\sin C}\right\}$$

$$= 2R\left\{\sin B \cos\theta - \sin B \cot A \sin\theta - \frac{\sin(B+C)\sin\theta}{\sin C}\right\},$$

$$= 2R \sin B \{\cos\theta - \sin\theta (\cot A + \cot B + \cot C)\},$$

$$= 2R \sin B \{\cos\theta - \sin\theta \cot\omega\},$$

$$= 2R \sin B \cdot \frac{\sin(\omega-\theta)}{\sin\omega}.$$

The ratio of similitude is therefore $\frac{\sin(\omega-\theta)}{\sin\omega}$.

In any triangle $\rho = 2\sigma \sin\omega$ and hence $\rho = 2\sigma \sin(\omega-\theta)$ for $\triangle s$ ABC & XYZ.

Question 827.

(A. C. L. WILKINSON.):—If a skew surface is defined by

$$x = z, \quad y = bx + \beta, \quad z = cx + \gamma$$

where b, c, β, γ are functions of t, and if the axis of x is the generator corresponding to $t = 0$, the origin the central point of this

generator, and $z=0$ the tangent plane at the origin; then the hyperboloid of closest contact along the generator $t=0$ is given by

$$2c'\,\beta'^2\,z=c'\,\gamma''\,y^2+2c'^2\,\beta'\,xy-b''\,\beta'\,z^2-(c'\,\beta''-c''\,\beta')\,yz$$

where the values of the differential co-efficients of b, c, β, γ are for $t=0$.

Solution by K. R. Rama Aiyar.

The hyperboloid of closest contact has three consecutive generators in common with the surface; and so the axis of x is a generator and the origin is the central point. So the equation to the hyperboloid assumes the form

$$\lambda\,y^2+\mu\,z^2+\nu\,z+f\,yz+h\,xy=0. \qquad \ldots \qquad \ldots \quad (A)$$

Evidently when $t=0$, $b=\beta=c=\gamma=0'$ and the generator adjacent to the x axis is given by $y=(b'\,x+\beta')\,\delta t$, $z=\delta t\,(c'\,x+\gamma')$ where b', β'. c', γ' are values corresponding to $t=0$. Since the central point is the origin, the x axis, a generator and the z axis, the normal at the origin the S. D between x axis and the adjacent generator is the y axis. Since, therefore, $x=0$, $z=0$ both intersects and is perpendicular to $y=dt\,(b'\,x+\beta')$, $z=dt\,(c'\,x+\gamma')$ we find

$$b'=\gamma'=0.$$

Since the surface has triple contact with the hyperboloid along the x axis we have the following equations satisfying A

(1) $y=0$, $z=0$

(2) $y=\beta'\,\delta t$, $z=\delta t.\ c'\,x.$

(3) $y=(\beta''+b''\,x)\dfrac{\delta t^2}{2}+\beta'\,\delta t$

$$z=(\gamma''+c''\,x)\frac{\delta t^2}{2}+c'x\,\delta t$$

Hence we find omitting higher powers of δt than necessary

$$(h\beta'+\nu c')\,x\,\delta t=0$$

and $x\,\delta t\,(h\beta'+\nu c')+\delta t^2\left\{x^2\,(h\frac{b''}{2}+\mu c'^2)+x\,(h\frac{\beta''}{2}+\nu\frac{c''}{2}+f\,\beta'\,c')\right.$

$$\left.+\lambda\beta'^2+\nu\frac{\gamma''}{2}\right\}=0$$

for all values of x.

$\therefore$ $\quad h\,\beta'+\nu c'=0,$ or $\quad -\dfrac{h}{c'}=\dfrac{\nu}{\beta'}=\kappa$ (say)

$$h\,\frac{\beta''}{2}+\mu c'^2=0,$$

$$h\,\frac{\beta''}{2}+\nu\,\frac{c''}{2}+f\,\beta'\,c'=0$$

$$\lambda \beta'^2 + \nu \frac{\gamma''}{2} = 0.$$

$$\therefore \quad \nu = \kappa\beta', \; h = -\kappa c', \; \mu = \kappa \frac{\beta}{2c'} \; \lambda = -\kappa \frac{\gamma}{2\beta'}$$

$$f = -\frac{\kappa}{\beta' c'} \left\{ \frac{c''\beta'}{2} - \frac{c'\beta''}{2} \right\}.$$

Hence the equation (A) reduces to

$$2c' \beta'^2 z = c' \gamma'' y^2 + 2c'^2 \beta' xy - b'' \beta' z^2 - (c' \beta'' - c'' \beta') yz$$

where the values of the differential co-efficients of b, c, β, γ are for $t=0$

Question 850.

(A. C. L. WILKINSON) :—Prove that

$$\int_0^\infty \frac{\sin x \sinh x}{\cosh x + \cos x} \frac{dx}{x} = \frac{\pi}{4},$$

and

$$\int_0^\infty \frac{\cos x \sinh x}{\cosh x + \cos x} \frac{dx}{x} = 0.$$

Solution by K. B. Madhava.

Consider the integral $\int \frac{e^{z(1+i)}}{e^z + e^{iz}} \frac{dz}{z}$ over the contour consisting of (i) the x—axis from O to R (ii) the quadrant of the circle centre the origin and radius R, and (iii) the y—axis from R to O ; and make R tend to infinity. On this contour there are no infinities of the integrand, for its only poles are given by $z = (2n+1)\, i\pi + iz$, *ie* the line $y + x = 0$. Putting now on (i) $z = x$; on (ii) $z = \mathrm{R}\, e^{i\theta}$ and on (iii) $z = iy$ we have as R becomes infinite

$$\int_0^\infty \frac{e^{x(1+i)}}{e^x + e^{ix}} \frac{dx}{x} + \int_0^{\frac{1}{2}\pi} \frac{e^{\mathrm{R}e^{i\theta}}}{1 + e^{\mathrm{R}(1-i)\, e^{i\theta}}} id\theta + \int_\infty^0 \frac{e^{ix(1+i)}}{e^{ix} + e^{-x}} \frac{dx}{x} = 0.$$

On the infinite quadrant, the second integral is in modulus $= 1$ and therefore, collecting the first and the third integrals together we have

$$\int_0^\infty \frac{e^{2ix}(e^x - e^{-x})}{e^{ix}(e^{ix} + e^{-ix} + e^x + e^{-x})} \frac{dx}{x} = \frac{\pi}{4}.$$

Hence separating the real and imaginary parts we have

$$\int_0^\infty \frac{\sinh x \cos x}{\cosh x + \cos x} \frac{dx}{x} = 0$$

and

$$\int_0^\infty \frac{\sinh x \sin x}{\cosh x + \cos x} \frac{dx}{x} = \frac{\pi}{4}.$$

Question 864.

(M. K. KEWALRAMANI):—Prove that the primitive of the differential equation

$$\frac{d^2y}{dx^2}-a^2y+\frac{p(1-p)}{x^2}y=0$$

can be put into the form

$$y=x^p\left[\mathrm{A}\int_0^1\frac{\cosh arx}{(1-r^2)^{1-p}}dr+\mathrm{B}\int_1^\infty\frac{e^{-arx}}{(r^2-1)^{1-p}}dr\right]$$

where p is always positive.

Solution by C. Krishnamachari and S. V. Venkatachala Iyer.

Let $$y=\int e^{-rt}\,x^m\,\mathrm{R}\,dr,$$

where t is a function of x alone, and R a function of r alone. Differentiating and substituting in the equation, we easily obtain

$$-\int r\frac{d^2t}{dx^2}x^m\,e^{-rt}\,\mathrm{R}\,dr-2m\int x^{m-1}\,e^{-rt}\frac{dt}{dx}\,\mathrm{R}\,rdr$$

$$+\int e^{-rt}\,\mathrm{R}\,x^m\,dr\left\{r^2\left(\frac{dt}{dx}\right)^2-a^2\right\}$$

$$+\{m(m-1)+p(1-p)\}\int e^{-rt}\,x^{m-2}\,\mathrm{R}\,dr=0.$$

Put $m=p$ and $t=ax$.

The first and last terms vanish. The equation now reduces to

$$-2pa\int x^{p-1}\,e^{-arx}\,\mathrm{R}\,rdr+\int e^{-arx}\,x^p\,\mathrm{R}\,dr\,(r^2-1)\,a^2=0.$$

Integrating the second term by parts, we have

$$-2pa\int x^{p-1}\,e^{-arx}\,\mathrm{R}\,rdr-\left[e^{-arx}\,x^{p-1}\,\mathrm{R}\,(r^2-1)\,a\right]$$

$$+\int e^{-arx}\,x^{p-1}\,a\frac{d}{dr}\{\mathrm{R}\,(r^2-1)\}\,dr=0,$$

i.e. $$a\int e^{-arx}\,x^{p-1}\,\mathrm{R}dr\left[\frac{d}{dr}\{\mathrm{R}\,(r^2-1)\}-2p\,\mathrm{R}\,r\right]$$

$$+[e^{-arx}\,x^{p-1}\,\mathrm{R}\,(r^2-1)\,a]=0 \qquad \ldots \quad (1)$$

Now R is found from the equation

$$\frac{d}{dr}\{\mathrm{R}\,(r^2-1)\}-2p\,\mathrm{R}\,k=0$$

i.e. $$\mathrm{R}=(r^2-1)^{p-1}.$$

The limits of integration should be so chosen that the last term in (1) vanishes. The values of r are ± 1 and ∞.

$\therefore$ The primitive is

$$Ax^p \int_{-1} \frac{e^{-arx}}{(1-r^2)^{1-p}}\,dr + Bx^p \int_1^\infty \frac{e^{-arx}}{(r^2-1)^{1-p}}\,dr.$$

The first can be written

$$A_x p \left[\int_{-1}^{0} \frac{e^{-arx}}{(1-r^2)^{1-p}}\,dr + \int_0^1 \frac{e^{-arx}}{(1-r^2)^{1-p}} dr \right].$$

Putting $-r$ for r in the first part, we get the required result.

QUESTIONS FOR SOLUTION.

1093. (Martyn M. Thomas, M. A.) :—A fixed ray of light falls on a plane mirror revolving about an axis in its plane. Show that the locus, in the plane, of the point of incidence is a conic, and that the reflected rays generate a ruled conicoid.

1094. (S. R. Ranganathan) :—If the number of A's : the number of B's : the number of C's as $a : b : c$ and if a_1 per cent. and a_2 per cent. of the A's are also B's and C's respectively, discuss the limits between which the percentage of the B's that are also C's should lie.

1095. (R. S. Narasimhan) :—Seven thieves A, B, C, ... secure a sum of rupees, but are obliged to conceal it without counting it. A returns alone, divides the sum into 7 parts, finds that there are 6 rupees over, takes the 6 rupees and one-seventh part and departs. B returns alone and does the same with the diminished sum. He divides it into 7 parts, find that there are 5 rupees over, takes the 5 rupees and one-seventh part and departs. Each does the same the successive remainders being 6, 5, 4, 3, 2, 1, 0. Finally, they all come together and divide the remainder which is a multiple of 7, into 7 equal parts and each takes one part. Find the least number of rupees stolen and the amount that each gets.

NOTICE.

A Conference of the Indian Mathematical Society will be held at Lahore in March next during the Easter Holidays and all those who desire to read papers at the Conference are requested to communicate their intention to the undersigned as early as possible.

D. D. KAPADIA,
2414, *East Street, Poona.*

5th December 1920.

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DOUBLE POINTS AND LINES.

By M. Bhimasena Rao.

THIS note is in continuation of the Author's Notes on the same subject published on pp. 19—21, Vol. IV and pp. 264—8, Vol. X of the Journal of the Indian Mathematical Society, and falls into three parts. The first part deals with orthologic triangles in perspective, and contains a simple construction for finding the double point of such triangles. Analytical formulæ for the double point and lines are given in the second part and applied to obtain the following interesting result :

Any two triangles ABC *and* A'B'C' *and the triangle formed by the radical axes of pairs of circles described on* BC', B'C; CA', C'A; AB', A'B, *have a common double point.*

In the case of two orthologic triangles a simple reciprocal relation is shown to exist between their double point and the point of concurrence of the perpendiculars from the vertices of one on the sides of the other. This relation is looked at from the view of correspondence in the last part wherein the present theory is made use of to discuss geometrically several properties of the invariant triangles occurring in the transformation of a given point into the double point of a given triangle and the pedal triangle of the point.

PART I.

Orthologic Triangles in Perspective.

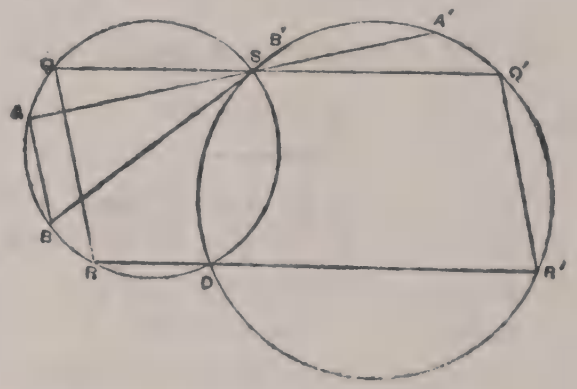
§ 1. *Construction for finding the Double Point.*

Let ABC and A′B′C′ be similar triangles in perspective but not homothetic, and S the centre of perspective. It follows by Elementary Geometry [1] that

(1) S lies on the circumcircles of ABC and A′B′C′ intersecting again at the double point D of the triangles.

(2) Any line through S cuts the circles in corresponding points Q and Q′.

(3) If QR and Q′R′ be parallel chords of the respective circles, RR′ passes through the double point D, and RA, RB, RC are respectively parallel to R′A′, R′B′, R′C′.



It is known [2] that triangles in perspective are conjugate with respect to a conic whose centre is the double point of the triangles.

As an immediate consequence we deduce the following theorem which results by projection :—

THEOREM I.

If two triangles ABC, A′B′C′ are conjugate with respect to a conic, the diameter of the conic through A passes through a point X such that XB′, XC′ are respectively parallel to AB, AC.

It has also been shown [3] that the double point of orthologic triangles ABC and A′B′C′ is the point of concurrence of the radical axes of pairs of circles described on BC′, B′C; CA′, C′A; AB′, A′B, as diameters.

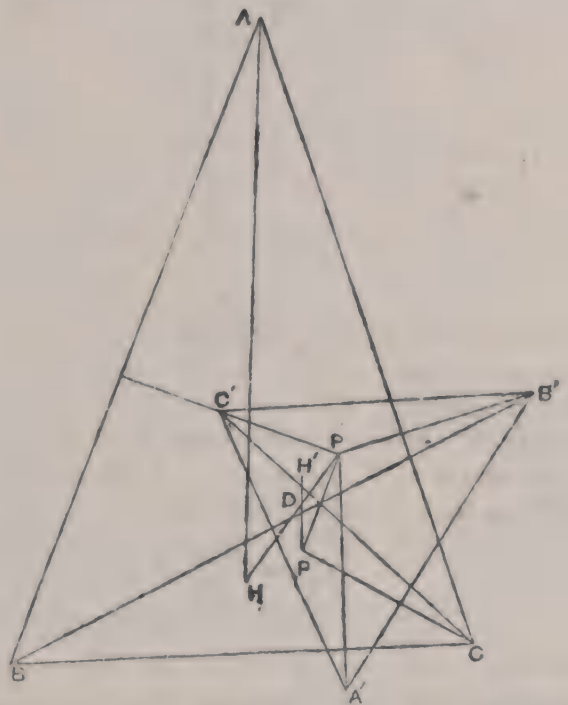
[1] Cf. Lachlan : *Modern Geometry*, p. 180.

[2] Cf. Author's paper "Double Lines"—Journal of the Indian Math. Soc., Vol. X, pp. 264—8.

[3] *Loc. cit.* : *Jour. Ind. Math. Soc.*

If the triangles are also in perspective, the following is a simple construction for finding the double point.

Let H, H′ be the orthocentres of ABC and A′B′C′, and P,P′, the points of concurrence of the perpendiculars from A,B,C on the corresponding sides of A′B′C′, and of the perpendiculars from A′,B′,C′ on the corresponding sides of ABC. Then HP′ and H′P intersect at the double point.



Proof.—It has been shown[4] that P,P′ are corresponding points of ABC and A′B′C′, and that PP′ passes through the centre of perspective. Since AH is parallel to A′P′, it follows from § 1. (3), that HP′ passes through the double point. Similarly H′P passes through the double point.

§ 2. *Some Theorems.*

We shall now proceed to obtain a condition for two orthologic triangles defined in a particular manner to be in perspective. Thus,

THEOREM II.

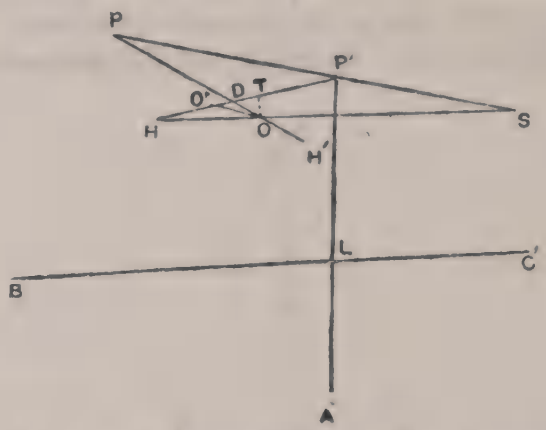
If LMN is the pedal triangle of P′ with respect to ABC, and P′L P′M, P′N be produced to A′B′C′ such that

$$\frac{P'A'}{P'L} = \frac{P'B'}{P'M} = \frac{P'C'}{P'N} = \lambda$$

[4] *Loc. cit.* p. 267—7 (3).

and the join of P′ to its isogonal conjugate meets the Euler's line (OH) in S, then if ABC and A′B′C′ are in perspective

$$\lambda = 2\frac{OS}{HS}.$$



Proof.—Let P be the isogonal conjugate of P′. Then P,P′ are corresponding points of ABC and A′B′C′. If H′ is the ortho-centre of A′B′C,′ then from the construction given in § 1, HP′ and H′P intersect in the double point D. Now the mid. point T of HP′ [5] is the centre of the orthopolar ellipse of P′. Again since LMN and A′B′C′ are homothetic having P′ for their double point, the point O′ on P′T defined by

$$\frac{P'O'}{P'T} = \frac{P'A'}{P'L} = \lambda,$$

corresponds to T of LMN, and therefore to O of ABC. Now O′ lies on HP′; therefore the circumcentre O lies on the corresponding line H′P. Since corresponding points on two corresponding lines through the double point are on parallel rays, OO′ is parallel to PP′.

$$\text{Hence } \frac{O'P'}{HP'} = \frac{OS}{HS}.$$

$$\text{But } HP' = 2TP',$$

$$\therefore \frac{O'P'}{2TP'} = \frac{OS}{HS}.$$

$$\text{Also } \frac{O'P'}{TP'} = \frac{P'A'}{P'L} = \lambda.$$

$$\therefore \lambda = 2\frac{OS}{HS}.$$

[5] The orthopolar ellipse of P′ with respect to ABC, considered as a figure of LMN, corresponds to the circumcircle of ABC. Cf. *Jour. Ind. Math. Soc.*, Vol. IV., p. 110.

The converse of this theorem is also true, *viz.*

THEOREM III.

If the line joining P to its isogonal conjugate with respect to ABC intersects the Euler's line OH in S, and if L, M, N, the feet of the perpendiculars from P on the sides of ABC be produced to A', B', C' such that $\frac{PA'}{PL} = \frac{PB'}{PM} = \frac{PC'}{PN} = 2\frac{OS}{HS} = \lambda$, *the triangle A'B'C' is in perspective with ABC.*

Some particular cases are interesting.

(i) Let $\lambda = 2$,

then S goes to infinity ; and the theorem becomes

THEOREM IV.

If the triangle formed by the reflexions of a point P in the sides of a triangle ABC is in perspective with ABC, the line joining P to its isogonal conjugate is parallel to the Euler's Line.

(ii) Let $\lambda = 1$.

In this case HS = 2OS and S is the orthocentre of the antimedial triangle of ABC. The theorem consequently becomes

THEOREM V.

If the pedal triangle of a point P with respect to a triangle ABC is in perspective with ABC, the line joining P to its isogonal conjugate passes through the orthocentre of the antimedial triangle of ABC.

(iii) Let $\lambda = -1$.

S is here the centroid and the theorem[6] as follows :—

THEOREM VI.

If P is the centre of a conic touching the sides of a triangle ABC at points where the normals are concurrent, the triangle formed by the reflexions of the feet of the perpendiculars from P on the sides of ABC, in P, is in perspective with ABC.

PART II.

§ 1. *Introductory Analytical Formulæ.*

Let ABC and A'B'C' be two triangles. Referred to the triangle ABC, let the normal co-ordinates of A', B', C' be (l_1, m_1, n_1), (l_2, m_2, n_2),

[6] When the line joining a point to its isogonal conjugate passes through the centroid, that point is the centre of a conic touching the sides of a triangle at points where the normals are concurrent. Cf. *Jour. of Ind. Math. Soc.*, Q. 480, Vol. IV.

(l_3, m_3, n_3). If D be the point (α, β, γ), the point corresponding to D for A′B′C′ is

$$\left(\frac{a\alpha l_1}{al_1 + bm_1 + cn_1} + \frac{b\beta l_2}{al_2 + bm_2 + cn_2} + \frac{c\gamma l_3}{al_3 + bm_3 + cn_3} ; \right.$$

with similar expressions for the other two co-ordinates).

On writing a' for $\dfrac{a}{al_1 + bm_1 + cn_1}$,

b' for $\dfrac{b}{al_2 + bm_2 + cn_2}$,

c' for $\dfrac{c}{al_3 + bm_3 + cn_3}$,

the A′B′C′ — correspondent of D is

$$(a'l_1\alpha + b'l_2\beta + c'l_3\gamma, \ldots\ldots)$$

Now if D is to be the double point, it must correspond to itself, that is to say, we should have

$$\begin{aligned} a'l_1\alpha + b'l_2\beta + c'l_3\gamma &= \alpha, \\ a'm_1\alpha + b'm_2\beta + c'm_3\gamma &= \beta, \\ a'n_1\alpha + b'n_2\beta + c'n_3\gamma &= \gamma. \end{aligned}$$

These equations of which only two are independent represent three lines through the double point. On solving, we find the *normal co-ordinates of the double point* to be

$$\begin{aligned} &(al_1 + bm_1 + cn_1)(al_2l_3 + bl_2m_3 + cl_3n_2), \\ &(al_2 + bm_2 + cn_2)(bm_3m_1 + cm_3n_1 + am_1l_3), \\ &(al_3 + bm_3 + cn_3)(cn_1n_2 + an_1l_2 + bn_2m_1). \end{aligned}$$

Similarly to find *the equation of the double lines*:—

Let $L \equiv p\alpha + q\beta + r\gamma = 0$ be a double line. If (XL) denotes the length of the perpendicular from X on L, we have

$$\frac{(A'L)}{(AL)} = \frac{(B'L)}{(BL)} = \frac{(C'L)}{(CL)}.$$

Consequently,

$$\begin{aligned} \frac{a}{p} \cdot \frac{pl_1 + qm_1 + rn_1}{al_1 + bm_1 + cn_1} &= \frac{b}{q} \cdot \frac{pl_2 + qm_2 + rn_2}{al_2 + bm_2 + cn_2} \\ &= \frac{c}{r} \cdot \frac{pl_3 + qm_3 + rn_3}{al_3 + bm_3 + cn_3} = \lambda \text{ say}. \end{aligned}$$

These equations may be written

$$a'(pl_1 + qm_1 + rn_1) = p\lambda \quad \ldots \quad \ldots \quad \ldots \quad (1)$$
$$b'(pl_2 + qm_2 + rn_2) = q\lambda \quad \ldots \quad \ldots \quad \ldots \quad (2)$$
$$c'(pl_3 + qm_3 + rn_3) = r\lambda \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

together with the equation of L,

$$p\alpha + q\beta + r\gamma = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

The elimination of p, q, r, λ from these equations gives the equation of the double lines.

The elimination is most simply effected as follows :—

Multiplying (1), (2), (3) by α, β, γ respectively and adding, we have

$$p(a'l_1\alpha + b'l_2\beta + c'l_3\gamma) + q(a'm_1\alpha + b'm_2\beta + c'n_3\gamma) + r(a'n_1\alpha + b'n_2\beta + c'n_3\gamma) = 0 \quad \ldots \quad (5)$$

If (f, g, h) is the double point, we have

$$pf + qg + rh = 0, \quad \ldots \quad (6)$$

because the double line passes through the double point.

Hence eliminating p, q, r from (4), (5), (6), the equation of the double lines is

$$\begin{vmatrix} a'l_1\alpha + \ldots & a'm_1\alpha + \ldots & a'n_1\alpha + \ldots \\ \alpha & \beta & \gamma \\ f & g & h \end{vmatrix} = 0.$$

or on expansion,

$$\alpha^2 a'(n_1 g - m_1 h) + \beta^2 b'(l_2 h - n_2 f) + \gamma^2 c'(m_3 f - l_3 g) + \beta\gamma(c'l_3 h - b'l_2 g + b'm_2 f - c'n_3 f) + \ldots\ldots\ldots = 0.$$

§ 2. *Co-ordinates of the Double Point (continued.)*

Now, the double point of two triangles is usually defined as the mean centre of the vertices of those triangles for a common system of multiples. The analytical expression for those multiples is obtained from the normal co-ordinates given in § 1. The geometrical intrepretation in terms of the powers with respect to circles described on some associated lines as diameters is interesting, and a convenient proof of it is obtained from the foregoing formulæ.

THEOREM VII.

The double point D of two triangles ABC, A'B'C', is the mean centre of the vertices of either triangle for the multiples D (BC') — D (B'C), D (CA')—D (C'A), D (AB')—D (A'B) *where* X (YZ) *denotes the power of* X *with respect to the circle described on* YZ *as diameter.*

Now if

$$S \equiv (BC'^2 + CA'^2 + AB'^2) - (B'C^2 + C'A^2 + A'B^2)$$

the vanishing of S is the well-known condition for the triangles to be orthologic.

But,

$$S = 2\Delta . \ 4R \sum \frac{m_1 \cos B - n_1 \cos C}{al_1 + bm_1 + cn_1},$$

where Δ is the area of ABC. (1)

$$\therefore \quad S = 4R \sum (m_1 \cos B - n_1 \cos C)$$
$$= 4R \sum \cos A \, (l_2 - l_3).$$

employing absolute co-ordinates.

Again, it is well-known, that the triangle formed by the perpendiculars from A', B', C' on the sides of ABC is similar to ABC, the ratio of similitude, ρ, being equal to $\frac{S}{4\Delta}$. (2)

It is enough, for establishing the theorem, to show that

$$D(AB') - D(A'B) = -\rho ch.$$

where h is the distance of D from AB.

Now the equation of the circle (AB') is

$$\left\{ \frac{\beta}{b} (l_2 + n_2 \cos B) + \frac{\gamma}{c} (l_2 + m_2 \cos C) \right\} (a\alpha + b\beta + c\gamma)$$
$$= \frac{1}{bc} (al_2 + bm_2 + cn_2) (a\beta\gamma + b\gamma\alpha + c\alpha\beta);$$

and that of (A'B) is

$$\left\{ \frac{\gamma}{c} (m_1 + l_1 \cos C) + \frac{\alpha}{a} (m_1 + n_1 \cos A) \right\} (a\alpha + b\beta + c\gamma)$$
$$= \frac{1}{ca} (al_1 + bm_1 + cn_1)(a\beta\gamma + b\gamma\alpha + c\alpha\beta).$$

Consequently

$$\mathrm{D}(\mathrm{AB}') - \mathrm{D}(\mathrm{A}'\mathrm{B})$$

$$= \frac{abc}{af + bg + ch}\left\{\frac{\frac{g}{b}(l_2 + n_2\cos \mathrm{B}) + \frac{h}{c}(l_3 + m_3\cos \mathrm{C})}{al_2 + bm_3 + cn_2}bc - \right.$$

$$\left.\frac{\frac{h}{c}(m_1 + l_1\cos \mathrm{C}) + \frac{f}{a}(m_1 + n_1\cos \mathrm{A})}{al_1 + bm_1 + cn_1}ca\right\}$$

where f, g, h are, as before, the normal co-ordinates of D.

Substitute for f and g in the expression within the brackets their values given on p 126 we find that

$$(l_2 + n_2\cos \mathrm{B})(bm_3 m_1 + cm_3 n_1 + am_1 l_3) -$$
$$(m_1 + n_1\cos \mathrm{A})(al_3 l_3 + bl_3 m_3 + cl_3 n_3)$$
$$= (m_3\cos \mathrm{B} - l_3\cos \mathrm{A})(cn_1 n_2 + an_1 l_2 + bn_2 m_1)$$
$$= \frac{(m_3\cos \mathrm{B} - l_3\cos \mathrm{A})h}{al_3 + bm_3 + cn_3}.$$

On taking h outside the bracket, we get

$$\frac{\mathrm{D}(\mathrm{AB}') - \mathrm{D}(\mathrm{A}'\mathrm{B})}{h} = \frac{abc}{af + bg + ch}\left\{\frac{b(l_3 + m_2\cos \mathrm{C})}{\Sigma al_2} - \right.$$

$$\left.\frac{a(m_1 + l_1\cos \mathrm{C})}{\Sigma al_1} - \frac{c(l_3\cos \mathrm{A} - m_3\cos \mathrm{B})}{\Sigma al_3}\right\}.$$

The first two terms within the brackets will be found to have c as a factor. Taking this outside, we have

$$\frac{\mathrm{D}(\mathrm{AB}') - \mathrm{D}(\mathrm{A}'\mathrm{B})}{ch} = \frac{-abc}{af + bg + ch}\left\{\sum\frac{n_1\cos \mathrm{B} - n_1\cos \mathrm{C}}{al_1 + bm_1 + cn_1}\right\}$$

$$= \frac{-abc}{af + bg + ch}\cdot\frac{\mathrm{S}}{8\triangle \mathrm{R}}, \text{ by (1)}$$

$$= \frac{-\mathrm{S}}{4\triangle}, \text{ if } f, g, h \text{ denote actual lengths.}$$

$$= -\rho, \text{ the ratio of similitude referred to in (2).}^{1}$$

Consequently $\mathrm{D}(\mathrm{AB}') - \mathrm{D}(\mathrm{A}'\mathrm{B}) = -\rho ch$

and the theorem is established.

§ 3. *Three triangles having a common double point.*

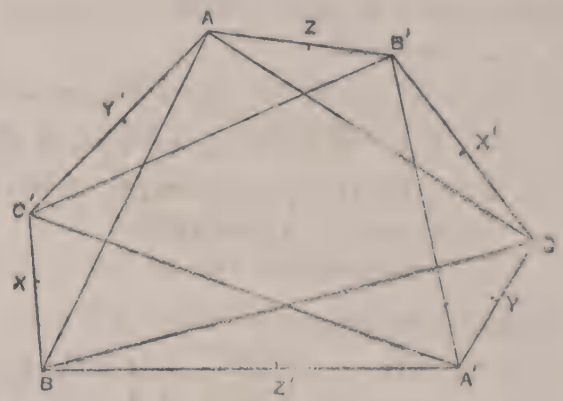
It has already been shown that in the case of two orthologic triangles ABC and A′B′C′, the radical axes of the pairs of circles described on BC′, B′C, etc. as diameters, intersect in the double point of the triangles. Now

[1] *Jour. Ind. Math. Soc. loc cit.* Vol. X

the elementary theorem in co-axal circles that the difference of the powers of a point with respect to two circles is twice the rectangle contained by its distance from their radical axis and the distance between their centres, serves to give a generalisation of this Theorem.

THEOREM VIII

Any two triangles ABC, A′B′C′ *and the triangle formed by the radical axes of the pairs of circles described on* BC′, B′C ; CA′, C′A ; AB′, A′B, *as diameters, have a common double point.*



Proof. Let X,X′ Y,Y′ Z,Z′ be the middle points of BC′, B′C etc. The triangles XYZ and X′Y′Z′ have the same median point which is the mean centre of A,B,C, A′,B′,C′. Therefore if λ, μ, ν are the angles of the triangle formed by XX′, YY′, ZZ′, we have

$$\frac{XX'}{\sin\lambda} = \frac{YY'}{\sin\mu} = \frac{ZZ'}{\sin\nu}. \qquad (1)$$

The radical axes being perpendicular to XX′, YY′, ZZ′ form a triangle, say LMN, with angles λ, μ, ν. If P be any point, and l, m, n be the distances of P from the sides of LMN, P is the mean centre of LMN for the multiples

$$l \sin\lambda,\ m \sin\mu,\ n \sin\nu\ ;$$

i. e., $l.\ XX',\ m.\ YY',\ n.\ ZZ'$, by (1).

But

$$2\,l.\ XX' = P\,(BC') - P\,(B'C) \text{ by co-axal theory.}$$

Therefore P is the mean centre of LMN for multiples P (BC′) − P (B′C), etc.

Therefore D, the double point of ABC and A′B′C′ is the mean centre of LMN for multiples D (BC′) − D (B′C), etc., and by Theorems VII, D is also the mean centre of ABC and A′B′C′ for the same multiples. Hence D is the common double point of ABC, A′B′C′ and LMN.

§ 4. *A Reciprocal Relationship.*

Let ABC and A′B′C′ be two orthologic triangles and the perpendiculars from A′, B′, C′ on BC, CA, AB meet them in L, M, N and concur at P. If P be the point (l, m, n) the normal co-ordinates of A′, B′, C′ are

$$l(1-\lambda),\ m+\lambda l\cos C,\ n+\lambda l\cos B;$$
$$l+\mu m\cos C,\ m(1-\mu),\ n+\mu m\cos A;$$
$$l+\nu n\cos B,\ m+\nu n\cos A,\ n(1-\nu);$$

where $\frac{PA'}{PL}=\lambda,\ \frac{PB'}{PM}=\mu,\ \frac{PC'}{PN}=\nu.$

Now the double point (α, β, γ) of ABC and A′B′C′ is found by substituting the above values of the co-ordinates of A′, B′, C′ in two of the equations in § 1 and solving them. Thus,

$$al(1-\lambda)\alpha+b(l+\mu m\cos C)\beta+c(l+\nu n\cos B)\gamma = \alpha(al+bm+cn), \qquad (1)$$

and a similar equation.

If the double point (α, β, γ) is given as also the quantities λ, μ, ν, P is uniquely determined by the following equations for l, m, n got by rearranging the above

Thus,

$$a\alpha(1+\lambda)l+bm(\alpha-\mu\beta\cos O)+cn(\alpha-\nu\gamma\cos B) = (a\alpha+b\beta+c\gamma)l \qquad (2)$$

and a similar equation.

If we examine the equations (1) and (2), we find that (2) can be derived from (1) by merely substituting

$$\left.\begin{matrix}\alpha & \beta & \gamma\\ l & m & n\\ \lambda & \mu & \nu\end{matrix}\right\} \text{ for } \left.\begin{matrix}l & m & n\\ \alpha & \beta & \gamma\\ -\lambda & -\mu & -\nu\end{matrix}\right\}.$$

Therefore if we are given the double point D of two orthologic triangles ABC and A′B′C′ and the ratios $\frac{PA'}{PL}, \frac{PB'}{PM}, \frac{PC'}{PN}$, P is determined by the following construction :—

From D drop DL′, DM′, DN′ perpendicular to the sides of ABC and find points A″, B″, C″ on them such that

$$\frac{DA''}{DL'}=-\frac{PA'}{PL},\ \frac{DB''}{DM'}=-\frac{PB'}{PM},\ \frac{DC''}{DN'}=-\frac{PC'}{PN}.$$

Then P is the double point of the triangles ABC and A″B″C″.

Designating the ratios $\frac{PA'}{PL}$, $\frac{PB'}{PM'}$ $\frac{PC'}{PN}$ by λ, μ, ν as before, when P, λ, μ, ν are given, the triangle A'B'C' is completely determined and may be demoted by (P, λ, μ, ν). If the statement that D is the double point of ABC and A'B'C' be denoted by the equation

$$(P, \lambda, \mu, \nu,) = D,$$

the above results show that

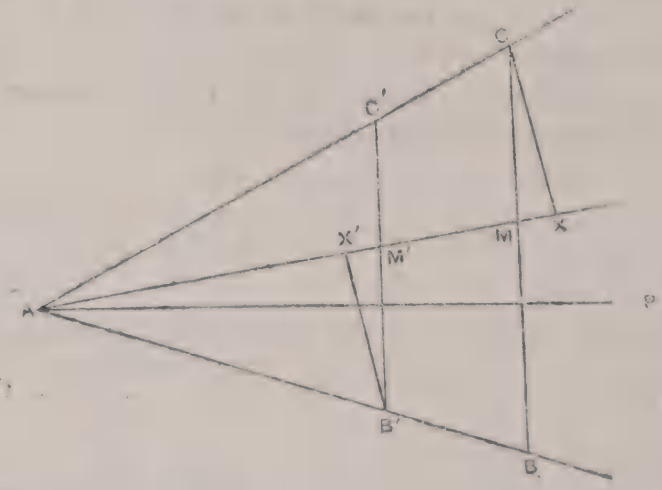
$$(D, -\lambda, -\mu, -\nu,) = P$$

that is to say, P and D are interchangeable with a change in the sign of λ, μ, ν.

§ 5 *Geometrical Proof.*

While examining some special cases, the reciprocal relationship between P and D, of two orthologic triangles mentioned above was noticed in its analytical aspect. The following is a geometrical proof of the same. We shall however, need the following Lemma at first.

Lemma.—In a triangle ABC, M,M' are the middle points of the base BC and an intercept B'C' parallel to BC. If the power of a point Z with respect to the circle described on XY as diameter be denoted by Z (XY), to show that M (PB') + M' (PC) = M (PC') + M' (PB) = MM'^2 where P is any point on the altitude from A.



Through C and B' drop CX, B'X' perp. to AM'M.

Since the triangles CMX and B'M'X' are similar,

$$\frac{MX}{M'X'} = \frac{CM}{C'M'} = \frac{BM}{B'M'} = \frac{AM}{AM'}$$

$\therefore$ $MX \cdot M'A = M'X' \cdot AM.$

$\therefore$ $(M'X - M'M)\ M'A = (MX' - MM')\ AM.$

But $M(AC) = M'X \cdot M'A$ and $M(AB') = -MX' \cdot AM.$

$\therefore$ $M'(AC) + M(AB') = M'M(AM - AM') = M'M^2$... (1)

Again since, B'C' is the radical axis of the circles (PB) and (AB'),

$$M(PB') - M(AB') = 2\delta l$$

where δ is the distance between BC and B'C' and l, the distance between the centres of the circles, and $AP = 2l$.

Hence, $M(PB') - M(AB') = \delta AP.$

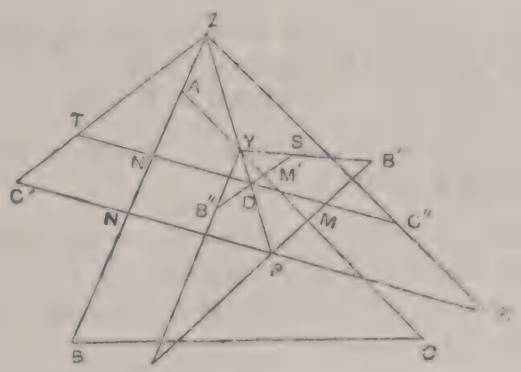
Similarly $M'(PC) - M'(AC) = (-\delta) AP.$

$\therefore$ $M(PB') + M'(PC) = M(AB') + M'(AC) = MM'^2.$ by (1)

Similarly, $M(PC') + M'(PB) = MM'^2.$

Hence the lemma is proved.

To revert to the original theorem, let YB''S and ZC'R be two triangles having the common median ZYDP; also ZAN B and AYMC be altitudes of ZC'R and YB''S, and TN'DC' be parallel to C'R and B'MP parallel to B''S.



Now, in the triangle ZC'R, TDC' is parallel to the base C'R and B is a point on the altitude from Z. Also, P is the mid point of CR and D that of TC''.

By the above lemma

$$P(BC') + D(BC') = PD^2.$$

Similarly by considering the triangle Y B''S and the altitude Y M'C, we get

$$P(CB'') + D(CB') = PD^2.$$

$\therefore$ $P(BC') + D(BC') = P(CB'') + D(CB').$... (1)

Now since when D is the double point of ABC and A'B'C', D lies on the radical axes of (BC') (CB'); (1) shows that P will lie on the radical axis of (BC'') and (CB'').

Remembering that D is the middle point of TC''

$$\frac{PC'}{PN} = \frac{DT}{DN'} = -\frac{DC''}{DN'} = \nu$$

since DT is parallel to PC'.

Similarly $$\frac{PB'}{PM} = \frac{DB''}{DM'} = \mu.$$

It follows therefore that if

$$(P, \lambda, \mu, \nu) = D,$$

then $$(D, -\lambda, -\mu, -\nu) = P.$$

§ 6 *A few Miscellaneous Theorems.*

Before we pass on to the next part, it may be convenient to give without proof, a few theorems which incidentally arose during the progress of the present paper.

THEOREM IX.

If the pedal triangle of P is in perspective with ABC (the centre of perspective being Q) and if OP and GQ (where O and G are the circumcentre and the centroid of ABC) interest in T,

(*i*) *A rectangular hyperbola through P and T can be drawn self-conjugate with respect to ABC.*

(*ii*) *If points A', B', C' be taken on the perpendiculars from P on the sides of ABC such that PA', PB', PC' are inversely proportional to the perpendiculars from T on the sides of ABC, the double point of ABC and A'B'C' lise on the rect. hyp.* (*i*).

(*iii*) *The perpendiculars from A,B,C on the sides of A'B'C' concur at T.*

THEOREM X.

(*i*) *If A', B', C' be taken on the altitudes of a triangle such that the treangle A'B'C' is similar to the ortho-centric triangle, the locus of the double point of ABC and A'B'C' is the isogonal transformation of the Euler's line OH.*

(*ii*) *If the altitudes of a triangle ABC meet the circumcircle in A',B',C', the double point of ABC and A'B'C' is the isogonal conjugate of the nine-point centre.*

THEOREM XI.

Let DEF be the pedal triangle of P with respect to ABC. If points A′,B′,C′ are taken on the perpendiculars from P on the sides of ABC, such that

$$\frac{PA'}{PD} = \frac{PB'}{PE} = \frac{PC'}{PF} = \frac{RQ}{QP'}$$

where RQ is the diameter through P, of the circumcircle of ABC, the double point of ABC and A′B′C′ is at infinity (being the isogonal conjugate of Q.; in other words, the mid-point lines of the quadrilaterals BCC′B′, CAA′C′, ABB′A′ are parallel to the pedal line of Q.

(To be concluded).

SHORT NOTES

An extended case of Hermite's Theorem

Hermite's theorem on partial fractions in Trigonometry states that

$$\frac{f(\sin\theta,\ \cos\theta)}{\prod_1^n \sin(\theta-\alpha_r)} = \sum_1^n \frac{f(\sin\alpha_r,\ \cos\alpha_r)}{P_r \cdot \sin(\theta-\alpha_r)}$$

where $f(\sin\theta,\ \cos\theta)$ is a homogeneous function of the $(n-1)$th degree in $\sin\theta$, $\cos\theta$; and P_r denotes $\sin(\alpha_r-\alpha_1)\times\sin(\alpha_r-\alpha_2)\ldots\sin(\alpha_r-\alpha_n)$. This theorem is capable of extension as follows when the degres of f is still lower, say equal to m where $m<n$. Substituting x for $e^{i\theta}$, a_r for $e^{i\alpha_r}$, etc

$$\frac{f_m(\sin\theta,\ \cos\theta)}{\prod_1^n \sin(\theta-\alpha_r)} = \frac{f_m\left(\frac{x^2-1}{2ix},\ \frac{x^2+1}{2x}\right)}{\frac{1}{(2i)^n}\prod_1^n\left(\frac{x}{a_r}-\frac{a_r}{x}\right)}$$

$$= (-)^n(2x)^{n-m}\frac{a_1a_2\ldots a_n f_m\left\{\frac{x^2-1}{i},\ x^2+1\right\}}{\prod_1^n(x^2-a_r^2)}$$

$$= (-)^n(2x)^{n-m}\frac{a_1a_2\ldots a_n\sum f_m\left[\frac{a_r^2-1}{i},\ a_r^2+1\right]}{(a_r^2-a_1^2)\ldots(a_r^2-a_n^2)}$$

$$= (-)^n\sum\left(\frac{x}{a}\right)^{n-m-1}\cdot\frac{f_m[(a_r^2-1)/2ia_r,\ (a_r^2+1)/2a_r]}{2^{n-1}\prod_{s=1}^{n}\left(\frac{a_r}{a_s}-\frac{a_s}{a_r}\right)(x_r^2-a_r^2)}$$

$$= \sum\frac{f_m(\sin\alpha_r,\cos\alpha_r)\left[\cos\overline{n-m-1}(\theta-\alpha_r)+i\sin\overline{n-m-1}(\theta-\alpha_r)\right]}{\sin(\theta-\alpha_r)\cdot P_r}.$$

Hence

$$\frac{f_m(\sin\theta,\ \cos\theta)}{\prod_1^n\sin(\theta-\alpha_r)} = \sum_1^n\frac{f_m(\sin\alpha_r,\ \cos\alpha_r)\cos\overline{n-m-1}(\theta-\alpha_r)}{\sin(\theta-\alpha_r)\cdot P_r}$$

which agrees of course, with Hermite's result when $m=n-1$.

G. S. M.

[The theorem is further capable of extension as follows :—

$$\frac{f_m(\sin\theta, \cos\theta)}{\prod_1^n \sin(\theta - \alpha_r)} = \sum_1^n \frac{f_m(\sin\alpha_r, \cos\alpha_r)}{P_r \sin(\theta - \alpha_r)},$$

$$\text{or} \quad = \sum_1^n \frac{f_m(\sin\alpha_r, \cos\alpha_r)}{P_r} \cot(\theta - \alpha_r),$$

according as $(n - m)$ is odd or even. To prove the latter case, which alone may be taken up in continuation of the above note, we observe that the left-hand member may be written $\dfrac{(\sin^2\theta + \cos^2\theta)^{\frac{n-m}{2}} f_m(\sin\theta, \cos\theta)}{\prod_1^n \sin(\theta - \alpha_r)}$

and can be expanded in a series of n co-tangents, viz., $\sum_1^n c_r \cot(\theta - \alpha_r)$, because the resulting homogeneous identity is consistent. The value of c_r is obviously the value of $\left[\dfrac{f_m(\sin\theta, \cos\theta)\sin(\theta - \alpha_r)}{\prod \sin(\theta - \alpha_r)}\right]$ when $\theta = \alpha_r$;

or $c_r = \dfrac{f_m(\sin\alpha_r, \cos\alpha_r)}{P_r}$.

Reference may be made to the paper "On the factorization of $\dfrac{\sin mx}{\sin nx}$ and allied theorems" by Prof. A. C. L. Wilkinson (*J. I. M. S.*, Vol. I, pp. 72 *et seq*), and to Questions 82 and 104 by Mr. V. Ramaswami Aiyar in Vol. I of the *J. I. M. S.*—EDITOR.]

Aberration in Latitude and Longitude.

[Ref. In Ball's *Spherical Astronomy*, page 259, § 87 is considered the effect of the elliptic motion of the earth on aberration in R. A. and declination, but not on aberration in latitude and longitude. The expressions giving the aberration in these co ordinates are very simple unlike the lengthy ones for R. A. and declination. Remarks similar to those on pages 260, 261 apply to equations (3) and (4) below ; and examples 1 and 2 on page 261 are obvious corollories of the equations (3) and (4). It has therefore been considered useful to give these expressions in this note.]

§ 1. It is well known that the velocity of the earth at any instant can be resolved into two constant velocities ;—

(i) $\frac{\mu}{h}$ perpendicular to the radius vector,

and (ii) $\frac{\mu e}{h}$ perpendicular to the major axis.

If $\odot$ = Longitude of the sun,

and ω = Longitude of perigee,

it is easily seen from a figure that the above velocitiees can be replaced by

$$\frac{\mu}{h}(\sin \odot + e \sin \omega) \qquad \text{along S}\Upsilon.$$

$$-\frac{\mu}{h}(\cos \odot + e \cos\omega) \qquad \text{perpendicular to S}\Upsilon.$$

If η_0 be the longitude of the earth's way, the velocity of the earth along the tangent to its path can be resolved into velocities

$v \cos \eta_0$ along SΥ.

$v \sin \eta_0$ perpendicular to SΥ;

so that we have

$$v \cos \eta_0 = \frac{\mu}{h}(\sin \odot + e \sin \omega) \qquad (1)$$

$$v \sin \eta_0 = -\frac{\mu}{h}(\cos \odot + e \cos \omega) \qquad (2)$$

§ 2. Now from (8) on page 252, Ball, we have if (λ, β) be the true longitude and latitude

(i) aberration in latitude $= -\frac{v}{u} \sin \beta \cos (\eta_0 - \lambda)$

$$= -\frac{\mu}{hu} \sin \beta \{\sin (\odot - \lambda) + e \sin (\omega - \lambda)\}$$

where v is the velocity of the observer, and u that of light; (3)

(ii) aberration in longitude

$$= -\frac{v}{u} \sec \beta \sin (\lambda - \eta_0)$$

$$= -\frac{\mu}{hu} \sec \beta \{\cos (\odot - \lambda) + e \cos (\omega - \lambda)\} \qquad (4)$$

and $$\frac{\mu}{h} = \frac{2\pi a}{T\sqrt{1-e^2}}$$ (Vide Q. 1004.)

C. KRISHNAMACHARI.

It will easily be verified that

$$\frac{d}{dx}\log\begin{bmatrix}\sin x\\ \operatorname{cosec} x\end{bmatrix} = \pm\cot x,$$

$$\frac{d}{dx}\log\begin{bmatrix}\cos x\\ \sec x\end{bmatrix} = \mp\tan x;$$

and

$$\frac{d}{dx}\log\begin{bmatrix}\tan x\\ \cot x\end{bmatrix} = \pm 2\operatorname{cosec} 2x;$$

so that

$$\left(\frac{d}{dx}\log\right)\left(\frac{d}{dx}\log\right)\begin{bmatrix}\sin x\\ \operatorname{cosec} x\end{bmatrix} = -2\operatorname{cosec} 2x, \text{ etc.};$$

and

$$\left(\frac{d}{dx}\log\right)\left(\frac{d}{dx}\log\right)\left(\frac{d}{dx}\log\right)\begin{bmatrix}\sin x\\ \operatorname{cosec} x\end{bmatrix} = -2\cot 2x.$$

In general, if θ denote the logarithmic derivate, viz., the operation $\left(\frac{d}{dx}\log\right)$; and $n > 1$.

$$\left|\begin{array}{l}\theta^{2n}(\sin x) = \theta^{2n}(\cos x) = \theta^{2n}(\operatorname{cosec} x) = \theta^{2n}(\sec x)\\ \qquad\qquad = -2^n\operatorname{cosec}(2^n x);\\ \theta^{2n}(\tan x) = \theta^{2n}(\cot x) = -2^n\cot 2^n x;\\ \theta^{2n+1}(\sin x) = \theta^{2n+1}(\cos x) = \theta^{2n+1}(\operatorname{cosec} x) = \theta^{2n+1}(\sec x)\\ \qquad\qquad = -2^n\cot 2^n x;\\ \theta^{2n+1}(\tan x) = \theta^{2n+1}(\cot x) = -2^{n+1}\operatorname{cosec}(2^{n+1}x).\end{array}\right.$$

* * * * *

It is perhaps well known that

The locus of the midpoint of the intercept of any tangent to a parabola on two fixed tangents is a tangent to the parabola (at the point whose parameter is the mean of the parameters of the points of contact of the fixed tangents).

If t_1, t_2 and t are respectively the parameters and X_1, X_2, the points where (t_1, t) (t_2, t) meet, the middle point of X_1X_2 will be

$$\left(at\ \frac{t_1 + t_2}{2};\ at + \frac{a}{2}t_1 + t_2\right)$$

whose locus is obviously

$$y = \frac{a}{2}(t_1 + t_2) + x\,\frac{2}{t_1 + t_2},$$

which demonstrates the theorem.

This theorem can be extended as follows :

The locus of the mean centre of the n points of intersection of any tangent to a parabola with n fixed tangents is also a tangent (at the point whose parameter is the mean of those of the n fixed points of contact).

With obvious notation, X_r is $(a\ t\ t_r,\ a\ t\overline{+\ t_r})$, and the mean centre of $X_1 \ldots\ldots X_n$ is $\xi = at\ \frac{\Sigma t_r}{n},\ \eta = at + a\ \frac{\Sigma t_r}{n}$.

Hence the locus of the mean centre is

$$y = a\ \Sigma(t^r)/n + xn/\Sigma(t_r)$$

which is the tangent at the point whose parameter is the artihemetic mean of those of the fixed points.

It will also be recognised that P is the point dividing in the ratio $1 : n - 1$ the line joining the mean centre of the points of contact of the n fixed tangents with the mean centre of their $n(n - 1)/2$ intersections.

SOLUTIONS TO QUESTIONS.

Question 791.

(K. Appukuttan Eraby, M.A.) The centres of three circles of radii a, b, c form a triangle of sides l, m, n and area Δ. If (r_1, r_1'), (r_2, r_2') (r_3, r_3') and (r_4, r_4') be the radii of the four pairs of circles tangential to the three circles (the circles belonging to any pair being inverses of each other with respect to the common orthogonal circle of the three original circles), show that $\frac{1}{r_1 r_1'} + \frac{1}{r_2 r_2'} + \frac{1}{r_3 r_3'} + \frac{1}{r_4 r_4'}$

$$= \frac{16\,(\Delta^2 - \Sigma a^2 l^2)}{\Sigma l^2 (a^2 - b^2)(a^2 - c^2) - \Sigma m^2 n^2 (b^2 + c^2) + \Sigma a^2 l^4 + l^2 m^2 n^2}$$

Solution by N. Sankara Aiyar and Martyn M. Thomas.

The numerator should be $16\,(4\,\Delta^2 - \Sigma a^2 l^2)$ as can be seen by making $a = b = c = 0$.

Let LMN be the triangle of centres, P the centre of the circle touching the three given circles internally and r_1 its radius.

Then $PL = r_1 + a$, $PM = r_1 + b$, $PN = r_1 + c$.

If $P\hat{L}M = \theta$ and $N\hat{L}M = A$, we get

$$(r_1 + b)^2 = n^2 + (r_1 + a)^2 - 2n(r_1 + a)\cos\theta$$

and $$(r_1 + c)^2 = l^2 + m^2 + (r_1 + a)^2 - 2m(r_1 + a)\cos(A - \theta)$$

or, $$2n(r_1 + a)\cos\theta = 2r_1(a - b) + n^2 + a^2 - b^2 = h \text{ (say)}$$

and $$2m(r_1 + a)\cos(A - \theta) = 2r_1(a - c) + m^2 + a^2 - c^2 = k \text{ (say)}$$

$$\therefore\ 4mn(r_1 + a)\sin\frac{A}{2}\sin\left(-\theta\frac{A}{2}\right) = nk - mh,$$

and $$4mn(r_1 + a)\cos\frac{A}{2}\cos\left(\frac{A}{2} - \theta\right) = nk + mh.$$

Multiplying by $\cos\frac{A}{2}$, $\sin\frac{A}{2}$, squaring and adding

$$16\,m^2 n^2 (r_1 + a)^2 \sin^2\frac{A}{2}\cos^2\frac{A}{2} = \cos^2\frac{A}{2}(nk - mh)^2 + \sin^2\frac{A}{2}(nk + mh)^2.$$

$$\therefore\ 4m^2n^2(r_1+a)^2\sin^2 A = \frac{1+\cos A}{2}(nk-mh) + \frac{1-\cos A}{2}(nk+mh)^2$$

$$= n^2k^2 + m^2h^2 - 2mnkh\cos A$$

$$= n^2\{2r_1\sqrt{a-c} + m^2 + a^2 - c^2\}^2$$

$$+ m^2\{2r_1\sqrt{a-b} + n^2 + a^2 - b^2\}^2$$

$$- 2mn\cos A\{2r_1\sqrt{a-c} + m^2 + a^2 - c^2\}$$

$$\{2r_1\sqrt{a-b} + n^2 + a^2 - b^2\}.$$

Expanding and writing down the term containing r_1^2 and the constant term we get

$$r_1^2\{4m^2n^2\sin^2 A - 4n^2(a-c)^2 - 4m^2(a-b)^2 + 8mn\cos A$$

$$(a-c)(a-b)\}$$

$$+ 4m^2n^2a^2\sin^2 A - n^2(m^2+a^2-c^2)^2 - m^2(n^2+a^2-b^2)^2$$

$$+ 2mn\,\text{Cos}\,A(m^2+a^2-c^2)(n^2+a^2-b^2) + r_1\{\ldots\ldots\} = 0.$$

$$\therefore\ r_1^2\{16\Delta^2 - 4n^2(a-c)^2 - 4m^2(a-b)^2 + 4(m^2+n^2-l^2)$$

$$(a-c)(a-b)\}$$

$$+ 4m^2n^2a^2\sin^2 A - n^2m^4 - 2n^2m^2(a^2-c^2) - (a^2-c^2)^2 - m^2n^4$$

$$- 2m^2n^2(a^2-b^2) - (a^2-b^2)^2 + (m^2+n^2-l^2)(m^2+a^2-c^2)$$

$$(n^2+a^2-b^2)$$

$$+ r_1\{\ldots\ldots\} = 0,$$

which becomes after some reductions

$$r_1^2\{16\Delta^2 - 4\Sigma a^2l^2 + 4\Sigma bc(l^2+m^2-n^2)\}$$

$$- l^2m^2n^2 + \Sigma m^2n^2(b^2+c^2) - \Sigma a^2l^4 - \Sigma l^2(a^2-b^2)(a^2-c^2)$$

$$+ r_1\{\ldots\ldots\ldots\} = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

The coefficient of r_1 being an odd function of a,b,c will change sign when a,b,c all change sign, *i.e.*, the equation for radius touching all internally will be the same as (i) with the sign of the middle term changed.

Hence the value of $\frac{1}{r_1 r_1'}$ is seen to be

$$\frac{16\Delta^2 - 4\Sigma a^2l^2 + 4\Sigma bc(l^2+m^2-n^2)}{\Sigma l^2(a^2-b^2)(a^2-c^2) - \Sigma m^2n^2(b^2+c^2) + \Sigma a^2l^4 + l^2m^2n^2}$$

Changing the signs of a,b,c one at a time, we get the values of

$$\frac{1}{r_2 r_2'},\ \frac{1}{r_3 r_3'},\ \frac{1}{r_4 r_4'},$$

adding we get $\frac{1}{r_1 r_1'} + \frac{1}{r_2 r_2'} + \frac{1}{r_3 r_3'} + \frac{1}{r_4 r_4'} =$

$$\frac{64\Delta^2 - 16 \Sigma a^2 l^2 + 4 \Sigma (l^2 + m^2 - n^2)(bc - bc - bc + bc)}{\Sigma l^2 (a^2 - b^2)(a^2 - c^2) - \Sigma m^2 n^2 (b^2 + c^2) + \Sigma a^2 l^4 + l^2 m^2 n^2},$$

which gives the required result.

Question 839

(R. VYTHYNATHASWAMY) :—In any system of parametric representation of the nodal cubic, the parameters of the points of inflexion are given to be the roots of the equation

$$At^3 + 3Bt^2 + 3Ct + D = 0.$$

Prove that the condition of collinearity of the points t_1, t_2, t_3 is

$$At_1t_2t_3 + B\Sigma t_1t_2 + C\Sigma t_1 + D = 0,$$

and that the parameters of the nodal points are the roots of the equation

$$(AC - B^2)x^2 + (AD - BC)x + BD - C^2 = 0.$$

Solution by Hemraj.

If a cubic has a double point, the co-ordinates are rational functions of one parameter. We may start with the expressions for the co-ordinates in terms of a parameter t, viz.

$$x = at^3 + 3bt^2 + 3ct + d,$$

$$y = a_1 t^3 + 3b_1 t^2 + 3c_1 t + d_1,$$

$$z = a_2 t^3 + 3b_2 t^2 + 3c_2 t + d_2.$$

$\therefore$ $\lambda x + \mu y + \nu z = 0$ must always have three roots.

The condition that three points t_1, t_2, t_3 of the curve should be collinear is

$$\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} = 0.$$

This is easily seen to be resolvable into partial determinants each of which is divisible by $(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$ and the condition is

$$At_1t_2t_3 + B\Sigma t_1t_2 + C\Sigma t_1 + D = 0, \quad \ldots \quad \ldots \quad (1)$$

A, B, C, D being the determinants of the system

$$\begin{Vmatrix} a & b & c & d \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{Vmatrix}$$

Given t_1, t_2, it usually determines t_3 uniquely, but if

$$A\,t_1t_2 + C\,(t_1 + t_2) + C = 0,$$

and $$B\,t_1t_2 + C\,(t_1 + t_3) + D = 0,$$

it is identical in t_3. In this case t_1, t_2 are the roots of

$$(At + B)\,(Ct + D) - (Bt + C)^2 = 0,$$

i.e. $$(AC - B^2)\,t^2 + (AD - BC)\,t + BD - C^2 = 0. \qquad (2)$$

These values of t belong to the same point on the curve which is, therefore, a double point, node, cusp, or conjugate point according to circumstances.

If t is an inflexion, t,t,t are collinear, and from (1), t is given by

$$At^3 + 3\,Bt^2 + 3\,Ct + D = 0. \qquad (3)$$

There are three points of inflexions and any cubic gives collinear points, if it is a polar to the cubic giving the inflexions.

If three points be determined by the cubic

$$A't^3 + 3\,B't^2 + 3\,C't + D' = 0,$$

then the condition that three points should be on a right line is

$$(AD' - A'D) - 3\,(BC' - B'C) = 0, \qquad \text{from (1).}$$

Hence the three inflexions are collinear.

The quadratic (2) giving the double point is the Hessian of the cubic giving the flexes.

Thus if a cubic has a crunode, it has one real point of inflexion.

If a cubic has an acnode, it has three real points of inflexion.

If a cubic has a spinode, the roots of the Hessian are equal, and the cubic has the same two roots equal corresponding to the cusp, and the third real root gives the inflexion. Hence a cuspidal cubic has only one flex and that must be real.

Question 940.

(R. Srinivasan, M.A., M.R.A.S.) :—Show that

$$\int_0^1 \frac{x^n \log x}{1 + x^2}\,dx = \frac{\pi^2}{4} \sec \frac{n\pi}{2} \tan \frac{n\pi}{2}.$$

Remarks by K. J. Sanjana, M.A.

The result is evidently inaccurate. If n is positive, the function $x^n/(1 + x^2)$ is positive and finite between the limits of integration. Hence the integrand is finite and negative throughout, since both at $x = 1$ and $x = 0$, $x^n \log x$ vanishes. But the right-hand side is positive from $n = 0$ to $n = 1$, and at $n = 1$ it is infinitely large.

Let $f(n) = \int_0^1 \frac{x^n dx}{1+x^2}$; then $f'(n) = \int_0^1 \frac{x^n \log x}{1+x^2}\, dx$.

Also $f(n) = \int_0^1 x^n(1 - x^2 + x^4 - \ldots)\, dx$, as the expansion is valid,

$$= \frac{1}{n+1} - \frac{1}{n+3} + \frac{1}{n+5} - \ldots\ldots$$

Hence the integral $f'(n) =$

$$-\left[\frac{1}{(n+1)^2} - \frac{1}{(n+3)^2} + \frac{1}{(n+5)^2} - \ldots \textit{ad inf.}\right]$$

The result may be obtained directly from the formula

$$\int_0^1 x^r \log x\, dx = -(r+1)^{-2}.$$

No finite form can be given to the sum except for odd integral positive values of n; for even integral values the sum has been calculated numerically by Dr. Glaisher and others.

Question 955.

(K. B. Madhava):—The 5th, 13th, 17th, 29th and 37th roots (if rational) of a number end in the same digit as the number itself, the numbers when fractional being expressed in decimals. Prove this property and illustrate how it enables one to give out at sight the above rational roots of big numbers.

Solution by Prof. N. B. Mitra, M.A.

Let all the letters in this solution, except N and n denote positive integers and let N and n denote any positive rational quantities.

Let d stand for any of the ten digits 0, 1, 2, ..., 9; and k for any of the four digits 1, 3, 7, 9.

Further let all the congruences in this solution be understood to be to the modulus 10.

Also, if a rational fraction converted to decimals be terminating then by the last digit of the fraction we shall mean the units figure of the number obtained by removing the decimal point from the equivalent decimal fraction written in its simplest form. Thus although $\cdot 456 = \cdot 4560$, by the last figure of this fraction we shall mean 6. Similarly by the last digit of a recurring decimal like $\cdot 4\dot{5}6\dot{7}$, we shall mean 7, although $\cdot 4\dot{5}6\dot{7} = \cdot 45\dot{6}75 = \cdot 456\dot{7}56$, &c.

Now $0^q \equiv 0$, $1^q \equiv 1$, $5^q \equiv 5$, $6^q \equiv 6$;

$\therefore$ if $d = 0, 1, 5$ or 6, $d^2 \equiv d$. (a)

Again $4^2 \equiv 6$; $4^{2q} \equiv 6^q \equiv 6$, $\therefore$ $4^{2q+1} \equiv 24 \equiv 4$;

also $9^2 \equiv 1$, $\therefore$ $2^{2q} \equiv 1$ and $9^{2q+1} \equiv 9$.

$\therefore$ if $d = 4$ or 9, $d^{2q+1} \equiv d$.... (b)

Lastly, $2^4 \equiv 6$, $\therefore$ $2^{4q} \equiv 6^q \equiv 6$; $\therefore$ $2^{4q+1} \equiv 12 \equiv 2$;

$3^4 \equiv 1$, $\therefore$ $3^{4q} \equiv 1$ and $3^{4q+1} \equiv 3$;

$7^4 \equiv 1$, $\therefore$ $7^{4q+1} \equiv 7$;

and $8^4 \equiv 6$, $\therefore$ $8^{4q+1} \equiv 8$.

$\therefore$ if $d = 2, 3, 7$ or 8, $d^{4q+1} \equiv d$ (c)

Hence from (a), (b) and (c), we see that in all cases $d^{4q+1} \equiv d$ and $k^{4q} \equiv 1$; in other words, the congruence $d^p \equiv d$ is satisfied if p is of the form $4q + 1$. (A)

Further the congruence $k^p \equiv 1$ is satisfied if p is of the form $4q$.

Now let $\mathcal{D}N = n$, so that $N = n^p$.

Two cases arise:

First: let n be an integer and let $n = d$ $(d < 10)$.

$\therefore$ $N = n^p \equiv d^p$.

Hence N and n will end in the same digit if $d^p \equiv d$, *i.e.*, if p is of the form $4q + 1$ by (A).

Next: let n be a fraction in its lowest terms and $= \frac{a}{b}$, so that a and b are relative primes.

Let $b = 2^l 5^m c$, where c is prime to 10, so that $c \equiv k$. ... (1)

Let f stand for either 2 or 5; the factors of 10 and let $l \sim m = g$.

Also let h stand for either l or m.

Now $n = \frac{a}{2^l 5^m c} = \frac{af^g}{10^h c}$.

Let $cx \equiv 9$ $(x < 10)$ (2)

and $af^g \equiv d$ $(d < 10)$ (3)

Then the last figure of n which equals $\frac{af^g x}{10^h cx}$ is the same as that of $af^g x$, *i.e.*, of dx.

Again $N = \frac{a^p f^{gp}}{10^{hp} c^p}$. Let $c^p y \equiv 9$ $y < 10)$ (4)

Then the last figure of N is the same as that of a^p f^{gp} y, *i.e.*, of $d^p y$ by (3).

Hence n and N will end in the same figure if $dx \equiv d^p y$, *i.e.*, if $dxc^p \equiv d^p y c^p$, *i.e*, if $dc^{p-1}\,9 \equiv d^p 9$ by (2) & (4) *i.e.*, if $dc^{p-1} \equiv d^p$, *i.e.*, if $dk^{p-1} \equiv d^p$ by (1) (B)

From (A) and (B), we see that this is satisfied if p is of the form $4q + 1$.

Hence, with the understanding set forth in the beginning, any number, integral or fractional and its $(4q + 1)$ th root (when rational) will end in the same digit, q being any positive integer.

Partial Solutions by Sâdanand and C. Krishnamachari.

Question 961.

(H. Br.) :—Show that

$$x^2 = y^2 + z^2,\ t^2 = y^2 + 4z^2$$

have no solutions in non-zero integers.

Solution by N. Sankara Aiyar.

We require that y can be expressed as $m^2 - n^2$ and $M^2 - N^2$ and z as MN and $2mn$.

Therefore we may suppose that since y and z have no common factors unless they are factors of x and t also, the same is true of m, n, M, N.

Let $\left.\begin{matrix} m = ab,\ \mathrm{M} = ac \text{ or } bc \\ n = cd,\ \mathrm{N} = 2bd \text{ or } 2ad \end{matrix}\right\}$

where a, b, c, d are prime to one another and d is even.

(1) If $\mathrm{M} = ac$, we get

$$a^2b^2 - c^2d^2 = a^2c^2 - 4b^2d^2$$

i.e. $$b^2(a^2 + 4d^2) = c^2(a^2 + d^2)$$

i.e. $$\frac{a^2 + 4d^2}{c^2} = \frac{a^2 + d^2}{b^2} = v \text{ (say)} = \frac{3d^2}{c^2 - b^2}.$$

Hence if $v = 3$, d is divisible by v and so also is a since $a^2 + d^2 = vb^2$.

Therefore $v = 1$, we get $a^2 + 4d^2 = c^2$ and $a^2 + d^2 = b^2$, a simple set of solution for the original equation where a, b, c, d are less than y, x, z, t.

If $v = 3$, then $d^2 = c^2 - b^2$, and $a^2 + c^2 - b^2 = 3b^2$,

i.e. $$a^2 + c^2 = 4b^2$$

which cannot be true unless both a and c are even.

(2) If $M = bc$, we get

$$a^2b^2 - c^2d^2 = b^2c^2 - 4a^2d^2$$

i.e. $$a^2(b^2 + 4d^2) = c^2(b^2 + d^2)$$

i.e. $$\frac{b^2 + 4d^2}{c^2} = \frac{b^2 + d^2}{a^2} = v \text{ (say)} = \frac{3d^2}{c^2 - a^2};$$

and as before this reduces to $b^2 + 4d^2 = c^2$, $b^2 + d^2 = a^2$.

In either case, we have to solve the equation for smaller values of x, y, z, t. This can be carried on till $y = 1$.

We get $x^2 = z^2 + 1$, $t^2 = 4z^2 + 1$, which is impossible.

Question 973.

(A. C. L. WILKINSON):—(i) Prove that

$$\text{D sn}(u_1 + u_2 + u_3) = \begin{vmatrix} s_1, & s_1', & s_1^3 \\ s_2, & s_2', & s_2^3 \\ s_3, & s_3', & s_3^3 \end{vmatrix},$$

$$\text{D cn}(u_1 + u_2 + u_3) = \begin{vmatrix} c_1, & c_1', & c_1^3 \\ c_2, & c_2', & c_2^3 \\ c_3, & c_3', & c_3^3 \end{vmatrix},$$

$$k^4 \text{ dn}(u_1 + u_2 + u_3) = -\begin{vmatrix} d_1, & d_1', & d_1^3 \\ d_2, & d_2', & d_2^3 \\ d_3, & d_3', & d_3^3 \end{vmatrix}$$

where $$\text{D} = \begin{vmatrix} 1, & s_1^2, & s_1 c_1 d_1 \\ 1, & s_2^2, & s_2 c_2 d_2 \\ 1, & s_3^2, & s_3 c_3 d_3 \end{vmatrix},$$ and $s_1 = \text{sn } u_1$ and $s_1' = \frac{d}{du_1}(\text{sn} u_1)$ etc.

(ii) Prove that

$$\text{E}(u_1) + \text{E}(u_2) + \text{E}(u_3) + \text{E}(u_4) - 2\,\text{E}(x)$$

$$= \frac{k^2 d_1 d_2 d_3 d_4}{k_1^2 + d_1 d_2 d_3 d_4}\left(\frac{s_1 c_1}{d_1} + \frac{s_2 c_2}{d_2} + \frac{s_3 c_3}{d_3} + \frac{s_4 c_4}{d_4}\right)$$

$$- \frac{2k^2 \text{ sn } x \text{ cn } x \text{ dn}(x - u_3 - u_4) \text{ dn}(x - u_2 - u_4) \text{ dn}(x - u_2 - u_3)}{k_1^2 + \text{dn } x \text{ dn}(x - u_3 - u_4) \text{ dn}(x - u_2 - u_4) \text{ dn}(x - u_2 - u_3)}$$

where $x = \frac{1}{2}(u_1 + u_2 + u_3 + u_4)$ and s_1, c_1, d_1 are written as usual for $\text{sn}\,u_1$, $\text{cn}\,u_1$, $\text{dn}\,u_1$

Solution by F. H. V. Gulasekharam.

(i) The results cited here for sn, cn, dn of $(u_1 + u_2 + u_3)$ are identically the same as mine published in J. I. M. S, Vol. VIII, p. 211. Professor Wilkinson expresses them in an elegant determinant form.

(ii) If $u_1 + u_2 + u_3 + u_4 = 2x$, we can easily prove

$$(a)\quad \frac{1}{k'^2}\,\text{dn}\,x\,\text{dn}\,(x-u_3-u_2)\,\text{dn}\,(x-u_3-u_4)\,\text{dn}\,(x-u_3-u_4)$$

$$= \frac{k'^2 + k^2 k'^2 s_1 s_2 s_3 s_4 + k^2 c_1 c_2 c_3 c_4 + d_1 d_2 d_3 d_4}{k'^2 - k^2 k'^2 s_1 s_2 s_3 s_4 - k^2 c_1 c_2 c_3 c_4 + d_1 d_2 d_3 d_4}.$$

$$\therefore \frac{1}{k'^2}(k'^2 - k^2 k'^2 s_1 s_2 s_3 s_4 - k^2 c_1 c_2 c_3 c_4 + d_1 d_2 d_3 d_4)$$

$$= \frac{2(k'^2 + d_1 d_2 d_3 d_4)}{k'^2 + \text{dn}\,x\,\text{dn}\,(x-u_3-u_2)\,\text{dn}\,(x-u_2-u_4)\,\text{dn}\,(x-u_3-u_4)}.$$

$$(b)\quad \frac{\theta(x).\ \theta(x-u_3-u_2).\ \theta(x-u_2-u_4).\ \theta(x-u_3-u_4)}{\theta(u_1).\ \theta(u_2).\ \theta(u_3).\ \theta(u_4)}$$

$$= \frac{1}{k'^2}(k^2 - k^2 k'^2 s_1 s_2 s_3 s_4 - k^2 c_1 c_2 c_3 c_4 + d_1 d_2 d_3 d_4)$$

$$= \frac{2(k'^2 + d_1 d_2 d_3 d_4)}{k'^2 + \text{dn}\,x\,\text{dn}\,(x-u_3-u_2)\,\text{dn}\,(x-u_2-u_4)\,\text{dn}\,(x-u_3-u_4)}.$$

where $\theta(u)$ stands for the Jacobian Theta fraction of u.

[*Vide*: Whitaker and Watson's *Modern Analysis*, Revised Edition, pp. 460-461].

[Now differentiating logarithmically with respect to u_1 both sides of the identity (b) and treating u_2, u_3, u_4 as constants, and remembering that

$$\frac{dx}{du_1} = \frac{1}{2}, \text{ and } \frac{\theta'(u)}{\theta(u)} = \text{E}(u) - u\frac{\text{E}}{\text{K}}, \text{ we get}$$

$$\text{E}(x) + \text{E}(x-u_3-u_2) + \text{E}(x-u_2-u_4) + \text{E}(x-u_3-u_4) - 2\,\text{E}(u_1)$$

$$= \frac{k^2\,\text{dn}\,x\,\text{dn}\,(x-u_3-u_2)\,\text{dn}\,(x-u_2-u_4)\,\text{dn}\,(x-u_3-u_4)}{k'^2 + \text{dn}\,x\,\text{dn}\,(x-u_3-u_2)\,\text{dn}\,(x-u_2-u_4)\,\text{dn}\,(x-u_3-u_4)}$$

$$\left[\frac{\text{sn}\,x\,\text{cn}\,x}{\text{dn}\,x} + \frac{\text{sn}\,(x-u_3-u_2)\,\text{cn}\,(x-u_3-u_2)}{\text{dn}\,(x-u_3-u_2)}\right.$$

$$+\frac{\text{sn}\,(x-u_2-v_4)\,\text{cn}\,(x-u_3-u_4)}{\text{dn}\,(x-u_2-u_4)}+\frac{\text{sn}\,(x-u_3-u_4)\,\text{cn}\,(x-u_2-u_4)}{\text{dn}\,(x-u_3-u_4)}\Big]$$

$$-\frac{k^2 s_1 c_1 d_2 d_3 d_4}{k'^2+d_1 d_2 d_3 d_4}.$$

Since it is obvious that in the above, x, $x-u_3-u_2$, $x-u_2-u_4$, $x-u_3-u_4$ are interchangeable with u_1, u_2, u_3, u_4 respectively, we get the required result.

Question 985.

(M. K. Kewalramani):—If A′, B′, C′ are points in the sides of the triangle ABC such that BA′ : A′C = CB′ : B′A = AC′ : C′B = M : 1, find an expression for the radius of the circumcircle of the triangle XYZ formed by the intersections of AA′, BB′, CC′.

Solution by N. B. Mitra, Sadanand, and G. V. Krishnasawmi.

Taking ABC for the triangle of reference, the equations of AA′, BB′, CC′ are

$$\begin{aligned} \text{M}b\beta - c\gamma &= 0 \\ -a\alpha + \text{M}c\gamma &= 0 \\ \text{M}a\alpha - b\beta &= 0. \end{aligned}$$

Hence the co-ordinates of Z are given by

$$a\alpha : b\beta : c\gamma = 1 : \text{M} : \text{M}^{-1},$$

and so for X and Y.

Now the area of the triangle whose vertices are $(\alpha_1, \beta_1, \gamma_1)$, &c., is

$$\frac{abc}{8\Delta}\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

$$\therefore \text{area of XYZ} = \frac{abc}{8\Delta^2}\cdot\frac{8\Delta_3}{(\text{M}^2+\text{M}+1)^3\,abc}\begin{vmatrix} 1 & \text{M} & \text{M}^2 \\ \text{M} & \text{M}^2 & 1 \\ \text{M}^2 & 1 & \text{M} \end{vmatrix}$$

$$= \frac{\Delta(\text{M}-1)^2}{1+\text{M}+\text{M}^2} \text{ (without regard to the sign.)}$$

Again by the formula for the sine of the angle between two lines,

$$\sin \text{X} = 2\Delta(1+\text{M}+\text{M}_2) \div \sqrt{\{(a^2+\text{M}^2c^2+2\,\text{M}\,a\,c\cos\text{B})(b^2+\text{M}^2a^2+2\,\text{M}\,a\,b\cos\text{C})\}}.$$

Similarly for sin Y & sin Z.

Hence the circum-radius of XYZ is

$$\sqrt{\frac{\text{area of XYZ}}{2\sin\text{X}\sin\text{Y}\sin\text{Z}}}$$

$$= \frac{M-1}{4\Delta(1+M+M^2)^2}\{+a^2+M^2c^2+2Mac\cos B)(b^2+M^2a^2+2ab\,M\cos C)\times(b^2+M^2b^2+2M\,bc\cos A)\}^{\frac{1}{2}}.$$

NOTE.—It can be shown that circum-radius of XYZ

$$= \frac{abc}{8\Delta}\,\frac{\sin(2A+\beta-\gamma)\cos\alpha+\sin(2B+\gamma-\alpha)\cos\beta+\sin(2C+\alpha-\beta)\cos\gamma}{\sin(A+\beta-\gamma)\sin(B+\gamma-\alpha)\sin(C+\alpha-\beta)}$$

where $\cot\alpha = \frac{\cot B - M\cot C}{1+M}$, $\cot\beta = \frac{\cot C - M\cot A}{1+M}$,

& $\cot\gamma = \frac{\cot A - M\cot B}{1+M}$.

Question 988.

(S. MALHARI RAO):--Complete the following magic square of 16 cells, by placing 8 other prime numbers in the vacant cells:

7	43	71	113
73	97	23	41

Solution by K. B Madhava, Jamuna Prasad Nigama, R. D. Karve and M. K. Kewalramani.

The following *three* squares satisfy the problem proposed:

7	43	71	113
A	83	37	B
C	11	103	D
73	97	23	41

where the corresponding values of A, B, C and D are

A = 101, 53, 47; B = 13, 61, 67

C = 53, 101, 107; D = 67, 19. 13.

Question 889.

(S. Malhari Rao) :—Give three solutions in positive integers of the equation $(x + y + z)^3 = xyz$.

Solution by N. B. Mitra, N. B. Pendse, and Sadanand.

Assume $x : y : z :: l : m : n$.

Then substituting in the proposed equation, we find

$$x = \frac{(l + m + n)^2}{mn}, \quad y = \frac{(l + m + n)^2}{nl}, \quad z = \frac{(l + m + n)^2}{lm}.$$

This is the most general solution in *rational* numbers, not necessarily integral.

To get integral solutions, we give particular values to two of the quantities l,m,n and determine the values of the third so that x,y,z may, if possible, be integral.

Thus if $l = 1$, $m = 1$, then $x = y = n + 4 + \frac{4}{n}$, so that, we may take $n = 1$, 2, or 4 ; and the corresponding solutions of (1) are

(i) 9,9,9. (ii) 8,8,16. (iii) 9,9,36.

If $l = 1$, $m = 2$, $x = \frac{1}{2}\left(x + 6 + \frac{9}{n}\right)$ and $\therefore$ $n = 1$, 3 or 9, giving the solutions

(ii) 8,16,8. (iv) 6,12,18 (v) 8,16,72.

If $l = 1$, $m = 3$, n must be 2 or 8 and the solutions are

(iv) 6,18,12. (vi) 6,18,48.

It may be observed that, if $l = 1$, then $(m + 1)^2 \equiv 0$, (mod. n) and in addition $n + 1 \equiv 0$, (mod. m), when m is a prime or the square of a prime.

Some more solutions together with the corresponding values of l, m & n are given below :—

	(vii)	(viii)	(ix)	(x)	(xi)	(xii)	(xiii)	(xiv)	(xv)	(xvi)	(xvii)
$l =$	1,	1,	1,	1,	1,	1,	3,	4,	5,	2,	2,
$m =$	4,	4,	5,	8,	9,	9,	8,	5,	9,	3,	9,
$n =$	5,	25,	9,	27,	20,	50,	121,	81,	196,	25,	121,
$x =$	5,	9,	5,	6,	5,	8,	18,	20,	25,	12,	16,
$y =$	20,	36,	25,	48,	45,	75,	48,	25,	45,	18,	72,
$z =$	25,	225,	45,	162,	100,	400,	726,	405,	980,	150,	968,

An indefinite number of solutions may thus be obtained.

Question 993.

(PROF. HEMRAJ):—From a point P perpendiculars PL, PM, PN are drawn to the sides of a triangle ABC. If AL, BM, CN meet at the point Q, find the locus of P, and prove that PQ passes through a fixed point.

Solution by Martyn M. Thomas.

Taking ABC as the triangle of refersnce, let P be (α,β,γ) and Q (l,m,n) so that eqn. to AQL is $\frac{y}{z} = \frac{m}{n}$. $\therefore$ L is $(o,m,n,)$ and eqn. to PL is

$$\begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ o & m & n \end{vmatrix} = 0. \quad \therefore x(\beta n - \gamma m) - y\alpha n + z\alpha m = 0.$$

The condition that PL may be perpendicular to BC is therefore

$$-m(\alpha \cos B + \gamma) + n(\alpha \cos C + \beta) = 0. \quad \ldots \quad \ldots (1)$$

Similarly, conditions of perpendicularity for (PM, AC) and (PN, AB) are

$$l(\beta \cos A + \gamma) - n(\beta \cos C + \alpha) = 0, \quad \ldots (2)$$

$$-l(\gamma \cos A + \beta) + m(\gamma \cos B + \alpha) = 0. \quad \ldots (3)$$

Eliminating l,m,n from (1), (2) and (3), the locus of P is

$$\begin{vmatrix} o, & -(\alpha \cos B + \gamma), & (\alpha \cos C + \beta) \\ \beta \cos A + \gamma, & o, & -(\beta \cos C + \alpha) \\ -(\gamma \cos A + \beta), & \gamma \cos B + \alpha, & o \end{vmatrix} = 0.$$

i.e. $$\left(\cos A + \frac{\gamma}{\beta}\right)\left(\cos B + \frac{\alpha}{\gamma}\right)\left(\cos C + \frac{\beta}{\alpha}\right) - \left(\cos A + \frac{\beta}{\gamma}\right)\left(\cos B + \frac{\gamma}{\alpha}\right)\left(\cos C + \frac{\alpha}{\beta}\right) = o$$

i.e. $$\sum \left\{ \left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)\left(\cos C - \cos A \cos B\right) \right\} = 0.$$

This locus has been verified to pass through the in-centre (1,1,1), the three ex-centres, $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$ the circum-centre (cos A, cos B, cos C) and the ortho-centre (sec A, sec B, sec C).

Equation to the line PQ is

$$\begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ l & m & n \end{vmatrix} = 0,$$

which is seen to pass through the fixed point

(cos A — cos B cos C) : (cos B — cos C cos A) : cos C — cos A cso B).

For, if we multiply (1), (2), (3) by cos A, cos B, cos C and add, we have Σ [cos A — cos B cos C) $(3n - \gamma m,] = 0$, whence the result follows.

Question 995.

(T. P. Trivedi, M.A., LL.B):—Prove that

$$\int_0^1 x^x\,dx = \sum_{n=0}^{n=\infty}(-1)^n\left\{\frac{1^n}{1^{n+1}} - n\cdot\frac{n+1}{2^{n+2}} + \frac{n^2}{2!}\cdot\frac{(n+1)(n+2)}{3^{n+3}}\right.$$

$$\left. - \frac{n^3}{3!}\cdot\frac{(n+1)(n+2)(n+3)}{4^{n+4}} + \ldots\ldots\right\}$$

Solution by Martyn M. Thomas, M.A.

Let $x = e^{-y}$, so that $dx = -e^{-y}\,dy$, and limits for y are ∞ to 0.

$$\therefore \int_0^1 x^{x^x}\,dx = \int_0^\infty e^{-ye^{-ye^{-y}}} \times e^{-y}\,dy$$

$$= \int_0^\infty \sum_{n=0}^{\infty}(-1)^n\left(\frac{y^n e^{-nye^{-y}}}{n!}\right)e^{-y}dy$$

$$= \int_0^\infty \sum_{n=0}^{\infty}(-1)^n\cdot\frac{y^n}{n!}\left\{1 - \frac{nye^{-y}}{1!} + \frac{n^2y^2e^{-2y}}{2!} - \frac{n^3y^3e^{-3y}}{3!} + \ldots\right\}e^{-y}dy$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\infty\left(e^{-y}y^n - \frac{n}{1!}e^{-2y}y^{n+1} + \frac{n^2}{2!}e^{-3y}y^{n+2} + \ldots\right)dy$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left[\frac{n!}{1^{n+1}} - \frac{n}{1!}\cdot\frac{(n+1)!}{2^{n+2}} + \frac{n^2}{2!}\cdot\frac{(n+2)!}{3^{n+3}} - \ldots\right]$$

$$= \sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{1^{n+1}} - \frac{n}{1!}\cdot\frac{(n+1)}{2^{n+2}} + \frac{n^2}{2!}\cdot\frac{(n+1)(n+2)}{3^{n+3}} + \ldots\right].$$

Question 1034.

(V. Ramaswami Aiyar, M.A.):—Prove that the locus of the double foci of a system of confocal bicircular quartics consists of the two circular cubics which form the extreme members of the system.

Solution by M. Bhimasena Rao.

Let A, B, C, H, be the centres of inversion, DEF the ortho-centric triangle of ABCH, and P, Q, R, S the foci situated on a circle of inversion. The circular cubics of the system have for their focal conics the two parabolas through P, Q, R, S. The axes of the parabolas are at right angles, for P, Q, R, S are concyclic. If C_1 and C_2 be the centres of the

parabolas, the circular cubic the centre of whose focal parabola is C_1 passes through C_2 and *vice versa*. Both the cubics pass through the seven points A, B, C, H, D, E, F (Basset: *Cubic and Quartic Curves*). Since A, B, C, H are the in- and the ex-centres of the triangle DEF, the cubics are the loci of isogonal conjugate points, with respect to DEF, on parallel rays through C_1 and C_2.

The double foci of a bicircular quartic being the foci of its focal conic, we have to find the locus of the foci of the pencil of conics PQRS. Since P, Q, R, S are concyclic, the axes of the conics are fixed in direction and hence pass through C_1 and C_2—the centres of the parabolas of the pencil. If H is the centre of the circle PQRS, ABC is the diagonal triangle of PQRS and the focal conics are self-conjugate with respect to ABC. The foci of each conic are therefore isogonal conjugate points with respect to DEF, the orthocen trictriangle of ABC. (Question 13, Progress Report, Indian Mathematical Club), and being situated on lines passing through fixed points C_1 and C_2 at infinity, lie on the two circular cubics ABCHDEFC, and ABCHDEFC$_2$. Hence the theorem.

Question 1046.

(S. Krishnaswmi Aiyangar):—At every point O of a plane curve, a parabola and a rectangular hyperbola of closest contact with it are drawn. If the common normal meets the conics at P and Q, then

$$9.QO^3 + 4\,PO\ \rho_1^{\,2} = 0,$$

where ρ_1 is the radius of curvature at the corresponding point of the evolute to the curve.

Solution by Martyn M. Thomas, Kewalramani, Hemraj and V. Tiruvenkatachari.

Choosing the tangent and normal at O as axes of x and y, the eqn. to conics having four point contact with it at O will be

$$2y = ax^2 + 2hxy + by^2,$$

where a and h will be the same for all the conics; for,

$$2\rho = Lt.\ \frac{x^2}{y} \text{ at } O = \frac{2}{a};$$

and the centres of all the conics lie on the line

$$ax + hy = 0,$$

which is inclined in front of the normal at O at an angle $\tan^{-1}\left(\frac{1}{3}\frac{d\rho}{ds}\right)$

so that

$$-\frac{h}{a} = \frac{1}{3}\left(\frac{d\rho}{ds}\right).$$

$$\therefore \quad h = -\frac{1}{3\rho}\left(\frac{d\rho}{ds}\right).$$

Now b is different for different conics ; for a parabola $h^2 - ab = 0$ and for a rect. hyperbola $a + b = 0$.

Where the common normal meets these two conics

$$OP = \frac{2}{b}, \text{ as obtained from the parabola}$$

$$= \frac{2a}{h^2} = 18\rho \Big/ \left(\frac{d\rho}{ds}\right)^2 = 18\rho^3 \Big/ \left(\frac{d\rho}{d\psi}\right)^2 = \frac{18\rho^3}{\rho_1^2};$$

and $$OQ = \frac{2}{b}, \text{ as obtained from the rect. hyperbola}$$

$$= -\frac{2}{a} = -2\rho.$$

Eliminating ρ, we have, $\quad 9OQ^3 + 4OP\,\rho_1^2 = 0.$

Question 1047.

(S. KRISHNASAWMI AIYANGAR) :—If ρ be the radius of curvature at any point of a cardioide and ρ_o, ρ_α, $\rho_{\alpha\beta}$ at the corresponding points of its evolute, α-evolute, and the β-evolute of the α-evolute, show that

$$\rho_{\alpha\beta} = \rho_\alpha \sin\beta + \rho_o \sin\alpha\cos\beta - \frac{1}{9}\rho\cos\alpha\cos\beta.$$

Solution by Martyn M. Thomas.

It has been shown, in Question 854, that for a curve in general,

$$\rho_\alpha = \rho\left(\frac{d\rho}{ds}\cos\alpha + \sin\alpha\right)$$

$$\rho_{\alpha\beta} = \rho\left(\frac{d\rho}{ds}\cos\alpha + \sin\alpha\right)\left(\frac{d\rho}{ds}\cos\beta + \sin\beta\right) + \rho^2\frac{d^2\rho}{ds^2}\cos\alpha\cos\beta.$$

$$\therefore \quad \rho_{\alpha\beta} = \rho_\alpha \sin\beta + \rho\frac{d\rho}{ds}\sin\alpha\cos\beta + \rho\cos\alpha\cos\beta\left\{\left(\frac{d\rho}{ds}\right)^2 + \rho\frac{d^2\rho}{ds^2}\right\}.$$

In the case of a cardioide,

$$r = a(1 - \cos\theta), \quad \rho = \frac{4}{3}a\sin\frac{\theta}{2},$$

$$\frac{d\rho}{ds} = \frac{1}{3}\cot\frac{\theta}{2}, \quad \rho\frac{d^2\rho}{ds^2} = -\frac{1}{9}\operatorname{cosec}^2\frac{\theta}{2}.$$

$$\therefore \quad \rho_{\alpha\beta} = \rho_\alpha\sin\beta + \frac{d\rho}{d\psi}\sin\alpha\cos\beta + \rho\cos\alpha\cos\beta\left(-\frac{1}{9}\right)$$

$$= \rho_\alpha\sin\beta + \rho_o\sin\alpha\cos\beta - \frac{1}{9}\rho\cos\alpha\cos\beta.$$

Question 1054.

(MARTYN M. THOMAS) :—If s_r denote the sum of the squares of the fractions $\frac{1}{1}, \frac{1}{2}, \ldots \frac{1}{r}$, and p_r denote the sum of their products taken two at a time, show that

$$ {}_nC_1 s_1 - {}_nC_2(s_2 + p_2) + {}_nC_3(s_3 + p_3) - \ldots + (-1)^{r-1} {}_nC_r(s_r + p_r) = \frac{1}{n^2}. $$

Solution by M. V. Ramakrishnan.

[The question is incorrect. The summation should extend to n terms.]

$$ {}_nC_1 s_1 - {}_nC_2(s_2 + p_2) + \ldots\ldots + (-1)^{n-1} {}_nC_n(s_n + p_n) $$
$$ = s_1 + (n-1)\{s_1 - (s_2 + p_2)\} $$
$$ - \frac{(n-1)(n-2)}{2!}\{(s_2 + p_2) - (s_3 + p_3)\} + \ldots\ldots \text{to } n \text{ terms.} $$

But

$$ (s_r + p_r) - (s_{r-1} + p_{r-1}) = \frac{1}{r^2} + \frac{1}{r} + \frac{1}{2 \cdot r} + \ldots + \frac{1}{(r-1)r}. $$

Hence if S denote the sum of the series

$$ S = s_1 - \frac{(n-1)}{2}\left(\tfrac{1}{1} + \tfrac{1}{2}\right) + \frac{(n-1)(n-2)}{3!}\left(\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3}\right) - \ldots\ldots $$

$$ = \frac{1}{n}\left\{ns_1 - \frac{n(n-1)}{2!}\left(\tfrac{1}{1} + \tfrac{1}{2}\right) + \frac{n(n-1)(n-2)}{3!}\left(\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3}\right) - \ldots\ldots\right\} $$

$$ = -\frac{1}{n}\left\{(-t_1) - (n-1)(t_1 - t_2) + \frac{(n-1)(n-2)}{2!}(t_2 - t_3) - \ldots\right\}, $$

where $t_r = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{r}$.

$$ = \frac{1}{n^2}\left\{{}_nC_1 - {}_nC_2 + \ldots + (-1)^{n-1} {}_nC_n\right\} = \frac{1}{n^2}. $$

Question 1062.

(C. KRISHNAMACHARI) :—Prove that

$$ 1 + \binom{n-1}{1}\frac{1}{3} + \binom{n-1}{2}\frac{1}{5} + \ldots\ldots + \frac{1}{2n-1} $$
$$ = 2^{n-1}\left\{\frac{1}{2n-1} + \frac{n-1}{(2n-1)(2n-3)} + \ldots + \frac{\lfloor n-1}{(2n-1)(2n-3)\ldots 1}\right\}. $$

Solution by M. V. Ramakrishnan.

The left side of the equality is equal to the coefft. of x^{2n-1} in the expansion of $\frac{1}{2}(1+x^2)^{n-1}\log\frac{1+x}{1-x}$.

Hence if

$$A_0+A_1x+\ldots+A_{2n-1}x^{2n-1}+\ldots=\tfrac{1}{2}(1+x^2)^{n-1}\log\frac{1+x}{1-x}, \quad (1)$$

then $A_{2n-1}=1+\binom{n-1}{1}\frac{1}{3}+\binom{n-1}{2}\frac{1}{5}+\ldots+\frac{1}{2n-1}$.

Differentiating (1) and equating coefft. of x^{2n-2}, we get

$$(2n-1)A_{2n-1}=\text{coefft. of } x^{2n-2} \text{ in } \frac{(1+x^2)^{n-2}}{1-x^2}$$

$$+\text{ coefft. of } x^{2n-2} \text{ in } (n-1)x(1+x^2)^{n-2}\log\frac{1+x}{1-x}$$

$$=2^{n-1}+\text{coefft. of } x^{2n-2} \text{ in } (n-1)x(1+x^2)^{n-2}\log\frac{1+x}{1-x}.$$

$$\therefore (2n-1)A_{2n-1}-2^{n-1}=\text{coefft. of } x^{2n-2} \text{ in } (n-1)x(1+x^2)^{n-2}\log\frac{1+x}{1-x}.$$

Hence, if $B_{2n-1}=(2n-1)A_{2n-1}-2^{n-1}$, we get

$$B_0+B_1x+\ldots+B_{2n-1}x^{2n-2}+\ldots=(n-1)x(1+x^2)^{n-2}\log\frac{1+x}{1-x},$$

or $$\frac{B_0}{x}+B_1+\ldots+B_{2n-1}x^{2n-3}+\ldots=(n-1)(1+x^2)^{n-2}\log\frac{1+x}{1-x}.$$

Differentiating and equating coefft. of x^{2n-3}, we get

$$(2n-3)B_{2n-1}=2^{n-1}+\text{coefft. of } x^{2n-3} \text{ in } 2(n-1)(n-2)x(1+x^2)^{n-3}\log\frac{1+x}{1-x}.$$

Repeating this process $(n-1)$ times, we get the result.

QUESTIONS FOR SOLUTION.

1113. [SELECTED]:—An ellipse of eccentricity $\sin 2\alpha$, passes through the focus of a parabola of latus rectum $4a$ and has its foci on the curve. Show that the major axis envelopes the parabola

$$y^2 = 4a(1 - \tan^2\alpha)(x - a\tan^2\alpha);$$

and that the minor axis envelopes the semicubical parabola

$$27a\sec^2\alpha\, y^2 = 4(x - a - a\sec^2\alpha)^3$$

1114. (T. P. TRIVEDI, M.A. LL. B):—If p, q are any two integers selected at random; prove that the probability that the fraction (p/q) is in its lowest terms is $6/\pi^2$.

1115. (T. P. TRIVEDI, M.A., LL. B):—Prove that

(i) $$\frac{2^2-1}{2^2+1}\cdot\frac{3^2-1}{3^2+1}\cdot\frac{5^2-1}{5^2+1}\cdot\frac{7^2-1}{7^2+1}\ldots ad\ inf. = \frac{2}{5},$$

(ii) $$\frac{2^4-1}{2^4+1}\cdot\frac{3^4-1}{3^4+1}\cdot\frac{5^4-1}{5^4+1}\cdot\frac{7^4-1}{7^4+1}\ldots ad\ inf. = \frac{6}{7},$$

where 2, 3, 5, etc. are all the prime numbers.

1116. (Prof. K. J. SANJANA):—If r be a positive integer (including zero) such that $(x - r\pi)$ always lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, shew that

(i) $$\int_{-\infty}^{\infty}\frac{x - r\pi}{x}\,dx = \frac{\pi}{2}\log 2,$$

and (ii) $$\int_0^{\infty}(-1)^r\frac{x - r\pi}{x}\,dx = 2\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots ad\ inf.\right)$$

1117. (Prof. K. J. SANJANA):—If a given function of x, y, z be transformed by the substitutions

$$r = \tfrac{1}{2}\log(x^2 + y^2 + z^2),\quad \phi = \tan^{-1}\frac{y}{x},\quad \theta = \tan^{-1}\frac{(x^2 + y^2)^{\frac{1}{2}}}{z},$$

prove that the operation $(x^2 + y^2 + z^2)\left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}\right)$ is transformed into $\left(\frac{\delta^2}{\delta r^2} + \frac{\delta^2}{\delta\theta^2} + \operatorname{cosec}^2\theta\,\frac{\delta^2}{\delta\phi^2} + \frac{\delta}{\delta r} + \cot\theta\,\frac{\delta}{\delta\theta}\right)$.

1118. (N. DORAI RAJAN):—Show that in the cubical parabola $x^3 = a^2y$, all tangent chords are trisected by the y —axis.

1119. (N. Dorai Rajan) :—A triangle having its angular points on a limacon has its orthocentre at the node. Show that its circumcircle touches a fixed circle.

1120. (N. Dorai Rajan) :—Show that the curve whose (p, r) equation is $r = a + p$, is an involute of an involute of a circle, *i.e.*, a 2nd involute of a circle. Find its polar equation and trace the curve.

1121. (N. Dorai Rajan) :—Constant lengths a are measured along the tangents to a curve. If the locus of the extremities be a circle of radius a, show that the curve is the inverse of an involute of a circle of radius $2a$, the radius of inversion being equal to $2a$.

1122. (V. Ramaswami Aiyar) :—If the joins of four concyclic points A,B,C,D taken in opposite pairs intersect in P,Q,R, prove that the nine points circle of PQR passes through the centroid of the points A, B, C, D.

1123. (G. S. M.) :—Find a formula for the number of regions into which space of three dimensions is divided by n planes.

1124. (G. S. M.) :—Prove that

$$1 + (r, 1)\frac{n}{m} + (r, 2)\frac{n(n-1)}{m(m-1)} + \ldots\ldots \text{ to } (n+1) \text{ terms}$$

$$= \frac{(m+1)(m+2)\ldots\ldots(m+r)}{(m-n+1)(m-n+2)\ldots\ldots(m-n+r)},$$

where $(r, 1)$, $(r, 2)$, ... are the *figurate* numbers of the rth order.

1125. (Communicated by G. S. M.) :—Prove that

$$n!\ \mathrm{D}^{-(n+1)}(u) = x^n\ \mathrm{D}^{-1}(u) - c_1\ x^{n-1}\ \mathrm{D}^{-1}(xu) +$$

$$\ldots\ldots + (-1)^n\ \mathrm{D}^{-1}(x^n u),$$

where $\mathrm{D} \equiv \frac{d}{dx}$ and $u \equiv f(x)$.

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AN ALGEBRA OF ARITHMETICAL FUNCTIONS.

By F. Hallberg.

(*Continued from page* 169).

Algebraical Analysis.

§ 8. *The binomial theorem.*

Consider the following formal development.

$$\left\{1 + \mathrm{F}(s)\right\}^k = 1 + \frac{k}{1}\mathrm{F}(s) + \frac{k(k-1)}{2!}\mathrm{F}^2(s) + \ldots$$

$$= \frac{1}{1^s}\cdot\{1 + f(1)\}^k + \frac{1}{2_s}\cdot\frac{k}{1!}\{1+f(1)\}^{k-1}f(2)$$

$$+ \frac{1}{3}\cdot\frac{k}{1!}\{1 + f(1)\}^{k-1}f(3)$$

$$+ \frac{1}{4^s}\cdot\left\{\frac{k}{1!}\{1+f(1)\}^{k-1}f(4) + \frac{k(k-1)}{2!}\{1 + f(1)\}^{k-2}f^2(2)\right\}$$

$$+ \frac{1}{5_s}\cdot\frac{k}{1!}\{1 + f(1)\}^{k-1}f(5)$$

$$+ \frac{1}{6_s}\cdot\left\{\frac{k}{1!}\{1 + f((1\}^{k-1}f(6)\right.$$

$$\left. + \frac{k(k-1)}{1!\ 1!}\{1_k + f(1)\}^{k-2}f(2)f(3)\right\} + \ldots$$

Changing $[f(n)]$ to $[f(n)] - 1$, we obtain

$$\mathrm{F}^k(s) = \frac{1}{1^s}\cdot f^k(1) + \frac{1}{2^s}\cdot\frac{k}{1!}f^{k-1}(1)f(2) + \frac{1}{3^s}\cdot\frac{k}{1!}f^{k-1}(1)f(3)$$

$$+ \frac{1}{4^s}\left\{\frac{k}{1!} f^{k-1}(1) f(4) + \frac{k(k-1)}{2!} f^{k-2}(1) f^2(2)\right\}$$

$$+ \frac{1}{5^s} \cdot \frac{k}{1!} f^{k-1}(1) f(5) +$$

$$+ \frac{1}{6^s} \cdot \left\{\frac{k}{1!} f^{k-1}(1) f(6) + \frac{k(k-1)}{1!\,1!} f^{k-2}(1) f(2) f(3)\right\} + \ldots$$

Consequently, using the general definition of a function of an **A. F.**, we deduce

$$[f(n)]^k = \left[f^k(1), \frac{k}{1!} f^{k-1}(1) f(2), \frac{k}{1!} f^{k-1}(1) f(3),\right.$$

$$\left\{\frac{k}{1!} f^{k-1}(1) f(4) + \frac{k(k-1)}{2!} f^{k-2}(1) f^2(2)\right\}, \frac{k}{1!} f^{k-1}(1) f(5),$$

$$\left.\left\{\frac{k}{1!} f^{k-1}(1) f(6) + \frac{k(k-1)}{1!\,1!} f^{k-2}(1) f(2) f(3)\right\}, \ldots\right]$$

$$= \left[\sum_n \frac{k\,k-1)\ldots(k-\lambda+1)}{\alpha_1!\,\alpha_2!\ldots\alpha_r!} f^{k-\lambda}(1) f^{\alpha_1}(a_1) f^{\alpha_2}(a_2)\ldots f^{\alpha_r}(a_r)\right] \ldots (8.1)$$

where the sum is extended to every decomposition of n in the form

$$n = a_1 a_2 \ldots a_r \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8.11)$$

$a_1, a_2, \ldots, a_r$ being all different, λ being given by the formula

$$\lambda = \alpha_1 + \alpha_2 + \ldots + \alpha_r \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8.12)$$

Since no assumption was made about the constant k, the formula (8.1) is perfectly general.

If k is a positive integer, the terms under the sign $\sum_n$, for which $\lambda > k$, vanish. In particular the case $k = 2$ gives

$$[f(n)]^2 = \left[\sum_{d:n} f(d) \cdot f\left(\frac{n}{d}\right)\right], \quad \ldots \quad \ldots \quad (8.2)$$

in agreement with (2.3).

In case the condition (5.1) is satisfied, we have, putting

$$[1]^k = [K(k, n)] = \left[\sum_n \frac{k(k-1)\ldots(k-\lambda+1)}{\alpha_1!\,\alpha_2!\ldots\alpha_r!}\right]$$

$$= [1, k, k, \tfrac{1}{2}k(k+1), k, k^2, k, \tfrac{1}{6}k(k+1)(k+2), \ldots] \ldots (8.3)$$

as equivalent to (8.1), the formula

$$[f(n)]^k = [K(k, n) \cdot f(n)], \quad \ldots \quad \ldots \quad \ldots \quad (8.4)$$

which, for $k = 2$, gives, by (7.1)

$$[f(n)]^2 = [t(n) \cdot f(n)], \quad \ldots \quad \ldots \quad \ldots \quad (8.41)$$

while, for $k = -1$, (5.3) is obtained. The latter fact enables us to give a definite expression for the A. F. $[\mu(n)]$. The result is

$$\mu(n) = \sum_{n} \frac{(-1^{\lambda})\,\lambda\,!}{\alpha_1!\,\alpha_2!\,\ldots\,\alpha_r!}, \qquad \ldots \qquad \ldots \quad (8.5)$$

(for $n > 1$).

It follows from our definitions, that the law of indices

$$[f(n)]^k \cdot [f(n)]^l = [f(n)]^{k+l} \qquad \ldots \qquad \ldots \quad (8.6)$$

is valid. The elemental theorem of order $n = 6$, for instance, obtained from this formula by the aid of (8.1), has the appearance

$$f^k(1).\left\{\frac{l}{1!}f^{l-1}(1)\,f(6) + \frac{l(l-1)}{1!\;1!}f^{l-2}(1)\,f(2)\,f(3)\right\}$$

$$+\frac{k}{1!}f^{k-1}(1)\,f(2).\frac{l}{1!}f^{l-1}(1)\,f(3) + \frac{k}{1!}f^{k-1}(1)\,f(3).\frac{l}{1!}f^{l-1}(1)\,f(2)$$

$$+\left\{\frac{k}{1!}f^{k-1}(1)\,f(6) + \frac{k(k-1)}{1!\;1!}f^{k-2}(1)\,f(2)\,f(3)\right\}.f^{l}(1)$$

$$= \frac{k+l}{1!}f^{k+l-1}(1)\,f(6) + \frac{(k+l)(k+l-1)}{1!\;1!}f^{k+l-2}(1)\,f(2)\,f(3),$$

a relation which is easily seen to be an identity.

Similarly we have

$$\left\{[f(n)]^k\right\}^{\frac{1}{k}} = \left\{[f(n)]^{\frac{1}{k}}\right\}^k = [f(n)] \ldots\ldots (8.7)$$

The first member of this equation gives, for instance, for $n = 6$:

$$\frac{1}{k}\cdot\frac{1}{1!}\left\{f^k(1)\right\}^{\frac{1}{k}-1}\cdot\left\{\frac{k}{1!}f^{k-1}(1)\,f(6) + \frac{k(k-1)}{1!\;1!}f^{k-2}(1)\,f(2)\,f(3)\right\}$$

$$+\frac{1}{k}\left(\frac{1}{k}-1\right)\cdot\frac{1}{1!\;1!}\left\{f^k(1)\right\}^{\frac{1}{k}-2}\cdot$$

$$\cdot\left\{\frac{k}{1!}f^{k-1}(1)\,f(2).\frac{k}{1!}f^{k-1}(1)\,f(3)\right\} = f(6),$$

an obvious identity.

It follows from (8·3) that

$$[1]\cdot[K(k,n)] = [K(k+1,n)] \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (8.8)$$

and hence, from a theorem of *Laguerre* (See I. S. and A. F., p. 255), by the aid of (6.1), (6.11), that

$$\sum_{n=1}^{\infty} K(k, n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} x^n K(k+1, n) \quad \ldots \quad \ldots \quad (8.81)$$

Differentiation of this series gives

$$\sum_{n=1}^{\infty} K(k, n) \frac{n x^n}{(1-x^n)^2} = \sum_{n=1}^{\infty} n x^n K(k+1, n), \quad \ldots \quad \ldots \quad (8.82)$$

whilst by integration we obtain

$$\prod_{n=1}^{\infty} (1-x^n)^{\frac{K(k, n)}{n}} = \operatorname{Exp} \left\{ -\sum_{n=1}^{\infty} \frac{x_n}{n} K(k+1, n) \right\}. \quad \ldots \quad 8.83)$$

It would be of the greatest importance for the Theory of Numbers, if the order of

$$\sum_{r=1}^{n} K(k, r)$$

could be obtained, when $n \to \infty$. This would, in particular, give us the order of $\sigma(n)$, defined by (6.5), and thus solve the difficulty of Riemann's proposition over the imaginary roots of the ζ-function. (See *Encycl.*, l.c., p. 351; or I. S. & A. F., p. 263, where this question is shortly discussed.) It should be noticed, that $K(k, n)$, and thus also the required sum, may be regarded as a continuous function of k, a fact which may be of use, if the solution of the above problem is attempted. The algebraical difficulties, however, are certainly very great.

As a verification of some of the above methods, consider the case $k = -1$.

We have

$$\left[f(n) \right]^{-1} = \frac{1}{1 - \{1 - [f(n)]\}}$$

$$= 1 + \{1 - [f(n)]\} + \{1 - [f(n)]\}^2 + \{1 - [f(n)]\}^3 + \ldots$$

If we now develop the brackets of this series by the aid of (8.1) and then add the series of A. F. obtained, the result is

$$\left[f(n)\right]^{-1} = \left[f^{-1}(1), \frac{-1}{1!} f^{-2}(1) f(2), \frac{-1}{1!} f^{-2}(1) f(3),\right.$$
$$\left.\left\{\frac{-1}{1!} f^{-2}(1) f(4) + \frac{(-1)(-2)}{2!} f^{-3}(1) f^2(2)\right\}, \ldots\right] \quad \ldots \quad (8.9)$$

in agreement with (8.1), provided $f(1) \pm 0$. Hence, if this condition is satisfied, division by $[f(n)]$ is always possible.

§ 9. *Maclaurin's theorem.*

We have, γ being an analytic function (subject to certain conditions)

$$\gamma(z) = \sum_{r=0}^{\infty} \frac{z^r}{r!} \gamma^{(r)}(0). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9.1)$$

In this formula, put $[f(n)]$ for z.

$$\therefore \quad \gamma[f(n)] = [g(n)] = \sum_{r=0}^{\infty} \frac{[f(n)]^r}{r!} \gamma^{(r)}(0). \quad \ldots \quad \ldots \quad (9.2)$$

If we now develop the powers of $[f(n)]$, occurring in the terms of the series, by (8·1), and afterwards reduce, we obtain the value of $[g(n)]$. The result is

$$[g(n)] = \left[\sum_{r=0}^{\infty} \frac{f^r(1)}{r!} \gamma^{(r)}(0), f(2) \sum_{r=0}^{\infty} \frac{r+1}{1!} \frac{f^r(1)}{(r+1)!} \gamma^{(r+1)}(0),\right.$$
$$f(3) \sum_{r=0}^{\infty} \frac{r+1}{1!} \frac{f^r(1)}{(r+1)!} \gamma^{(r+1)}(0), \left\{ f(4) \sum_{r=0}^{\infty} \frac{r+1}{1!} \frac{f^r(1)}{(r+1)!} \gamma^{(r+1)}(0)\right.$$
$$\left.\left. + f^2(2) \sum_{r=0}^{\infty} \frac{(r+1)(r+2)}{2!} \frac{f^r(1)}{(r+2)!} \gamma^{(r+2)}(0) \right\}, \ldots \right], \quad \ldots (9.21)$$

which may be written

$$\gamma\left[f(n)\right] = \left[\gamma\{f(1)\}, \frac{f(2)}{1!} \gamma'\{f(1)\}, \frac{f(3)}{1!} \gamma'\{f(1)\},\right.$$
$$\left\{\frac{f(4)}{1!} \gamma'\{f(1)\} + \frac{f^2(2)}{2!} \gamma''\{f(1)\}\right\}, \frac{f(5)}{1!} \gamma'\{f(1)\},$$
$$\left.\left\{\frac{f(6)}{1!} \gamma'\{f(1)\} + \frac{f(2) \cdot f(3)}{1!\,1!} \gamma''\{f(1)\}, \ldots\right]\right.$$
$$= \left[\sum_{n} \frac{f^{\alpha_1}(a_1)}{\alpha_1!} \frac{f^{\alpha_2}(a_2)}{\alpha_2!} \cdots \frac{f^{\alpha_r}(a_r)}{\alpha_r!} \gamma^{(\lambda)}\{f(1)\}\right], \quad \ldots (9.3)$$

the notation being explained in (8.11), (8.12).

We see at once, that formula (8 . 1) is a special case of (9 . 3), obtained for $\gamma(z) = z^k$. The reason, while the binominal theorem is considered before the general case, is that in Maclaurin's theorem powers of the A. F. employed occur, for the development of which the former theorem is necessary. It should be noticed, however, that Maclaurin's theorem could also have been obtained directly from the fundamental definition (by the aid of Dirichlet's series, as in § 8, or otherwise). The above method has been chosen, because it makes the reciprocal nature of the two theorems apparent.

Assume the condition (5 . 1) to be satisfied. The formula (9 . 3) is then reduced to the following :—

$$\gamma[f(n)] = [f(n) \,.\, \eta(n)], \qquad \ldots \qquad \ldots \quad (9.4)$$

where

$$[\eta(n)] = \gamma[1], \qquad \ldots \qquad \ldots \qquad \ldots \quad (9.41)$$

and thus

$$\eta(n) = \sum_{n} \frac{\gamma^{(\lambda)}(1)}{\alpha_1 ! \; \alpha_2 ! \; \ldots \; \alpha_r !}$$

$$= \sum_{r=1}^{\infty} \frac{K(r, n)}{r !} \gamma^{(r)}(0), \qquad \ldots \qquad \ldots \quad (9.411)$$

for $n > 1$, $K(k, n)$ being the A. F. defined by (8 . 3). The last expression is obtained directly from (9 . 2) by putting $[f(n)] = [1]$, which method gives

$$\gamma[1] = \gamma(0) + \sum_{r=1}^{\infty} \frac{[K(r, n)]}{r !} \gamma^{(r)}(0).$$

The identity between the two expressions for $\eta(n)$ may readily be established in each special case. For instance, $n = 6$ gives

$$\sum_{r=1}^{\infty} \frac{K(r, 6)}{r !} \gamma^{(r)}(0) = \sum_{r=1}^{\infty} \frac{r^2}{r !} \gamma^{(r)}(0)$$

$$= \sum_{r=1}^{\infty} \frac{1}{(r-1) !} \gamma^{(r)}(0) + \sum_{r=1}^{\infty} \frac{1}{(r-2) !} \gamma^{(r)}(0)$$

$$= \sum_{r=0}^{\infty} \frac{1}{r !} \gamma^{(r+1)}(0) + \sum_{r=0}^{\infty} \frac{1}{r !} \gamma^{(r+2)}(0)$$

$$= \frac{1}{1!}\gamma'(1) + \frac{1}{1!\,1!}\gamma''(1) = \eta\,(6).$$

As an example of (9 . 4), let $\gamma(z) = (1 + z)^k$, and consider $z = [1]$.

$$\therefore \quad (1 + [1])^k = [2, 1, 1, 1, \ldots]^k$$

$$= \left[2^k,\ \frac{k}{1!}\,2^{k-1},\ \frac{k}{1!}\,2^{k-1},\ \left\{\frac{k}{1!}\,2^{k-1} + \frac{k(k-1)}{2!}\,2^{k-2}\right\},\right.$$

$$\left.\frac{k}{1!}\,2^{k-1},\ \left\{\frac{k}{1!}\,2^{k-1} + \frac{k}{1!\,1!}\,2^{k-2}\right\},\ \ldots\right]$$

$$= \left[2^k,\ \sum_{r=1}^{\infty} r\ \mathrm{C}_r^k,\ \sum_{r=1}^{\infty} r\ \mathrm{C}_r^k,\ \sum_{r=1}^{\infty} \frac{r(r+1)}{2}\,\mathrm{C}_r^k,\right.$$

$$\left.\sum_{r=1}^{\infty} r\ \mathrm{C}_r^k,\ \sum_{r=1}^{\infty} r^2\,\mathrm{C}_r^k,\ \ldots\right],$$

the nth element ($n > 1$) being

$$\sum_{r=1}^{\infty} K(r, n)\,\mathrm{C}_r^k.$$

§ 10. *The exponential and logarithmic functions.*

Put in (9 . 3) $\gamma(z) = e^z$.

We find

$$e^{[f(n)]} = e^{f(1)}\left[1,\ \frac{f(2)}{1!},\ \frac{f(3)}{1!},\ \left\{\frac{f(4)}{1!} + \frac{f^2(2)}{2!}\right\},\right.$$

$$\left.\frac{f(5)}{1!},\ \left\{\frac{f(6)}{1!} + \frac{f(2)\cdot f(3)}{1!\,1!}\right\},\ \ldots\right]$$

$$= e^{f(1)}\left[\sum_n \frac{f^{\alpha_1}(a_1)}{a_1!}\,\frac{f^{\alpha_2}(a_2)\ \ldots\ f^{\alpha_r}(a_r)}{a_2!\ \ldots\ a_r!}\right], \qquad \ldots (10\,.\,1,$$

which, *in case* $f(n)$ *satisfies the relation* (5 . 1), becomes

$$e^{[f(n)]} = e^{[f(n)\,.\,\varepsilon(n)]}, \qquad \ldots\ \ldots\ \ldots\ (10\,.\,2)$$

where

$$[\varepsilon(n)] = e^{[1]-1}; \qquad \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ (10.21)$$

$$\varepsilon(1) = 1; \qquad \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ (10.210)$$

$$\varepsilon(n) = \sum_{n} \frac{1}{a_1!\, a_2! \ldots a_r!} =$$

$$= \frac{1}{e} \sum_{r=1} \frac{K(r, n)}{r!}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10.211)$$

for $n > 1$

In a similar manner we treat the logarithmic function

$$\gamma(z) = \log z.$$

We obtain

$$\log [f(n)] = \left[\log f(1), \frac{1}{1!} \frac{f(2)}{f(1)} \frac{1}{1!} \frac{f(3)}{f(1)}, \right.$$

$$\left\{ \frac{1}{1!} \frac{f(4)}{f(1)} - \frac{1!}{2!} \left\{ \frac{f(2)}{f(1)} \right\}^2 \right\}, \frac{1}{1!} \frac{f(5)}{f(1)},$$

$$\left. \left\{ \frac{1}{1!} \frac{f(6)}{f(1)} - \frac{1!}{1!\ 1!} \frac{f(2)}{f(1)}, \frac{f(3)}{f(1)} \right\}, \ldots \right] \quad \ldots \quad (10.3)$$

the nth element for $n > 1$ of the A. F. being

$$\sum_{n} \frac{f^{\alpha_1}(a_1) f^{\alpha_2}(a_2) \ldots f^{\alpha_r}(a_r)}{a_1!} \frac{}{a_2! \quad \ldots \quad a_r!} \frac{(-1)^{\lambda-1} \cdot (\lambda-1)!}{f^{\lambda}(1)}, \quad \ldots \quad (10.31)$$

In case (5·I) is satisfied, this reduces to

$$\log [f(n)] = [f(n) \cdot l(n)], \quad \ldots \quad \ldots \quad (10.4)$$

where

$$[l(n)] = \log [1]; \quad \ldots \quad \ldots \quad \ldots \quad (10.41)$$

$$l(1) = 0; \quad \ldots \quad \ldots \quad \ldots \quad (10.410)$$

$$l(n) = \sum_{n} \frac{(-1)^{\lambda-1} (\gamma-1)!}{a_1!\, a_2!\, \ldots\, a_r!}, \ldots \quad \ldots \quad (10.411)$$

for $n > 1$, the second expression for $l(n)$ being in this case unvalid.

The A. F. defined by (10.1), (10.3) now obey all the laws of the ordinary exponential and logarithmic functions. We have, for instance,

$$e^{[f(n)]} \cdot e^{[g(n)]} = e^{[f(n) + g(n)]}; \quad \ldots \quad \ldots \quad \ldots \quad (10.5)$$

$$\log [f(n)] + \log [g(n)] = \log \{ [f(n)] \cdot [g(n)] \}; \quad \ldots \quad (10.6)$$

$$\log e^{[f(n)]} = e^{\log [f(n)]} = [f(n)] \quad \ldots \quad \ldots \quad \ldots \quad (10.7)$$

As an example, it will be sufficient to consider the first part of (10.7). We get

$$\log e^{[f(n)]} = \log e^{f(1)} + \log \left[\sum_{n} \frac{f^{\alpha_1}(a_1)\ f^{\alpha_2}(a_2) \ldots f^{\alpha_r}(a_r)}{a_1!} \frac{}{a_2!} \frac{}{\ldots} \frac{}{a_r!} \right]$$

$$= f(1) + \left[\log 1,\ \frac{1}{1!}\frac{f(2)}{1!},\ \frac{1}{1!}\frac{f(3)}{1!},\ \left\{\frac{1}{1!}\frac{f(4)}{1!} + \frac{1}{1!}\frac{f^2(2)}{2!}\right.\right.$$

$$\left.\left. - \frac{1!}{2!}\left\{\frac{f(2)}{1!}\right\}^2\right\},\ \frac{1}{1!}\frac{f(5)}{1!},\ \left\{\frac{1}{1!}\frac{f(6)}{1!} + \frac{1}{1!}\frac{f(2).f(3)}{1!\ 1!}\right.\right.$$

$$\left.\left. - \frac{1!}{1!\ 1!}\frac{f(2)}{1!}\frac{f(3)}{1!}\right\},\ \ldots\right] \quad [f(1), f(2), f(3), f(4), f(5), f(6), \ldots]$$

$$= [f(n)].$$

§ 11. *Extended definition of a power.*

As yet we have only dealt with powers of the form

$$[f(n)]^k,\ e^{[f(n)]}.$$

We are now in a position to extend our definitions to the most general case. We define

$$[f(n)]^{[g(n)]} = e^{[g(n)].\log[f(n)]}. \quad \ldots \quad \ldots \quad \ldots \quad (11.1)$$

Now, by (1.03)

$$[g(n)].\log[f(n)] = \left[g(1)\log f(1),\ \left\{g(2)\log f(1) + \frac{g(1)}{1!}\frac{f(2)}{f(1)}\right\},\right.$$

$$\left\{g(3)\log f(1) + \frac{g(1)}{1!}\frac{f(3)}{f(1)}\right\},\ \left\{g(4)\log f(1) + \frac{g(2)}{1!}\frac{f(2)}{f(1)} + g(1)\left\{\frac{1}{1!}\frac{f(4)}{f(1)} - \frac{1!}{2!}\left(\frac{f(2)}{f(1)}\right)^2\right\}\right\},\ \left.\ldots\right].$$

Hence, by (10.1)

$$[f(n)]^{[g(n)]} = f(1)\,e^{g(1)}\left[1, \frac{1}{1!}\left\{g(2)\log f(1) + \frac{g(1)}{1!}\frac{f(2)}{f(1)}\right\},\right.$$

$$\frac{1}{1!}\left\{g(3)\log f(1) + \frac{g(1)}{1!}\frac{f(3)}{f(1)}\right\},\ \left\{\frac{1}{1!}\left\{g(4)\log f(1) + \frac{g(2)}{1!}\frac{f(2)}{f(1)}\right.\right.$$

$$\left.\left. + g(1)\left\{\frac{1}{1!}\frac{f(4)}{f(1)} - \frac{1!}{2!}\left(\frac{f(2)}{f(1)}\right)^2\right\}\right\} + \frac{1}{2!}\left\{g(2)\log f(1) + \right.\right.$$

$$\left.\left. + \frac{g(1)}{1!}\frac{f(2)}{f(1)}\right\}^2\right\},\ \ldots\left.\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11.2)$$

§ 12. *The trigonometrical functions.*

If in (9.3) we assume

$$\gamma(z) = \sin z,$$

we obtain

$$\sin[f(n)] = \Big[\sin f(1),\ \frac{f(2)}{1!}\cos f(1),\ \frac{f(3)}{1!}\cos f(1),$$

$$\Big\{\frac{f(4)}{1!}\cos f(1) - \frac{f^2(2)}{2!}\sin f(1)\Big\},\ \frac{f(5)}{1!}\cos f(1),$$

$$\Big\{\frac{f(6)}{1!}\cos f(1) - \frac{f(2)\,.\,f(3)}{1!\quad 1!}\sin f(1)\Big\},\ \ldots\Big]$$

$$= \Big[\sum_n \frac{f^{\alpha_1}(a_1)\,f^{\alpha_2}(a_2)\,\ldots\,f^{\alpha_r}(a_r)}{\alpha_1!\quad \alpha_2!\quad \ldots\quad \alpha_r!}\sin\Big\{f(1) + \lambda\frac{\pi}{2}\Big\}\Big]\,.\ \ldots \qquad (12.1)$$

Similarly for

$$\gamma(z) = \cos z$$

$$\cos[f(n)] = \Big[\cos f(1), \Big\{-\frac{f(2)}{1!}\sin f(1)\Big\}, \Big\{-\frac{f(3)}{1!}\sin f(1)\Big\},$$

$$\Big\{-\frac{f(4)}{1!}\sin f(1) - \frac{f^2(2)}{2!}\cos f(1)\Big\}, \Big\{-\frac{f(5)}{1!}\sin f(1)\Big\},$$

$$\Big\{-\frac{f(6)}{1!}\sin f(1) - \frac{f(2)\,.\,f(3)}{1!\quad 1!}\cos f(1)\Big\},\ \ldots\Big] =$$

$$= \Big[\sum_n \frac{f^{\alpha_1}(a_1)\,f^{\alpha_2}(a_2)\,\ldots\,f^{\alpha_r}(a_r)}{\alpha_1!\quad \alpha_2!\quad \ldots\quad \alpha_r!}\cos\Big\{f(1) + \lambda\frac{\pi}{2}\Big\}\Big]. \qquad (12.2)$$

These expressions evidently become much simplified, if $f(1)$ is a multiple of $\frac{\pi}{4}$.

If (5.1) *is satisfied*, (12.1) becomes

$$\sin[f(n)] = [f(n)\,.\,s(n)]. \quad \ldots \quad \ldots \quad (12.3)$$

where

$$[s(n)] = \sin[1]; \quad \ldots \quad \ldots \quad (12.31).$$

$$s(1) = \sin 1; \quad \ldots \quad \ldots \quad (12.310)$$

$$s(n) = \sum_n \frac{\sin\Big(1 + \lambda\frac{\pi}{2}\Big)}{\alpha_1!\,\alpha_2!\,\ldots\,\alpha_r!}$$

$$= \sum_{r=1}^{\infty}(-1)^{r-1}\frac{K(2r-1,\,n)}{(2r-1)!}, \quad \ldots \quad (12.311.)$$

for $n > 1$.

Similarly, (5.1) *being satisfied*, we have

$$\cos[f(n)] = [f(n)\,.\,c(n)], \quad \ldots \quad \ldots \quad (12.4)$$

where

$$[c(n)] = \cos[1]; \quad \ldots \quad \ldots \quad \ldots \quad (12.41)$$

$$c(1) = \cos 1; \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12.410)$$

$$c(n) = \sum_{n} \frac{\cos\left(1 + \lambda \frac{\pi}{2}\right)}{a_1!\, a_2! \ldots a_r!}$$

$$= \sum_{r=1}^{\infty} (-1)^r \frac{K(2r, n)}{2r!}, \quad \ldots \quad = \quad (12.411)$$

for $n > 1$.

We may now define and obtain expressions for other trigonometrical functions as in ordinary trigonometry, and thus carry over the entire trigonometry to our extended algebraical domain.

We note in particular

$$\cos[f(n)] + i \sin[f(n)]$$

$$= \{\cos f(1) + i \sin f(1)\} \left[\sum_{n} \frac{f^{\alpha_1}(a_1) f^{\alpha_2}(a_2) \ldots f^{\alpha_r}(a_r)}{\alpha_1!\ \alpha_2!\ \ldots\ \alpha^r!} i^{\lambda}\right]$$

$$= e^{i[f(n)]}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12\,5)$$

from which we deduce the equivalent of De Moivre's theorem

$$\{\cos[f(n)] + i \sin[f(n)]\}^{[g(n)]} =$$

$$= \cos[f(n)] \cdot [g(n)] + i \sin[f(n)] \cdot [g(n)], \quad \ldots \quad (12.6)$$

leaving out the discussion of multiple values. Interesting, but complicated, identities may be obtained by developing the left side of this formula by the aid of (11.2).

As an example of (12.6), let $[g(n)] = k$, $n = 6$

$$\therefore \frac{k}{1!} \{\cos f(1) + i \sin f(1)\}^{k-1} \left\{ -\frac{f(6)}{1!} \sin f(1) \right.$$

$$\left. - \frac{f(2) \cdot f(3)}{1!\ 1!} \cdot \cos f(1) + i \frac{f(6)}{1!} \cos f(1) - i \frac{f(2) \cdot f(3)}{1!\ 1!} \sin f(1) \right\}$$

$$+ \frac{k(k-1)}{1!\,1!} \cdot \{\cos f(1) + i \sin f(1)\}^{k-2}$$

$$\cdot \left\{ -\frac{f(2)}{1!} \sin f(1) + i \frac{f(2)}{1!} \cos f(1) \right\} \cdot \left\{ -\frac{f(3)}{1!} \sin f(1) + \right.$$

$$\left. + i \frac{f(3)}{1} \cos f(1) \right\}$$

$$= -\frac{k f(6)}{1!} \sin\{k f(1)\} - \frac{k f(2) \cdot k f(3)}{1!\ \ 1!} \cos\{k f(1)\} +$$

$$+ i \frac{k f(6)}{1!} \cos\{k f(1)\} - i \frac{k f(2) \cdot k f(3)}{1!\ \ 1!} \sin\{k f(1)\},$$

which relation is easily seen to be an identity.

§ 13. *The inverse circular functions.*

Let in (9 . 3)

$$\gamma(z) = \tan^{-1} z,$$

which assumption gives

$$\tan^{-1}[f(n)] = \left[\tan^{-1} f(1),\ f(2)\frac{1}{1+f^2(1)},\ f(3)\frac{1}{1+f^2(1)},\right.$$

$$\left\{f(4)\frac{1}{1+f^2(1)} - f^2(2)\frac{f(1)}{[1+f^2(1)]^2}\right\},\ f(5)\frac{1}{1+f^2(1)},$$

$$\left.\left\{f(6)\frac{1}{1+f^2(1)} - f(2)f(3)\frac{2f(1)}{[1+f^2(1)]^2}\right\},\ \ldots\right]$$

$$= \left[\sum_n \frac{f^{\alpha_1}(a_1)\ f^{\alpha_2}(a_2) \ldots f^{\alpha_r}(a_r)}{\alpha_1!\ \ \alpha_2!\ \ldots\ \ \alpha_r!} (-1)^{\lambda-1} (\lambda-1)! \sin\lambda\theta \cdot \sin^{\lambda}\theta\right],$$

... ... (13 . 1)

where

$$\theta = \frac{\pi}{2} - \tan^{-1} f(1), \qquad \ldots \qquad \ldots \ (13 . 11)$$

the first element being $\tan^{-1} f(1)$.

When the relation (5 . 1) *is satisfied,* (13 . 1) gives

$$\tan^{-1}[f(n)] = [f(n) \cdot \tau(n)]$$

$$= \left[\frac{\pi}{4},\ \frac{f(2)}{2},\ \frac{f(3)}{2},\ \frac{f(4)}{4},\ \frac{f(5)}{2},\ 0,\ \ldots\right] \qquad \ldots \qquad \ldots \ (13 . 2)$$

where

$$[\tau(n)] = \tan^{-1}[1]; \qquad \ldots \qquad \ldots \ (13 . 21)$$

$$\tau(1) = \tan^{-1} 1 = \frac{\pi}{4}; \qquad \ldots \qquad \ldots \ (13 . 210)$$

$$\tau(n) = \sum_n \frac{(-1)^{\lambda-1} 2^{-\frac{\lambda}{2}} (\lambda-1)!\ \sin^{\lambda}\frac{\pi}{4}}{\alpha_1!\ \ \alpha_2!\ \ldots\ \ \alpha_r!} =$$

$$= \sum_{r=1}^{\infty} (-1)^{r-1} \frac{K(2r-1, n)}{2r-1}, \qquad \ldots \qquad \ldots \ (13 . 211)$$

Here it should be noticed, however, that the series in the last expression for $\tau(n)$ converges only for $n = 1$. For higher n-values we must use Borel's method of summing an oscillatory series. Thus we have, for instance, for $n = 4$

$$\sum_{r=1}^{\infty} (-1)^{r-1} \frac{K(2r-1, 4)}{2r-1} = \sum_{r=1}^{\infty} (-1)^{r-1} \frac{(2r-1) \cdot 2r}{2(2r-1)}$$

$$= \sum_{r=1}^{\infty} (-1)^{r-1} r = \tfrac{1}{4},$$

and for $n = 6$

$$\sum_{r=1}^{\infty} (-1)^{r-1} \frac{K(2r-1, 6)}{2r-1} = \sum_{r=1}^{\infty} (-1)^{r-1} \frac{(2r-1)^2}{2r-1}$$

$$= \sum_{r=1}^{\infty} (-1)^{r-1} (2r-1) = 0,$$

which is evidently correct.

Other inverse circular functions may be treated in the same manner and the theory of these functions may be built up exactly as in ordinary analysis.

§ 14. *The ζ-function.*

It is to be noticed, that the formula (9 · 3) is not a consequence of Maclaurin's theorem, but much more general. To realize this, it is necessary to consider some other devolopment of an analytic function. We choose the ζ-function as an example.

$$\zeta[f(n)] = \sum_{r=1}^{\infty} r^{-[f(n)]} = \sum_{r=1}^{\infty} e^{[-f(n) \log r]}$$

$$= \Big[\sum_{r=1}^{\infty} r^{-f(1)}, -\frac{f(2)}{1!} \sum_{r=2}^{\infty} r^{-f(1)} \log r, -\frac{f(3)}{1!} \sum_{r=2}^{\infty} r^{-f(1)} \log r,$$

$$\Big\{-\frac{f(4)}{1!} \sum_{r=2}^{\infty} r^{-f(1)} \log r + \frac{f^2(2)}{2!} \sum_{r=2}^{\infty} r^{-f(1)} \log^2 r\Big\}, \ \ldots\Big]$$

$$= \left[\zeta\{f(1)\}, \frac{f(2)}{1!}\zeta'\{f(1)\}, \frac{f(3)}{1!}\zeta'\{f(1)\}, \right.$$
$$\left. \left\{ \frac{f(4)}{1!}\zeta'\{f(1)\} + \frac{f^2(2)}{2!}\zeta''\{f(1)\} \right\}, \ \ldots \right]$$

$$= \left[\sum_n \frac{f^{\alpha_1}(a_1)\, f^{\alpha_2}(a_2) \ldots f^{\alpha_r}(a_r)}{\alpha_1!\quad \alpha_2!\ \ldots\ \alpha_r!} \zeta^{(\lambda)}\{f(1)\} \right], \qquad \ldots (14.1)$$

in agreement with formula (9 . 3).

In a similar manner we may treat a Dirichlet's series in general.

We may now extend our fundamental definition as follows:

If an arithmetical function $[f(n)]$ *and a function* $\gamma(z)$ *are given, then the arithmetical function* $\gamma[f(n)]$ *is defined by formula (9 . 3), provided* $\gamma(z)$ *is analytic for* $z = f(1)$.

The Infinitesimal Calculus.

§ 15. *The Differential Calculus.*

The process of differentiation of an A. F. may be arrived at in various ways.

Put in the formula (9.3) γ' instead of γ.

We obtain

$$\gamma'[f(n)] = \left[\sum_n \frac{f^{\alpha_1}(a_1)\ f^{\alpha_2}(a_2) \ldots f^{\alpha_r}(a_r)}{\alpha_1!\quad \alpha_2!\quad \ldots\quad \alpha_r!}\ \gamma^{(\lambda+1)}\{f1)\} \right]$$

$$= \left[\frac{d\{g(n)\}}{d\{f(1)\}} \right] = \frac{d\,\gamma[f(n)]}{d[f(n)]}, \qquad \ldots \qquad \ldots \qquad \ldots \quad (15{\cdot}1)$$

by definition.

As examples, consider the particular functions, examined in detail above. We get

from (8.1): $\dfrac{d[f(n)]^k}{d[f(n)]} = k[f(n)]^{k-1}$; (15·11)

from (10·1): $\dfrac{d\, e^{[f(n)]}}{d[f(n)]} = e^{[f(n)]}$; (15·12)

from (10·3), and (8·9): $\dfrac{d \log [f(n)]}{d[f(n)]} = [f(n)]^{-1}$; (15·13)

from (12·1), and (12·2): $\dfrac{d \sin [f(n)]}{d[f(n)]} = \cos[f(n)]$, ... (15·14)

$\dfrac{d \cos [f(n)]}{d[f(n)]} = -\sin[f(n)]$, ... (15·15)

etc.

It is not difficult to see, that we may build up the whole theory, following the course generally adopted in the differential calculus.

We may also arrive at these results in the following manner.

Assume that the elements of the A. F. considered are continuous functions of the variable x.

We define

$$\frac{d\,[f(x,n)]}{dx} = \underset{\Delta x=0}{Lt} \frac{[f(x+\Delta x, n)] - [f(x,n)]}{\Delta x}$$

$$= \underset{\Delta x=0}{Lt} \left[\frac{f(x+\Delta x, n) - f(x,n)}{\Delta x}\right] = \left[\frac{d f(x,n)}{dx}\right] \quad \ldots (15.2)$$

Let further

$$\gamma\,[f(x,n)] = [g(x,n)], \quad \ldots \quad \ldots \quad \ldots \quad (15.3)$$

and define

$$\frac{d\,[g(x,n)]}{d\,[f(x,n)]} = \gamma'\,[f(x,n)]$$

$$= \left[\frac{d\,g(x,n)}{dx}\right] \cdot \left[\frac{d f(x,n)}{dx}\right]^{-1}. \quad \ldots \quad (15.4)$$

In order to establish the identity between the two definitions, consider (9.3), where instead of $f(n)$, we put $f(x,n)$. If our proposition is correct, we must have identically (by (15.1), (15.4)).

$$\gamma'\,[f(x,n)] = [\gamma'\{f(x,1)\}, \frac{f(x,2)}{1!}\gamma''\{f(x,1)\}, \frac{f(x,3)}{1!}.$$

$$.\gamma''\{f(x,1)\}, \left\{\frac{f(x,4)}{1!}\gamma''\{f(x,1) + \frac{f^2(x,2)}{2!}\gamma'''\{f(x,1)\}\right\}, \ldots]$$

$$= \Big[\gamma'\{f\,x,1)\}.f'(x,1), \left\{\frac{f(x,2)}{1!}\gamma''\{f(x,1)\}f'(x,1) + \frac{f'(x,2)}{1!}\gamma'\{f(x,1)\}\right\}, \left\{\frac{f(x,3)}{1!}\gamma''\{f(x,1)\}f'(x,1) + \frac{f'(x,3)}{1!}\gamma'\{f(x,1)\}\right\}, \left\{\frac{f(x,4)}{1!}\gamma''\{f(x,1)\}f'(x,1) + \frac{f'(x,4)}{1!}\gamma'\{f(x,1)\} + \frac{f^2(x,2)}{2!}\gamma'''\{f(x,1)\}f'(x,1) + \frac{2f(x,2)f'(x,2)}{2!}\gamma''\{f(x,1)\}\right\}, \ldots\Big] \cdot \Big[\{f'(x,1)\}^{-1},$$

$$\frac{-1}{1!}\{f'(x,1)\}^{-2}f'(x,2), \frac{-1}{1!}\{f'(x,1)\}^{-2}f'(x,3),$$

$$\left\{\frac{-1}{1!}\{f'(x,1)\}^{-2}f'(x,4) + \frac{(-1)(-2)}{2!}\{f'(x,1)\}^{-3}\{f'(x,2)\}^2\right\}, \ldots\Big] \quad \ldots (15.5)$$

This is evidently the case, as may be seen by actually carrying out the multiplication to the right of the sign of equality.

It is clear, that an A. F. may also be regarded as a constant during the process of differentiation. As an example, consider

$$\frac{d\,[f(n)]^k}{d\,k} = [f(n)]^k \,.\, \log\,[f(n)]$$

$$= \Big[\, f^k(1) \log f(1),\ \frac{f(2)}{1!} f^{k-1}(1)\ \{1 + k \log f(1)\},$$

$$\frac{f(3)}{1!} f^{k-1}(1)\ \{1 + k \log f(1)\},\ \Big\{\frac{f(4)}{1!} f^{k-1}(1)\{1 + k \log f(1)\}$$

$$+ \frac{f^2(2)}{2!} f^{k-2}(1)\ \{2k-1 + k(k-1) \log f(1)\},\ \ldots\Big],\ \ldots \quad (15{\cdot}6)$$

by (8·1) (10·3)

§ 16. *The integral calculus.*

As in § 15, we may introduce the process of integration in the following manner:

$$\gamma\,[f(x, n)] + [C(n)] = \int \gamma'\,[f(x, n)] \,.\, d\,[f(x, n)]$$

$$= \int \gamma'\,[f(x, n)] \,.\, \Big[\frac{d\,f(x, n)}{dx}\Big]\, dx = \int [G(x, n)]\, dx, \text{ (say)},$$

$$= \Big[\int G(x, n)\, dx\Big], \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (16.1)$$

$[C(n)]$ being an arbitrary constant A. F., or we may use the inverse process to that employed in (15.1). For the purpose of identifying the two results, use (15.5).

The theory of definite integration with arithmetical functions as limits is probably of interest, and has eventually a direct application to geometrical problems of great complexity.

(*To be continued.*)

SINGULARITIES OF PEDAL CURVES.

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The pedal of a curve with respect to a point (or origin) is usually defined as the locus of the foot of the perpendicular from the point, on a variable tangent to the curve. The inverse of the pedal with respect to a circle having its centre at the origin is the reciprocal polar of the curve with respect to the circle. The pedal is thus the inverse of the reciprocal polar. Reciprocation changes point singularities into line singularities and *vice versa*. Since a point inverts into a point and a line into a circle, point singularities, in general, remain the same by inversion, while line singularities are altered.

§ 2. The effect of inversion on singularities is best studied by replacing the circle by a conic, called the base, and the origin by any point, not necessarily the centre of the conic. Points collinear with the origin and conjugate with respect to the base may be said to be inverse. This process is merely a generalised circular inversion known as *quadric inversion* and establishes a correspondence of point to point and straight line to conic.

Let O be the origin and OI, OJ, tangents to the base. The inverse of any point is determinate except when the point is O, I or J in which case the inverse is any point on the line I J, O I or O J. These points are therefore called the special points of the transformation, and the lines, the special lines. The degree of the inverse of a curve of class n is $2n$, except when the curve passes through a special point, in which case the inverse degenerates. For every point of intersection of the curve with a special line the corresponding special point is repeated in the inverse, that is to say, the inverse has nodes of the nth order at the special points.* It will be seen that contact with a special line gives rise to a cusp at the corresponding special point, and a node on the special line becomes a tacnode. If the tangent to the curve at a point of inter-

* The multiple points at O, I, J are the only singularities due to the special nature of the transformation. For a curve of degree $2n$ contains $\frac{2n\,(2n+3)}{2}$ constants and can therefore satisfy as many conditions. But a node of order n at a given point is equivalent to $1+2+3+\ldots n = \frac{n\,(n+1)}{2}$ conditions. With three such nodes, the number of disposable constants is $\frac{2n\,(2n+3)}{2} - \frac{3n\,(n+1)}{2}$ $= \frac{n\,(n+3)}{2}$, which is just the number of conditions that a curve of degree n can satisfy.

section Z, with the special line I J passes through O, the inverse has a point of inflexion at O with O Z for the inflexional tangent. An account of the singularities of inverse curves is given in detail in Miss Scott's *Modern Analytical Geometry* to which the reader is referred.

§ 3. By reciprocation, we obtain at once the following results :—

(1) The pedal transformation* is a case of correspondence of point to conic and of straight line to point.

(2) The pedal of a curve of class m is of order $2m$ with nodes of order m at the special points.

(3) To a multiple tangent corresponds in the pedal a multiple point of the same order.

(4) For every passage of a curve through a special point, the pedal developes a cusp at that point.

(5) Nodes on a special line transform into multiple tangents of the same order passing through the corresponding special point.

§ 4. Other results may be obtained in the same way, Thus†, where the curve has inflexional contact with the line I J, the inverse curve does not differ in appearance from an ordinary branch, though there is really a triple point at O. Hence when the curve has a cusp at O, the pedal has two vanishing loops at O, looking like an ordinary point, perhaps with a certain sharpness of bend as shown in figure 1. Again‡, the inverse of a cusp with the line I J as tangent, has at O a triple point with two evanescent loops but looking like an ordinary point of inflextion. Hence the pedal of a curve having an inflexion at O has really a triple point at O, though this appears just like an ordinary point of inflexion. Figure 2 exhibits the loop just before disappearance and also the final form on the pedal.

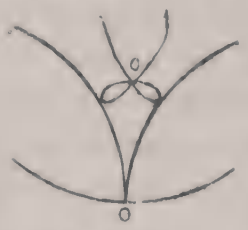


FIG. 1.



FIG. 2.

* The most general quadratic relation connecting point and line co-ordinates in two planes and birational in both sets of variables may be shown to be a generalised pedal transformation on the lines given in § 230, pages 217—19, Scott's *Analytical Geometry*.

† § 240, Scott's *Analytical Geometry*.

‡ Do. do.

§ 5. *Plucker's constants for the Pedal.*

Let us denote by o, i, j, the number of times a curve passes through O, I, J respectively and by o', i', j' their reciprocal singularities, *viz.* the number of times that it touches IJ, OI, OJ. Hence with the usual notation, (the capitals referring to the pedal) we have

$$N = 2m - o' - i' - j'$$

$$\Delta = T + \sum_{o', i', j'} \frac{(m - i' - j')(m - i' - j' - 1)}{2}$$

$$K = i + o + i' + j',$$

and the others (M, I & T) could be found in terms of these.

§ 6. A table is appended giving a list of simpler cases occurring in the usual Pedal transformation. The pedals of an ellipse, hyperbola and parabola are given in figures 3, 4 and 5.

No.	Singularity on The Curve C.	Singularity on The Pedal C'	Remarks.
1	One pt. P at infinity ...	The asymptote to C at P is normal to the Pedal at the corresponding pt. P'	
2	Node ...	Circle through O touching the Pedal at two distinct points.	
3	Cusp at P. ...	Circle of curvature at the corresponding pt. P' passes thro' O.	
4	Multiple point of order r...	Circle through O touching the Pedal at r points	
5	Parallel tangents to O ... (Multiple point at infinity)	Multiple tangent thro' O	The circle corresponding to (4) degenerates into a straight line.

No.	The Curve C.	The Pedal C.	Remarks.
6	Double tangent ...	Node.	
7	Stationary tangent ...	Cusp.	
8	Class m. ...	Order 2 m.	
		Multiple points of order m at O and the circular points equivalent to $\frac{3}{2}m(m-1)$ nodes.	
	No. of Double tangents T.	nodes T.	
	No. of stationary tangents i. ...	cusps i.	
	Deficiency. $\frac{1}{2}(m-1)(m-2 - T - i$. ...	D $= \frac{1}{2}(2m-1)(2m-2) - 3\frac{m}{2}(m-1) - T - i = \frac{1}{2}(m-1)(m-2), - T - i$.	The deficiency of the Pedal is the same as that of C.
9	C passing through the circular points ...	Cusps at the circular points.	The pedal of a circle has cusps at the circular pts.
10	O on C.	Cusp at O.	
11	O on an asymptote of C.	O is a point of inflexion (Flecnode) on C', the asymptote being the normal at O.	Prove by forming the eqn. of the pedal of a conic w, r, t a point on an asymptote, e.g., the pedal of a central conic with respect to its centre is $(x^2+y^2)^2 = a^2x^2 + b^2y^2$ has a biflecnode at the centre. (Page 171. Basset: *Cubic and Quartic Curves.*)

No.	The Curve C.	The Pedal C.	Remarks.
12	O a focus.	The pedal curve degenerates into the circular lines thro' O and a residual curve of degree two less.	The pedal of a conic w, r, t a focus is the auxiliary circle and the line pair OI. OJ, which together constitute a bicircular quartic with a node at O.

§ 7. *Pedals of Conics.*

It follows from the preceding that the pedal of a central conic, is a rational bi-circular quartic, with a node at O. If O be the centre, this is a

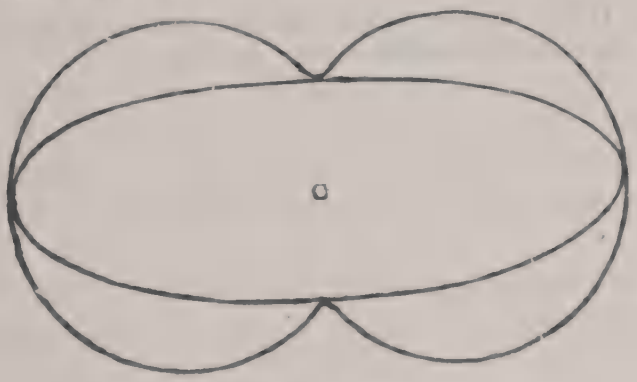


Fig. 3.

biflecnode, real for the hyperbola and imaginary for the ellipse. The pedal touches the conic at the feet of the 4 normals from O and has in

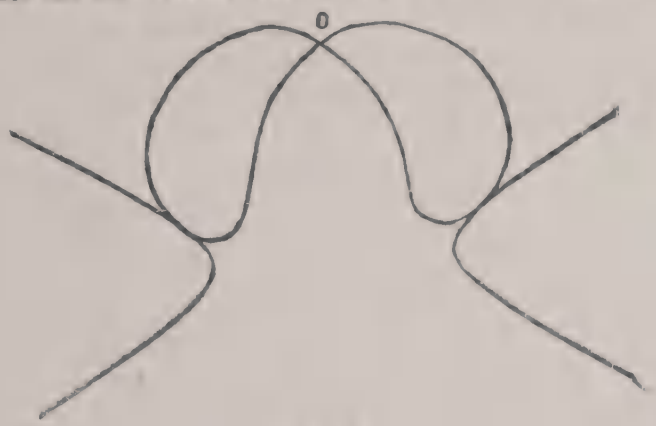


Fig. 4.

consequence no other point in common with it. When the conic is a circle, the nodes at I and J are replaced by cusps, and there is also a cusp at O if O be on the conic (*cf.* cardioid). The pedal degenerates

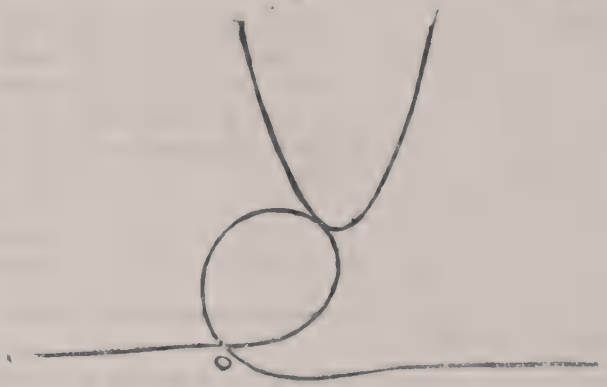


FIG. 5.

either when O is a focus or when C touches the line at infinity. In the latter case the pedal is a circular cubic. See Fig. 5.

SHORT NOTES.

Further Note on Question 653.

Q. 653. (Selected) :—To find a cube the sum of whose divisors is a perfect square.

Let $\sigma(n)$ denote the sum of the divisors of n, including 1 and n we have to solve $\sigma(x^3) = y^2$.

I have succeeded in getting the following values of x and y.

	x	y
(1)	7.	20.
(2)	2.3.5.13.41.47.	$2^7.3^2.5^3.7.13.17.29.$
(3)	2.3.5.13.17.31.41.191.	$2^{12}.3^3.5^3.7.13.17.29^2.37.$
(4)	$2^3.3.5.11.19.31.43.743.$	$2^{18}.3^2.5^3.11.13.31.37.61.181.$
(5)	$2^5.5.23.47.211.241.577.$	$2^9.3^3.5^3.11.13^3.17^2.53.113.197.257.$
(6)	$2^8.5.17.23.31.191.211.241.577.$	$2^{14}.3^3.5^2.11.13^2.17^3.29.37.53.113.197.257.$
(7)	3.13.17.23.31.83.191.239.	$2^{16}.3^3.5^3.7..13^3.17.29.37.53.$
(8)	13.17.23.31.41.43.47.83.307.	$2^{16}.3^3.5^5.7^2.11.13^2.17.29^2.37.53.$
(9)	3.13.23.47.83.239.	$2^{11}.3^2.5^3.7.13^3.17.53.$
(10)	3.17.31.41.43.239.307.	$2^{17}.3^2.5^4.7.11.13^3.29^2.37.$ [257.
(11)	3.31.43.47.211.241.659.787.	$2^{14}.3.5^3.11^2.13.17.37.53.113.197.241.$

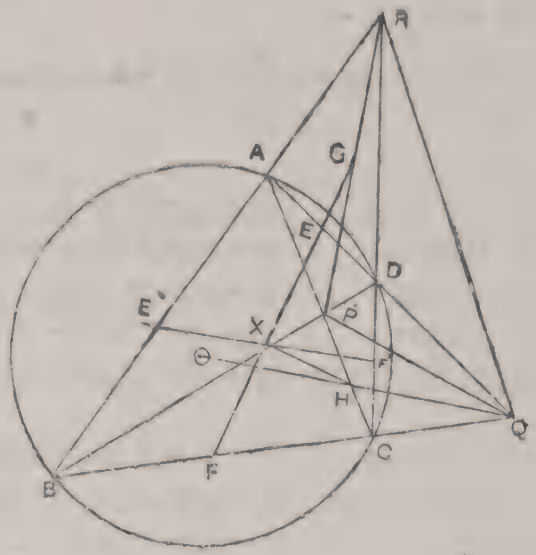
Of these, solutions () and (2) were given by Lucas (*Theorie de Nombres*, p. 380). He gives, however, no indication as to how he got them. The remaining nine are believed to be new. [For a method of obtaining solution (1) see J. I. M. S., Vol. XI, (Dec. 1919), page 228]. If in (7), (9), (10), (11) we multiply x by $2 . 3^2 . 11$ and y by $2 . 3 . 11 . 61$, we get four other solutions.

N. B. Mitra.

Note on Question 1122.

Q. 1122. (V. Ramaswami Aiyar):—If the joins of four concyclic points A, B, C, D taken in opposite pairs intersect in P, Q, R, prove that the nine-point circle of PQR passes through the centroid of the points A, B, C, D.

Let P, Q, R be the points of intersection of (AC, BD), (AD, BC), (AB, CD) respectively; O the centre of the circle ABCD; E, F, G, H, X the mid. points of AD. BC, PR, OQ, EF respectively. Then, obviously, X is the centroid of the quadrilateral ABCD.



Since the triangle PQR is self-polar with respect to the circle ABCD, O is the orthocentre of the triangle PQR and hence its nine-point circle is the circle on GH as diameter.

Now, the mid. points G, F, E of the diagonals of the complete quadrilateral BRCP are collinear, so that G, F, X are collinear.

Again, because OQ is the diameter of the circle QEF and X the mid. point of the chord EF, HX is perpendicular to XF, *i.e.*, $H\hat{X}G$ is right.

Hence X lies on the circle with HG for diameter, *i.e.* on the nine-point circle of the triangle PQR.

Cor. 1. In the cyclic quadrilateral ABCD, the perpendiculars from the mid. point of each side on the opposite side are concurrent, the point of concurrence being on the circum-circle of the triangle PQR.

Join OX and produce it to O′ such that O′X = XO.

Then O′ is a point on the circum-circle of the triangle PQR.

Again, as EF and OO′ bisect each other, O′E is parallel to OF and therefore perpendicular to BC, and O′F is parallel to OE and therefore perpendicular to AD.

Thus the perpendiculars from the mid-points of AD, BC on the opposite sides BC, AD meet at O′ on the circum-circle of triangle PQR.

Similarly by joining the mid-points E′, F′ of AB, CD to O′ it can be shown that these joins are perpendicular to CD, AB respectively.

Hence the proposition.

2. Incidentally, we may note that the point of concurrence O′ of the four perpendiculars is the common ortho-centre of the two triangles QEF and RE′F′.

3. When the diagonals AC, BD cut at right-angles, it is obvious that the four perpendiculars pass through the point P which is the intersection of the diagonals. Thus in this case O′ coincides with P.

A. A. Krishnaswami Aiyangar.

Astronomical Notes.

1. *Nova Cygni III*:—One of the most important astronomical events of the year 1920 was the appearance of a new star in the Constellation Cygnus. The star was first discovered by Mr. Denning at Bristol on the night of 20th August. The magnitude at that time was 3·5, and since then, the brightness increased rapidly (like all other Novæ); the maximum (magnitude 2·2) was reached on August 24. The decrease in brightness was rather more rapid than that of some of the Novæ which appeared recently. Dr. Lockyer has estimated the decline to be on the average five times more rapid than that of Nova Aquilæ (1918) or Nova Persei (1901).

The position of the Nova is

R. A. $19^h\ 56^m$ and Decl. 53° 24′ North.

Forty-one new stars have been discovered to have made their appearance till now, since Tycho Brahe's star of 1572, excluding the faint novæ which have appeared in the midst of Nebulæ and Clusters; the number recorded during the last thirty years alone is about thirty-one per annum on the average.

2. *Pons-Winnecke's Comet*:—This is one of the short period comets which are due to perihelion during 1921. The last observed return was on Sep. 1, 1915 and, its period being a little over $5\frac{1}{2}$ years the next return will happen about the end of June. As the comet will be very near the earth at that time, the return is likely to be interesting and we may probably be treated to a brilliant meteoric shower.

3. *Saturn*:—This planet has been under continuous observation recently, the interest attached to the occasion being due to the fact that the earth passed through the plane of his rings in Nov. 1920. This happens once every fifteen years and at the time of passage, the minor axis of the ring ellipse continues diminishing slowly and the rings themselves remain invisible for some short time. The actual passage occurred on Nov. 7, 1920, and the phenomenon was well observed by several astronomers. Prof. Barnard estimates that the rings are probably not more than fifty miles thick. For an interesting discussion of the condition of the earth passing through the plane of Saturn's rings, reference may be made to Prof. Turner's paper in the *Monthly Notices of the R.A.S.* 1908 April. The problem is also briefly dealt with in Ball's *Spherical Astronomy*, Ex. 23 page 428.

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SOLUTIONS.

Question 1059.

(K. J. SANJANA, M.A.) :—Prove that the sums of the products of the quantities $(2^2-1)^{-1}$, $(4^2-1)^{-1}$, $(6^2-1)^{-1}$ ad inf., taken three at a time and four at a time, are respectively $\frac{10-\pi^2}{32}$ and $\frac{1680-180\pi^2+\pi^4}{6144}$; and show how to find the sums of the products taken five, six at a time.

Solutions by Hemraj, N. G. Leather and G. S. Mahajani.

$$\frac{\sin \pi y/2}{\sin \pi z/2} = \frac{y}{z} \prod_1^\infty \frac{(2r)^2-y^2}{(2r)^2-z^2} = \frac{y}{z} \prod \left(1 + \frac{x}{(2r^2)-z^2}\right) \text{ where}$$

$y^2 = z^2 - x$. Put $z = 1$ and we get

$$(1-x)^{-\frac{1}{2}} \sin \frac{\pi}{2}(1-x)^{\frac{1}{2}} = \prod \left(1 + \frac{x}{(2r)^2-1}\right) = 1 + \sum_1^8 A_r \, x^r \ldots\ldots (1)$$

where A_r denotes the sum of the products of $(2^2-1)^{-1}$, $(4^2-1)^{-1}$... taken r at a time.

Now $(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{1\cdot3}{2\cdot4}x^2 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^3 + \ldots\ldots\ldots$

$$\sin \frac{\pi}{2}(1-x)^{\frac{1}{2}} = \sin\left[\frac{\pi}{2}\left(1 - \frac{1}{2}x - \frac{1}{2^2}\frac{x^2}{2!} - \frac{3}{2^3}\frac{x^3}{3!} - \frac{15}{2^4}\frac{x^4}{4!} \ldots\right)\right]$$

$$= \sin\left(a + bx + c\frac{x^2}{2!} + d\frac{x^3}{3!} + e\frac{x^4}{4!} + \ldots\ldots\ldots\right)$$

$$= A + Bx + C\frac{x^2}{2!} + D\frac{x^3}{3!} + E\frac{x^4}{4!} + \ldots\ldots\ldots$$

By Arbogast's Method of Derivations (Art 88, *Diff. : Cal.* : Williamson).

$A = \sin a$, $B = \delta . \sin a = b \cos a$, $C = \delta . b \cos a = c \cos a - b^2 \sin a$ etc.

But $a = \frac{\pi}{2}$, $\therefore A = 1$, $B = 0$, $C = -\frac{\pi^2}{16}$, $D = -\frac{3\pi^2}{32}$,

$$E = -\frac{15\pi^2}{64} + \frac{\pi^4}{256} \text{ etc.}$$

Hence $(1-x)^{-\frac{1}{2}} \sin\left[\frac{\pi}{2}(1-x)^{\frac{1}{2}}\right]$

$$= \left\{1 + \frac{1}{2}x + \frac{1\cdot3}{2\cdot4}x^2 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^3 + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}x^4 + \ldots\ldots\right\}$$

$$\left\{1 - \frac{\pi^2}{32}x^2 - \frac{\pi^2}{64}x^3 - \left(\frac{15\pi^2}{64} - \frac{\pi^4}{256}\right)\frac{x^4}{4!} \ldots\ldots\ldots\right\}$$

$= 1 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots\ldots + A_r x^r + \ldots\ldots$ from (1)

Comparing co-efficients of x^r on both sides, we get A_r.

In particular $A_1 = \frac{1}{2}$; $A_2 = \frac{12 - \pi^2}{32}$; $A_3 = \frac{10 - \pi^2}{32}$;

$$A_4 = \frac{1680 - 180\pi^2 + \pi^4}{6144}.$$

Question 1080.

(K. J. SANJANA, M.A.) :—Show how to find the sum of the products r at a time of the quantities $(3^2-1)^{-1}$, $(5^2-1)^{-1}$, $(7^2-1)^{-1}$, ... ad inf.; and find the values for $r = 3$ and $r = 4$.

Solutions by Hemraj, N. G. Leather and G. S. Mahajani.

$\frac{\cos \pi y/2}{\cos \pi z/2} = \prod_1^\infty \left[1 + \frac{x}{(2n-1)^2 - z^2}\right]$, where $y^2 = z^2 - x$.

Put $z = 1$ and we get

$$\frac{4}{\pi x} \cos \frac{\pi}{2} (1-x)^{\frac{1}{2}} = \left(1 + \frac{x}{3^2-1}\right)\left(1 + \frac{x}{5^2-1}\right)\left(1 + \frac{x}{7^2-1}\right) \ldots\ldots$$

$= 1 + \sum_1^\infty B_r x^r$ where B_r stands for the sum of the products of $(3^2 - 1)^{-1}$, $(5^2 - 1)^{-1}$ etc., taken r at a time.

$$\cos \frac{\pi}{2} (1-x)^{\frac{1}{2}} = \cos \frac{\pi}{2}\left[\left(1 - \frac{1}{2}x - \frac{1}{2^2}\frac{x^2}{2!} - \frac{3}{2^3}\frac{x^3}{3!} \ldots\ldots\right)\right]$$

$$= \cos (a + bx + c\frac{x^2}{2!} + d\frac{x^3}{3!} + e\frac{x^4}{4!} + \ldots,$$

$$= A + Bx + C\frac{x^2}{2!} + D\frac{x^3}{3!} + E\frac{x^4}{4!} + F\frac{x^5}{5!} + \ldots$$

By Art. 88, *Diff. Cal.*: Williamson, we have

$$A = 0;\ B = -b = \frac{\pi}{4};\ C = -c = \frac{\pi}{8};$$

$$D = -d + b^3 = \frac{3\pi}{16} - \frac{\pi^3}{64};\ E = -e + 6b^2c = \frac{15\pi}{32} - \frac{3\pi^3}{64}$$

$$F = -f + 5(2b^2d + 3bc^2) - b^5 = \frac{105\pi}{64} - \frac{135}{256}\pi^3 + \frac{\pi^5}{1024}, \text{ etc.}$$

$$\therefore \frac{4}{\pi\sqrt{2}} \cos\frac{\pi}{2}(1-x)^{\frac{1}{2}} = 1 + \tfrac{1}{4}x + \frac{12-\pi^2}{96}x^2 + \frac{10-\pi^2}{128}x^3 + \frac{1680-540\pi^2+\pi^4}{30720}x^4 + \ldots$$

$$= 1 + B_1 x + B_2 x^2 + \ldots + B_r x^r + \ldots$$

Equating co-efficients of x^r on both sides, we get B_r.

In particular $B_1 = \frac{1}{4}$; $B_2 = \frac{12-\pi^2}{96}$; $B_3 = \frac{10-\pi^2}{128}$;

$$B_4 = \frac{1680-540\pi^2+\pi^4}{30720}.$$

Question 1062.

(C. KRISHNAMACHARI) :—Prove that

$$1 + \left({}_{n-1}C_1\right)\frac{1}{3} + \left({}_{n-1}C_2\right)\frac{1}{5} + \ldots \frac{1}{2n-1}$$

$$= 2^{n-1}\left\{\frac{1}{2n-1} + \frac{(n-1)}{(2n-1)(2n-3)} + \ldots + \frac{(n-1)!}{(2n-1)(2n-3)\ldots 1}\right\}.$$

Solutions by N. G. Leather, Vasavada and Kewalramani.

$$1 + {}_{n-1}C_1\frac{1}{3} + {}_{n-1}C_2\frac{1}{5} + \ldots + \frac{1}{2n-1} = \int_0^1 (1+x^2)^{n-1}\,dx$$

$$= \int_0^{\pi/4} \sec^{2n}\theta\, d\theta$$

$$= \left[\sec^{2n-2}\theta\tan\theta\right]_0^{\pi/4} - (2n-2)\int_0^{\pi/4}\sec^{2n-2}\theta\tan^2\theta\, d\theta$$

$$= 2^{n-1} - (2n-2)\int_0^{\pi/4}(\sec^{2n}\theta - \sec^{2n-2}\theta)\,d\theta$$

$$= \frac{1}{2n-1}\left\{2^{n-1} + 2(n-1)\int_0^{\pi/4}\sec^{2n-2}\theta\, d\theta\right\}$$

$$= \left[\frac{2^{n-1}}{2n-1} + \frac{2^{n-1}(n-1)}{2n-1\,.\,2n-3} + \frac{2^{n-1}(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)} + \ldots + \frac{2^{n-1}(n-1)!}{(2n-1)(2n-3)\ldots 1}\int_0^{\pi/4}\sec^2\theta\, d\theta\right]$$

$$= 2^{n-1}\left[\frac{1}{2n-1} + \frac{n-1}{2n-1\,.\,2n-3} + \frac{(n-1)(n-2)}{2n-1\,.\,2n-3\,.\,2n-5} + \ldots \frac{(n-1)!}{(2n-1)(2n-3)\ldots 3\,.\,1}\right].$$

Question 1008.

(C. KRISHNAMACHARI) :—Show that

$$(a)\quad \frac{1}{2^{n+1}}\left\{\frac{1}{n}+\frac{(2n-1)}{n(n-1)}+\ldots\frac{(2n-1)(2n-3)\ldots 3}{n(n-1)\ldots 1}\cdot\frac{\pi}{4}\right\}$$
$$=1-(n+1)\frac{1}{3}+\frac{(n+1)(n+2)}{1\,.\,2}\cdot\frac{1}{5}-\ldots$$

$$(b\quad 1-\binom{n-\frac{1}{2}}{1}\frac{1}{3}+\binom{n-\frac{1}{2}}{2}\frac{1}{5}\ldots$$
$$=2^{n+\frac{1}{2}}\left[\frac{1}{2n+1}-\frac{1}{2}\binom{n-\frac{1}{2}}{1}\cdot\frac{1}{2n+3}+\ldots\right].$$

Solutions by N. G. Leather, Vasavada, and Kewalramani.

$$(a)\quad \text{R. H. S.}=\int_0^1(1+x^2)^{-(n+1)}\,dx=\int_0^{\pi/4}\cos^{2n}\theta\,d\theta$$
$$=\left[\cos^{2n-1}\theta\sin\theta\right]_0^{\pi/4}+(2n-1)\int_0^{\pi/4}\cos^{2n-2}\theta\sin^2\theta\,d\theta$$
$$=\frac{1}{2^n}+(2n-1)\int_0^{\pi/4}(\cos^{2n-2}\theta-\cos^{2n}\theta)\,d\theta$$
$$=\frac{1}{2n}\left[\frac{1}{2^n}+(2n-1)\int_0^{\pi/4}\cos^{2n-2}\theta\,d\theta\right]$$
$$=\frac{1}{2^{n+1}}\left[\frac{1}{n}+\frac{2n-1}{n\,.\,n-1}+\frac{2n-1\,.\,2n-3}{n\,.\,n-1\,.\,n-2}\ldots\right.$$
$$\left.+\frac{2n-1\,.\,2n-3\,.\,2n-5\ldots 3\,.\,1}{n!}\cdot\int_0^{\pi/4}d\theta\right]$$
$$=\frac{1}{2^{n+1}}\left[\frac{1}{n}+\frac{2n-1}{n\,.\,n-1}+\frac{2n-1\,.2n-3}{n\,.\,n-1\,.\,n-2}+\ldots\right.$$
$$\left.+\frac{2n-1\,.\,2n-3\,.\,2n-5\ldots 3\,.\,1}{n!}\cdot\frac{\pi}{4}\right].$$

$$(b)\ \text{L. H. S.}=\int_0^1(1-x^2)^{n-\frac{1}{2}}\,dx=\tfrac{1}{2}\int_0^1(1-y)^{n-\frac{1}{2}}y^{-\frac{1}{2}}\,dy\ (y=x^2)$$
$$=\tfrac{1}{2}\,\frac{\Gamma(n+\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(n+1)}.$$

The series on the right-hand side.

$$=2^{2n+1}\left[\frac{1}{2n+1}\left(\frac{1}{\sqrt{2}}\right)^{2n+1}-{}_{n-\frac{1}{2}}C_1\left(\frac{1}{\sqrt{2}}\right)^{2n+3}\frac{1}{2n+3}+\ldots\right]$$
$$=2^{2n+1}\int_0^{\frac{1}{\sqrt{2}}}\left[x^{2n}-{}_{n-\frac{1}{2}}C_1\,x^{2n+2}+\ldots\right]dx$$

$$= 2^{2n+1} \int_0^{\frac{1}{\sqrt{2}}} x^{2n} \left(1 - x^2\right)^{n-\frac{1}{2}} dx$$

$$= 2^{2n+1} \int_0^{\pi/4} \sin^{2n}\theta \cos^{2n}\theta \, d\theta = 2^{2n} \int_0^{\pi/2} \sin^{2n}\theta \cos^{2n}\theta \, d\theta$$

$$= 2^{2n-1} \frac{\{\Gamma(n+\frac{1}{2})\}^2}{\Gamma(2n+1)} = \frac{2^{2n-1}\,\Gamma\left(n+\frac{1}{2}\right)\left[\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\ldots\frac{1}{2}\right]\Gamma\left(\frac{1}{2}\right)}{2n!}$$

$$= \frac{1}{2} \frac{\Gamma(n+\frac{1}{2})\,\Gamma(\frac{1}{2})}{n(n-1)(n-2)\ldots 1} = \frac{1}{2} \frac{\Gamma(n+\frac{1}{2})\,\Gamma\frac{1}{2}}{\Gamma(n+1)} = \text{L. H. S.}$$

Question 1065.

(S. KRISHNASWAMI AIYANGAR):—If ρ be the radius of curvature of the curve $r^m = a^m \sin m\theta$ at the point whose distance measured along the curve from a fixed point is s, prove that

$$(m-1)(m+1)^2 \rho \frac{d^2\rho}{ds^2} - m(m+1)^2 \left(\frac{d\rho}{ds}\right)^2 - m(m-1)^2 = 0.$$

Solution by P. R. Venkatakrishna Iyer and Kewalramani.

Using the notion given in Edward's *Diff. Calculus*, for the curve $r^m = a^m \sin m\theta$,

$$\varphi = m\theta \text{ and } \psi = (m+1)\theta \text{ . and}$$

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = r^2 \operatorname{cosec}^2 m\theta = a^2 (\sin m\theta)^{\frac{2}{m}-2}$$

$$= a^2 \left(\sin \frac{m}{m+1}\psi\right)^{\frac{2}{m}-2}.$$

$$\therefore \frac{ds}{d\theta} = a \left(\sin \frac{m}{m+1}\psi\right)^{\frac{1}{m}-1}.$$

$$\therefore \rho = \frac{ds}{d\psi} = \frac{ds}{d\theta} \Big/ \frac{d\psi}{d\theta} = \frac{1}{m+1} \cdot \frac{ds}{d\theta}$$

$$= \frac{a}{m+1} \left(\sin \frac{m}{m+1}\psi\right)^{\frac{1}{m}-1}.$$

Taking the logarithmic differential of ρ with respect to s, we have

$$\frac{d\rho}{ds} = -\frac{m-1}{m+1} \cot \frac{m}{m+1}\psi. \quad \ldots \quad \ldots \quad (1)$$

Differentiating (1) we get

$$\rho \frac{d^2\rho}{ds^2} = \frac{(m-1)m}{(m+1)^2} \operatorname{cosec}^2 \frac{m}{m+1}\psi. \quad \ldots \quad (2)$$

Eliminating ψ between (1) and (2), we get the result

$$(m-1)(m+1)^2\,\rho\,\frac{d^2\rho}{ds^2} - m(m+1)^2\left(\frac{d\rho}{ds}\right)^2 - m(m-1)^2 = 0.$$

Now to solve the differential equation, we can write it in the form

$$a\rho\,\frac{d^2\rho}{ds^2} - b\left(\frac{d\rho}{ds}\right)^2 - c = 0.$$

$$\frac{d\rho}{ds} = \left(\frac{ds}{d\rho}\right)^{-1} \text{ and } \frac{d^2\rho}{ds^2} = \left(\frac{ds}{d\rho}\right)^{-1}\cdot\frac{d}{d\rho}\left(\frac{ds}{d\rho}\right)^{-1} = -\frac{d^2s}{d\rho^2}\Big/\left(\frac{ds}{d\rho}\right)^3.$$

Substituting these in the differential equation,

we get $$a\rho\,.\,\frac{d^2s}{d\rho^2} + b\,\frac{ds}{d\rho} + c\left(\frac{ds}{d\rho}\right)^3 = 0.$$

In this, put $p = \frac{ds}{d\rho}$, then $\frac{d^2s}{d\rho^2} = \frac{dp}{d\rho}$.

Thus we have $a\rho\,.\,\frac{dp}{d\rho} + bp + cp^3 = 0,$

i.e., $$\frac{a}{b}\left\{\frac{1}{p} - \frac{cp}{cp^2+b}\right\}dp + \frac{d\rho}{\rho} = 0.$$

Integrating, we get

$$\frac{a}{b}\log\frac{p}{\sqrt{cp^2+b}} + \log\rho = \text{a constant} = k$$

$$\therefore\ \frac{p^2}{cp^2+b} = k^2\rho^{-\frac{2b}{a}}.$$

$$\therefore\ p^2 = \frac{bk^2\rho^{-\frac{2b}{a}}}{1 - ck^2\rho^{-\frac{2b}{a}}}.$$

i.e. $$\frac{ds}{d\rho} = \frac{b^{\frac{1}{2}}k\rho^{-\frac{b}{a}}}{\sqrt{1 - ck^2\rho^{-\frac{2b}{a}}}}.$$

Integrating $$s = k' + \int \frac{b^{\frac{1}{2}}k\rho^{-\frac{b}{a}}}{\sqrt{1 - ck^2\rho^{-\frac{2b}{a}}}}\,d\rho.$$

Question 1067.

(V. TIRUVENKATACHARI) : Show that

$$\int_0^{\pi/2} \frac{\log \sin x . \log \cos x}{\sqrt{\sin x \cos x}}\, dx = \tfrac{1}{8} \frac{\{\Gamma(\frac{1}{4})\}^2}{\sqrt{\pi}} \left\{ (\log 2)^2 + \pi \log 2 - \frac{\pi^2}{4} \right\}.$$

Solution by Hemraj and Kewalramani.

$$\int_0^{\pi/2} \sin^{2\alpha-1} x \cos^{2\beta-1} x\, dx = \tfrac{1}{2} \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Differentiate 1st with respect to α and then with respect to β

$$4 \int_0^{\pi/2} \sin^{2\alpha-1} x \cos^{2\beta-1} x \log \sin x . \log \cos x\, dx$$

$$= \tfrac{1}{2} \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \{\psi(\alpha) - \psi(\alpha+\beta)\} \left\{ \psi(\beta) - \psi(\alpha+\beta) - \frac{\psi'(\alpha+\beta)}{\psi(\alpha) - \psi(\alpha+\beta)} \right.$$

where $\psi(x) = \dfrac{d}{dx} \{\log \Gamma(x)\}$.

Put $\alpha = \beta = \frac{1}{4}$ and we get

$$4 \int_0^{\pi/2} \frac{\log \sin x \log \cos x}{\sqrt{(\sin x \cos x)}}\, dx = \frac{\{\Gamma(\frac{1}{4})\}^2}{2\sqrt{\pi}} \left\{ [\psi(\tfrac{1}{4}) - \psi(\tfrac{1}{2})]^2 - \psi'(\tfrac{1}{4}) \right\}$$

$$= \frac{\{\Gamma(\frac{1}{4})\}^2}{2\sqrt{\pi}} \left\{ (\log 2)^2 + \pi \log 2 - \frac{\pi}{4} \right\} \text{ from}$$

Bromwich ; *Theory of Infinite Series*, p. 475 Exs. 42 and 43.

Question 1069.

(A. C. L. WILKINSON) :—Normals PA, PB, PC, PD are drawn to a hyperbola ; PA meets the hyperbola again in A', PB in B', PC in C', PD in D'. If A'B'C'D' are concyclic, prove that the locus of P is an ellipse of which the equi-conjugate diameters coincide with the asymptotes of the hyperbola.

Solution by Hemraj.

The equation of the hyperbola referred to its asymptotes as axes is $xy = c^2$ and the normal at (t) is $y(1 - t^2 \cos\omega) - x(t^2 - \cos\omega) = \dfrac{c}{t}(1 - t^4)$ where ω is the angle between the asymptotes.

It passes through the point $P(x, y)$ if

$$ct^4 - (y \cos\omega + x)\, t^3 + (y + x \cos\omega)\, t - c = 0.$$

$$\therefore \Sigma t_1 = \frac{y\cos\omega + x}{c}, \Sigma \frac{1}{t_1} = \frac{y + x\cos\omega}{c}, t_1t_2t_3t_4 = -1 \quad \ldots (1)$$

$$\Sigma t_1t_2 = 0 \quad \therefore \Sigma t_1^2t_2^2 = 2\Sigma t_1 \Sigma \frac{1}{t_1} - 2\,; \; (\Sigma t_1)^2 = \Sigma t_1^2\,;$$

$$\left(\Sigma \frac{1}{t_1}\right)^2 = \Sigma \frac{1}{t_1^2} \quad \ldots (2)$$

Hence from (1)

$$x^2+y^2 = \frac{c^2}{\sin^4\omega}\left\{(1+\cos^2\omega)\left[\Sigma t_1^2 + \Sigma \frac{1}{t_1^2}\right] - 4\cos\omega\,\Sigma t_1 \Sigma \frac{1}{t_1}\right\} \ldots (3)$$

where A, B, C, D are t_1, t_2, t_3, t_4.

The normal meets the hyperbola again in a point (T) where

$$T = \frac{-t(t^2 - \cos\omega)}{1 - t^2\cos\omega} \text{ and } (T) \text{ lies on a circle}$$

$$x^2 + y^2 + 2xy\cos\omega + 2gx + 2fy + k = 0 \text{ if}$$

$$c^2T^4 + 2fcT^3 + (2c^2\cos\omega + k)T^2 + 2gcT + c^2 = 0.$$

Hence if A′, B′, C′, D′ be T_1, T_2, T_3, T_4, then $T_1T_2T_3T_4 = 1$

$$\therefore \; t_1(t_1^2 - \cos\omega)\,.\; t_2(t_2^2 - \cos\omega)\,.\; t_3(t_3^2 - \cos\omega)\,.\; t_4(t_4^2 - \cos\omega)$$

$$= (1 - t_1^2\cos\omega)(1 - t_2^2\cos\omega)(1 - t_3^2\cos\omega)(1 - t_4^2\cos\omega),$$

i.e., $\cos\omega(1+\cos^2\omega)\left[\Sigma t_1^2 + \Sigma \frac{1}{t_1^2}\right] - 4\cos^2\omega\left[\Sigma t_1 \Sigma \frac{1}{t_1} - 1\right]$

$$= 2(1 + \cos^4\omega), \text{ from (1) and (2).}$$

Substituting in (3) we get

$$x^2 + y^2 = \frac{2c^2}{\sec\omega}$$

which is an ellipse referred to its equi-conjugate diameters as axes.

Question 1071.

(T. P. Trivedi):—If $S_r = \sum_{p=1}^{p=r+1} p^{r+1-p}\,{}_{n+1-p}C_{r+1-p}$

prove that $\sum_{r=0}^{n} S_r = 1^{n+1} + 2^n + 3^{n-1} + \ldots + (n+1)$.

Solution by Hemraj and K. B. Madhava.

Obviously

$$S_r = 1^r\,.\,{}_nC_r + 2^{r-1}\,.\,{}_{n-1}C_{r-1} + 3^{r-2}\,.\,{}_{n-2}C_{r-2} + \ldots$$

$$\therefore \sum_{r=0}^{n} S_r = \sum_{r=0}^{n} {}_nC_r + \sum_{r=0}^{n-1} 2^r \cdot {}_{n-1}C_r + \sum_{r=0}^{n-2} 3^r \cdot {}_{n-2}C_r + \ldots$$

$$= (1 + 1)^n + (1 + 2)^{n-1} + (1 + 3)^{n-2} + \ldots$$

$$= 2^n + 3^{n-1} + 4^{n-2} + \ldots (n + 1) + (n + 2)^0.$$

$$= 1^{n+1} + 2^n + 3^{n-1} + \ldots (n + 1)$$

for the sake of symmetry.

Question 1072.

(T. P. Trivedi, M.A., LL.B.) :—Prove that

$$\int_0^{\pi/2} \sqrt{\tan x} \log \tan x \, dx = \frac{\pi^2}{2\sqrt{2}}.$$

$$\int_0^{\pi/2} \sqrt[5]{\tan x} \log \tan x \, dx = \frac{\pi^2}{4}\left(\frac{3}{\sqrt{5}} - 1\right).$$

Solutions by Hemraj and M. K. Kewalramani.

We know that $\int_0^\infty \frac{y^{k-1}}{1+y} dy = \pi \operatorname{cosec} k\pi$, where $1 > k > 0$.

Put $y = \tan^2 x$ and differentiate with respect to k :

$$4\int_0^{\pi/2} \tan^{2k-1} x \log \tan x \, dx = -\pi^2 \operatorname{cosec} k\pi \cdot \cot k\pi.$$

Putting $k = \frac{3}{4}$ and $\frac{5}{5}$ respectively, we get the results ; the first result correctly printed is therefore $\frac{\pi^2}{2\sqrt{2}}$.

Question 1080.

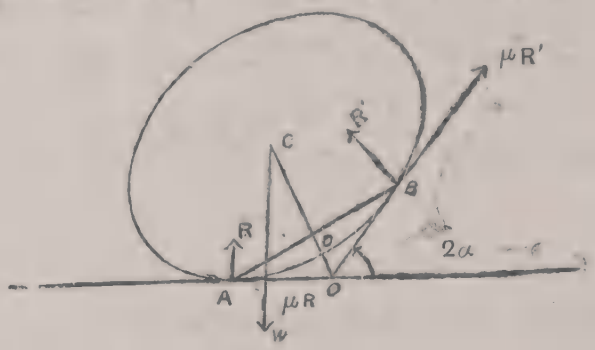
(K. J. Sanjana, M.A.) :—An elliptic cylinder of eccentricity e rests on a horizontal plane and against a plane inclined at an angle 2α to the horizon, touching them along lines parallel to and equidistant from their line of intersection. If the coefficient of friction (μ) be the same for each plane and the cylinder be just on the point of slipping, prove that μ is the positive root of the equation

$$\mu^2(\tan \alpha - e^2 \sin \alpha \cos \alpha) + \mu = e^2 \sin \alpha \cos \alpha,$$

Solution by Hemraj.

Take a section by a vertical plane.

Since the tangents OA and OB are equal, O lies on either axis. But



it is easy to show that in this case it lies on the minor axis. Take O (o, y_1), then the equation of AB is $yy_1 = b^2 \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ being that of the ellipse.

$$\therefore \cot \alpha = \frac{AD}{OD} = \frac{a}{\sqrt{y_1^2 - b^2}}, \text{ i.e., } y_1 = a\sqrt{(\sec^2 \alpha - e^2)}.$$

$$\therefore OA = OD \operatorname{cosec} \alpha = \frac{a \tan \alpha \sec \alpha}{\sqrt{\sec^2 \alpha - e^2}}. \text{ [2}\alpha \text{ may be acute or obtuse].}$$

Now the system of forces acting on the body is in limiting equilibrium,

Resolving horizontally and vertically and taking moments about O, we get,

$$\mu R = R' (\sin 2\alpha - \mu \cos 2\alpha) \quad \ldots \quad \ldots \quad (1)$$

$$W = R + R' (\mu \sin 2\alpha + \cos 2\alpha) \quad \ldots \quad \ldots \quad (2)$$

$$W \cdot OC \cdot \sin \alpha = (R - R') \cdot OA \quad \ldots \quad \ldots \quad (3)$$

Substituting for R and R' in (3) from (1) and (2) we get

$$\sin \alpha \cdot \sqrt{\sec^2 \alpha - e^2} = \frac{\sec \alpha (\sin \alpha - \mu \cos \alpha) \tan \alpha}{\sin \alpha (1 + \mu^2) \sqrt{\sec^2 \alpha - e^2}},$$

i.e., $\cos \alpha \sin \alpha (\sec^2 \alpha - e^2)(1 + \mu^2) = (\tan \alpha - \mu)$,

i.e., $\mu^2 (\tan \alpha - e^2 \sin \alpha \cos \alpha) + \quad = e^2 \sin \alpha \cos \alpha$

which gives μ.

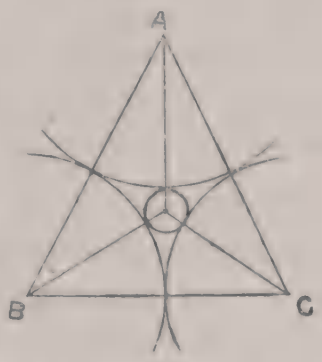
Question 1091.

(Enquirer) :—If a, b, c be the radii of three circles which touch one another externally, and r_1 and r_2 be the radii of the two circles that can be drawn to touch these three, prove that

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{2}{a} + \frac{2}{b} + \frac{2}{c}.$$

Solution and Remarks by K. J. Sanjana.

Let A, B, C be the centres of the respective circles, O the centre and



x the radius of the circle with which they have external contact. $\angle BAC = A$, we have

$$\cos A = \frac{a^2 + ab + ac - bc}{(a+b)(a+c)}, \quad \sin A = \frac{2\sqrt{abc(a+b+c)}}{(a+b)(a+c)}.$$

If $\angle OAB = \lambda$, $\angle OAC = \mu$, we have similarly

$$\cos\lambda = \frac{a^2 + ab + ax - bx}{(a+b)(a+x)}, \quad \cos\mu = \frac{a^2 + ac + ax - cx}{(a+c)(a+x)}.$$

But since $A = \lambda + \mu$, $\cos A = \cos\lambda\cos\mu - \sin\lambda\sin\mu$; transposing, squaring and simplifying, we get

$$\sin^2 A = \cos^2\lambda + \cos^2\mu - 2\cos A\cos\lambda\cos\mu.$$

Substituting and multiplying throughout by $(a+b)^2(a+c)^2(a+x)^2$, this gives

$$(a+c)^2(a^2+ab+ax-bx)^2 + (a+b)^2(a^2+ac+ax-cx)^2 - 2(a^2+ab+ac-bc)(a^2+ab+ax-bx)(a^2+ac+ax-cx) = 4abc(a+b+c)(a+x)^2.$$

This is a quadratic in x, which shows that the radius of the circle (with sign changed if necessary with which the given circles have internal contact is also obtained.

Simplifying, we get

$$x^2\{b^2c^2 + c^2a^2 + a^2b^2 - 2abc(a+b+c)\} - 2abc(bc+ca+ab)x + a^2b^2c^2 = 0.$$

If r_1, r_2 are the two roots, $\frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1+r_2}{r_1r_2}$

$$= \frac{2abc(bc+ca+ab)}{a^2b^2c^2} = 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

The proposer has taken the numerical value of the second root. The result obtained is that of Ex. 62, Chap. XII, Hobson's *Plane Trigonometry*, which has I think already been asked in our columns.

When the equation is solved, we get

$$x = \frac{abc\{bc + ca + ab \pm 2\sqrt{abc(a + b + c)}\}}{b^2c^2 + c^2a^2 + a^2b^2 - 2abc(a + b + c)};$$

on rationalizing, this becomes $\dfrac{abc}{bc + ca + ab \mp 2\sqrt{abc(a + b + c)}}$,

which is the result of Ex. 29, Chap. XII. *loc. cit.*

For internal contact this process gives the equation

$$(a + c)^2(a^2 + ab - ax + bx)^2 + (a + b)^2(a^2 + ac - ax + cx)^2$$
$$- 2(a^2 + ab + ac - bc)(a^2 + ab - ax + bx)(a^2 + ac - ax + cx)$$
$$= 4abc(a + b + c)(x - a)^2,$$

which is the first equation with the sign of x changed.

The six distances of the four points A, B, C, O are a, b, c, $a \pm x$, $b \pm x$, $c \pm x$; and the formula connecting such distances may be employed. It is found to give lengthy expressions involving x^4 and x^3, the coefficients of which, however, vanish.

By the help of *Frobenius's Theorem*, Casey has deduced the following equation for the radius of a circle cutting three circles at angles ϕ_1, ϕ_2, ϕ_3 (Eqn. 276, *Analytical Geometry*) :—

$$\begin{vmatrix} 0 & \alpha & \beta & \gamma & \rho \\ \alpha & -1 & \cos\overline{12} & \cos\overline{13} & \cos\phi_1 \\ \beta & \cos\overline{21} & -1 & \cos\overline{23} & \cos\phi_2 \\ \gamma & \cos\overline{31} & \cos\overline{32} & -1 & \cos\phi_3 \\ \rho & \cos\phi_1 & \cos\phi_2 & \cos\phi_3 & -1 \end{vmatrix} = 0,$$

where I have put α, β, γ, ρ for the reciprocals of a, b, c, x as taken above. As $\cos\overline{12}$, &c. are each -1, and $\cos\phi_1$, &c. are each either -1 or $+1$, the determinant equation is found to reduce to

$$\rho^2 - 2\rho(\alpha + \beta + \gamma) + \alpha^2 + \beta^2 + \gamma^2 - 2\beta\gamma - 2\gamma\alpha - 2\alpha\beta = 0.$$

Hence $\rho_1 + \rho_2$ or $\dfrac{1}{r_1} + \dfrac{1}{r_2} = 2\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right)$, and

$$\rho_1\rho_2 \text{ or } \frac{1}{r_1r_2} = \frac{b^2c^2 + c^2a^2 + a^2b^2 - 2abc(a + b + c)}{a^2b^2c^2}$$

giving the same results as before.

QUESTIONS FOR SOLUTION.

1141. (V. Ramaswami Aiyar):—If A, B, C, D be four concyclic points prove that the in- and ex-centres of the triangles ABC, BCD, CDA, DAB lie in a quartic which passes through A, B, C, D and touches the circle ABCD at the circular points at infinity.

If the equations of the circle and the rectangular hyperbola through A, B, C, D be $x^2 + y^2 - R^2 = 0$ and $xy + \lambda x + \mu y + \nu = 0$, show that the equation of the quartic is

$$\frac{x^2 + y^2 - R^2}{2R^2} = \frac{xy + \lambda x + \mu y + \nu}{xy - \nu}.$$

1142. (K. Ananda Rao):—Let $\omega(n)$ and $\mu(n)$ be defined as follows:* $\omega(n) = 0$ if n has any prime factor repeated more than twice; $\omega(n) = (-1)^r 2^r$ otherwise, r being the number of non-repeated prime factors of n; $\mu(n) = 0$ if n has any repeated prime factors; $\mu(n) = (-1)^r$ otherwise, r having the same meaning as above.
Show that for $|x| < 1$

$$\sum_1^\infty \mu(n)\, x^n = \sum_1^\infty \frac{\omega(n)\, x^n}{1 - x^n}$$

1143. (K. Ananda Rao):—If (i) $\lambda(n) = (-1)^s$, where s is the number of prime factors of n (each prime factor being counted according to its degree of multiplicity), (ii) the function $\phi(x)$ satisfies certain conditions, and (iii) $\psi(x) = \sum_1^\infty \lambda(n)\, \phi(nx)$, then show that $\phi(n) = \sum_q \psi(qx)$, the latter summation being taken over all the integers of q which have no repeated prime factors.

1144. (K. J. Sanjana, M.A.):—Show that

(a) $$\frac{1}{20}\left\{\frac{20}{3^3 \cdot 4^2} + \frac{47}{6^3 \cdot 5^2} + \frac{84}{10^3 \cdot 6^2} + \frac{131}{15^3 \cdot 7^2} + \ldots \text{ad inf.}\right\} = \pi^2 - 9{\cdot}86\dot{}.$$

(The value of the right-hand side is less than ·002938).

(b) $$\frac{1}{25}\left\{\frac{19}{3^3 \cdot 4^2} + \frac{46}{6^3 \cdot 5^2} + \frac{79}{10^3 \cdot 6^2} + \frac{118}{15^3 \cdot 7^2} + \ldots \text{ad inf.}\right\} = 9{\cdot}87\dot{8} - \pi^2.$$

(The value of the right-hand side is less than ·009285).

* 1 is not regarded as a prime number. The functions $\mu(n)$ and $\lambda(n)$ are well-known in the Analytic Theory of Numbers. See, for example, Landau, *Handuch der Lehre von der Verteilung des Primzahlen.*

1145. (N. Doraiarajan):—n rods $OA_1, OA_2, \ldots OA_n$ are hinged together at O, which is a plane joint. Show that the area of the plane polygon $A_1A_2 \ldots A_n$ is a maximum when the circles on A_1A_2, A_2A_3, ... A_nA_1 as diameters have a common orthogonal circle and that the perimeter is a maximum when all the sides teach a circle. When there are only 3 rods, can the triangle be constructed with a ruler and compasses ?

1146. (A. Narasinga Rao):—If three integers be chosen at random, the probability that no two of them have a common factor is

$$\frac{1}{\zeta(3)}\left\{1 + \frac{3}{\zeta(2)} - \frac{3}{\zeta(3)}\right\}$$

where $\zeta(s)$ is the sum of the series $\sum_1^\infty n^{-s}$.

1147. (A. Narasinga Rao):—(a_n) is a monotonic sequence of positive terms and a set S is constructed every member of which is the sum of *any* number of *distinct* a's. Show that if S is to be dense everywhere in some interval,

(1) zero must be the limit pt. of (a_n)

and (2) $a_n \not> a_{n+1} + a_{n+2} + \ldots$ to ∞ for sufficiently large values of n.

Conversely if (1) and (2) are satisfied by all the terms of the sequence, the set S is everywhere dense in the interval $(0 \text{ to } \sum_1^\infty a_n)$, if Σa_n be convergent, and in $(0 \text{ to } \infty)$ if Σa_n diverges.

1148. (A. A. Krishnaswami Aiyangar):—Given two vertices of a cyclic quadrilateral and the loci of the other two vertices, discuss the loci of the points of intersection of the three diagonals taken two by two. Give a purely geometrical investigation in the particular case when the given loci are concentric circles.

1149. (S. Anantarama Aiyar):—P is a variable point in the plane of a $\triangle ABC$ which moves so that, if PD, PE, PF are drawn perpendicular to the sides, DEF remains constant in area. Prove geometrically that P moves on a circle.

1150. (Martyn M. Thomas, M.A.):—If ρ and ρ_n be the corresponding radii of curvature of the curve $p = f(r)$ and its nth negative pedal show that

$$\frac{nr^2 - (n-1)\,p\,\rho_n}{(n+1)r^2 - np\,\rho_n} = \underset{\text{to } n+1 \text{ fractions.}}{\frac{1}{2-}\,\frac{1}{2-}\,\frac{1}{2-}\,\cdots - \frac{1}{2-}\,\frac{p\rho_n}{r^2}} = \frac{r^{n-2}\rho}{p^{n-1}}.$$

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17	11	$y/(y^2+4n^2\pi^2)$	$x/(x^2+4n^2\pi^2)$
18	4 from bottom	$x2n$	x^{2n}
20	2 „	If s	If n is
21	7	e^xz	e^{ax}
23	11	semi circles	semi-circles
25	6	$4n$	$4m$
29	3 from bottom	Second	second
30	15	Fortyth	Forsyth
31	3 from bottom	$f(x)$	$f(z)$
32	8	March 11	on March 11
36	3	form	for
36	4 from bottom	Δs ABC & XYZ	the triangle ABC
39	2 „	$2p\mathrm{R}k$	$2p\mathrm{R}r$
42	8 „	$[z-a)^n$ + etc.	$[(z-a)^n$ + etc.
50	4 „	$(-)^{\frac{n}{2}}-s$ in the numerator.	$(-)^{\frac{n}{2}-s}$
52	17	$[y'']_{z=d}$	$[y']_{z=d}$
64	9	Question 357	Question 857
71	12	Qn^2+gIk^2	Qx^2+gIk^2
78	2	$e^{\frac{4}{3}}$	$e^{4/3}$
92	11	and given	and are given
101	5 from last	$8x^3+8ax^2+\ldots\ldots$	$\{8x^3+8ax^2+\ldots\}$
102	8	8	
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113	9 & 10	$l2$	l^2
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AN ALGEBRA OF ARITHMETIC FUNCTIONS.

By F. Hallberg.

(*Continued from Vol. XII, page* 216.)

The Units of the Algebra.

§ 17. *Definitions.*

From the development so far, it is evident that our algebra may be considered as a quasi-linear algebra, differing from ordinary linear algebras in having an infinite number of units. Such an algebra may conveniently be called an *infinitary* algebra.

There is evidently a principal unit or modulus

$$e_1 = [1, 0, 0, 0, \ldots] = 1, \quad \ldots \quad \ldots \quad (17.1)$$

the remaining units being defined by

$$\left.\begin{aligned} e_2 &= [0, 1, 0, 0, \ldots] \\ e_3 &= [0, 0, 1, 0, \ldots] \\ e_4 &= [0, 0, 0, 1, \ldots] \\ \ldots &\quad \ldots \quad \ldots \quad \ldots \end{aligned}\right\} \quad \ldots \quad \ldots \quad (17.11)$$

Hence every A. F., $[x_n]$, can be expressed in the form

$$X = [x_n] = \sum_{n=1}^{\infty} x_n e_n, \quad \ldots \quad \ldots \quad \ldots \quad (17.2)$$

the elements of the A. F. being regarded as co-ordinates of an infinitary complex number X, and belonging to a given number field, with respect to which the units e_n are linearly independent.

From the definition of the process of multiplication in formula (2 . 3), it follows that the units are subject to the multiplication table

$$e_m e_n = e_{mn}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17.3)$$

so that the product of two infinitary complex numbers, X and X′, may be written

$$XX' = X'X = \sum_{m,\,n=1}^{\infty} x_m x'_n e_{mn}, \quad \ldots \quad \ldots \quad (17.31)$$

Thus the product is itself an infinitary complex number.

§ 18. *Some characteristic features of the infinitary algebra considered.*

Assume that for two infinitary numbers, X and X′, with *real* co-ordinates, x_n and x'_n

$$XX' \equiv 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18.1)$$

Excluding the trivial case in which both the given infinitary numbers vanish, we assume further that

$$x'_1 \equiv x'_2 \equiv x'_3 \equiv \ldots x'_{r-1} \equiv 0, \quad \ldots \quad \ldots \quad (18.11)$$

$$x'_r \not\equiv 0, \quad \ldots \quad \ldots \quad \ldots \quad (18.111)$$

r being a positive integer.

By (18.11), the co-ordinate with index r in the product is reduced to

$$x_1 x'_r \equiv 0, \quad \ldots \quad \ldots \quad (18.12)$$

from which follows, by (18.111)

$$x_1 \equiv 0. \quad \ldots \quad \ldots \quad (18.121)$$

Next, let p be a prime, and consider the co-ordinate with index pr in the product. By the aid of (18.11), (18.121) we obtain

$$x_p x'_r \equiv 0, \quad \ldots \quad \ldots \quad (18.13)$$

and hence, by (18.111)

$$x_p \equiv 0. \quad \ldots \quad \ldots \quad (18.131)$$

If now q is another prime, different or not from p, the co-ordinate with index pqr in the product becomes, by (18.11), (18.121), (18.131)

$$x_{pq} x'_r \equiv 0, \quad \ldots \quad \ldots \quad (18.14)$$

and again, by (18.111)

$$x_{pq} \equiv 0. \quad \ldots \quad \ldots \quad (18.141)$$

Continuing in this way, we see that each co-ordinate x_n of X is identically zero. This result may be expressed as follows :—

The necessary and sufficient condition that a product of numbers with real co-ordinates in the infinitary algebra considered shall be zero, is that at least one factor is zero.

Consequently the theorem of Frobenius, that the only linear associative algebras over the field of reals in which a product is zero only when one factor is zero, are the algebra of reals, the algebra of ordinary complex numbers, and the algebra of real quaternions, cannot be extended to infinitary algebras. (For the above theorem, with references, see L. E. Dickson, *Linear Algebra* (Cambr. Math. Tracts, No. 16), p. 10—12.)

It is clear that the above proof is valid also if the co-ordinates of our algebra are ordinary complex numbers.

It follows, that our algebra is not nilpotent, since no number X exists, such that

$$X \neq 0, \qquad \ldots \qquad \ldots \qquad (18.2)$$

$$X^k = 0, \qquad \ldots \qquad \ldots \qquad (18.21)$$

k being a positive integer.

One of the most interesting questions in connection with our algebra is that of the process of division. We noted already in § 2, that if the co-ordinate for the modulus in the divisor vanishes, while the corresponding co-ordinate in the dividend does not vanish, the operation is not possible. This followed at once from formula (8.9), from which we also deduced, that if the first mentioned co-ordinate did not vanish, the quotient always existed. But there are other possibilities. Thus

$$\frac{e_m}{e_n} = e_{\frac{m}{n}} \qquad \ldots \qquad \ldots \qquad (18.3)$$

provided m is a multiple of n; otherwise the quotient does not exist Similarly, from an identity such as

$$(ae_m + be_n)(\alpha e_r + \beta e_s) \equiv a\alpha e_{mr} + a\beta e_{ms} + b\alpha e_{nr} + b\beta e_{ns}, \quad \ldots \quad (18.4)$$

it is easily seen under what conditions an expression like that on the right may be factorised. A discussion of the general case would carry us too far. It may be observed, however, that if e_m and e_n are the units with non-zero co-efficients of lowest order in the dividend and divisor respectively then the condition for the validity of (18.3) is a necessary (but not sufficient) condition for the existence of the quotient.

[It is perhaps worth mentioning that some of the above anomalies with respect to the operation of division may be avoided by the admission of units with *fractional* indices, subject to the multiplication table (17.3). Since other complications are introduced simultaneously however, the algebra obtained will not be further considered here.]

In § 3, we utilised a connection between our algebra and the theory of ordinary Dirichlet's series to attain brevity and clearness in our representation. It will now be readily understood that the process indicated

furnishes the results obtained because of the fact, that we may, provisionally, put $e_n \sim n^{-s}$, ... (18.5)
where s is an *arbitrary* complex number, the function on the right behaving in a manner similar to that of the unit on the left with respect to the fundamental operations of ordinary algebra. In the place mentioned, we remarked that the convergence or non-convergence of the D. S. was of no account. This we now propose to prove.

For this purpose, assume the more general equivalence

$$e_n \sim \{f(n)\}^{-s}, \quad \ldots \quad \ldots \quad (18.6)$$

$f(n)$ satisfying the functional relation (5.1). This implies

$$f(1) = 1, \quad \ldots \quad \ldots \quad (18.61)$$

Let now an infinitary number X be given. It is clear, that a sequence of primes q_r may be chosen, corresponding to the natural sequence of all primes p_r, such that

$$f(p_r) = q_r, \quad \ldots \quad \ldots \quad (18.62)$$

and that the D. S.

$$\sum_{n=1}^{\infty} x_n \{f(n)\}^{-s} \quad \ldots \quad \ldots \quad (18.63)$$

possesses a half-plane of absolute convergence, the primes q_r being taken all different, and arranged in order of magnitude. It is further evident, that if some of the primes q_r are chosen larger, the new D. S. will still possess a half-plane of absolute convergence, with a limiting abscissa not larger than that of the former. If now a set of infinitary numbers be given, while others are derived from this set by means of functional operations, the sequence q_r may be so chosen, that every infinitary number corresponds to a D. S. of the form (18.63), with a half-plane of absolute convergence since any function of one or more such D. S. may be written in the form of a D. S. of the same type, $f(n)$ satisfying the relation (5.1). In the region common to all those half-planes the arbitrary s' is supposed to be situated. Then the terms may be rearranged so that the D.S. become of the ordinary type, with this peculiarity, that there is a set of zero coefficients of constant order common to all the D. S.

Let now two equivalent functional processes be applied to series of our set. The results must be identical, *i.e.*, the difference between the results must vanish identically. But then all the coefficients in this difference must be identically zero, otherwise we could find a value for s in the common half-plane of absolute convergence for which the D. S. in question has a sum different from zero. [Vide E. Landau *Handbuch der Lehre von der Verteilung der Primzahlen*, p. 747]. Passing over to our algebra, we have the important theorem :

Equivalent functional processes, applied to the same infinitary numbers of our algebra, furnish identical results.

This theorem justifies the definition of inverse functions, such as that of the logarithm by means of the identity

$$\text{Exp. } (\log \mathrm{X}) \equiv \mathrm{Y}, \quad \ldots \quad \ldots \quad (18.7)$$

§ 19. *Conditions for functionality.*

In the present paragraph we propose to deduce the conditions that the infinitary variable

$$\mathrm{Y} = \sum_{n=1}^{\infty} y_n e_n \quad \ldots \quad \ldots \quad (19.1)$$

be an analytic function of the independent infinitary variable

$$\mathrm{X} = \sum_{n=1}^{\infty} x_n e_n \quad \ldots \quad \ldots \quad (19.11)$$

We obtain by differentiation

$$\frac{\partial \mathrm{Y}}{\partial x_r} = \frac{d\mathrm{Y}}{d\mathrm{X}} \cdot \frac{\partial \mathrm{X}}{\partial x_r} = e_r \frac{d\mathrm{Y}}{d\mathrm{X}} = \sum_{n=1}^{\infty} e_n \frac{\partial y_n}{\partial x_r}. \quad \ldots \quad (19.2)$$

Assume

$$r = kl, \quad \ldots \quad \ldots \quad (19.21)$$

when (19.2) gives

$$e_l \sum_{n=1}^{\infty} e_n \frac{\partial y_n}{\partial x_{kl}} = e_{kl} \sum_{m=1}^{\infty} e_m \frac{\partial y_m}{\partial x_l}, \quad \ldots \quad (19.22)$$

or by (17.3)

$$\sum_{n=1}^{\infty} e_n \frac{\partial y_n}{\partial x_{kl}} = \sum_{m=1}^{\infty} e_{km} \frac{\partial y_m}{\partial x_l}. \quad \ldots \quad \ldots \quad (19{\cdot}221)$$

It follows that the indices of the units in the sum on the left must be multiples of k.

Putting

$$n = kml, \ldots \quad \ldots \quad \ldots \quad (19.222)$$

we obtain the necessary (but not sufficient) condition for functionality

$$\frac{\partial y_{km}}{\partial x_{kl}} = \frac{\partial y_m}{\partial x^l} \ldots \quad \ldots \quad \ldots \quad (19{\cdot}3)$$

In addition, if n is not a multiple of k, (19·221) gives

$$\frac{\partial y_n}{\partial x_{kl}} = 0, \quad \ldots \quad \ldots \quad \ldots \quad (19{\cdot}31)$$

from which follows, that the derivatives in (19·3) vanish, unless m is a multiple of l, or, by (19·21), (19·222)

$$n = rs \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19{\cdot}4)$$

As the *necessary and sufficient conditions for functionality*, we thus finally obtain, by the aid of (19·3)

$$\frac{\partial y^n}{\partial x_r} = \left\{ \begin{array}{l} = \dfrac{\partial y}{\partial x_1} \\ = 0 \end{array} \right\}, \quad \ldots \quad \ldots \quad (19{\cdot}5)$$

according as (19·4) is, or is not, satisfied.

If these conditions are satisfied, we have

$$\frac{dy}{dx} = \frac{\partial y}{\partial x_1}, \quad \ldots \quad \ldots \quad (19{\cdot}6)$$

by putting $r = 1$ in (19·2).

The relation (19·5), (19·6) may be at once vertified by the aid of (9·3), which may be written

$$f\left\{\sum_{n=1}^{\infty} x_n\, e_n\right\} = f(x_1)\, e_1 + \frac{x_2}{1!} f'(x_1)\, e_2 + \frac{x_3}{1!} f'(x_1)\, e_3$$

$$+ \left\{ \frac{x_4}{1!} f'(x_1) + \frac{x^2_2}{2!} f''(x_1) \right\} e_4 + \frac{x_5}{1!} f'(x_1)\, e_5$$

$$+ \left\{ \frac{x_6}{1!} f'(x_1) + \frac{x_2\, x_3}{1!\,1!} f''(x_1) \right\} e_6 + \ldots . \quad (19{\cdot}7)$$

§ 20. *Transformation of units.*

Lex X be any given number of our infinitary algebra, which from its connection with Dirichlet's series may be said to be *of Dirichlet's type.*

We may now construct a new infinitary algebra with the units

$$E_n = X^{n-1}, \quad \ldots \quad \ldots \quad (20{\cdot}1)$$

subject to the multiplication table

$$E_m E_n = E_{m+n-1}, \quad \ldots \quad (20{\cdot}2)$$

We may, for example, let

$$X = [\,1\,], \quad \ldots \quad \ldots \quad (203).$$

when

$$\left.\begin{aligned} &E_1 = e_1 = 1\,, \\ &E_2 = \sum_{n=1}^{\infty} e_n = \left[\,1\,\right], \\ &E_3 = \sum_{n=1}^{\infty} t\,(n).\,e_n = \left[\,1\,\right]^2, \\ &\quad\ldots\ldots\ldots \end{aligned}\right\} \quad \ldots \quad \ldots \quad (20\ 31)$$

If now the e's are supposed solved with respect to the E's, we may express any number

$$A = \sum_{n=1}^{\infty} a_n\, e_n \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (20.4)$$

of our initial algebra in the form

$$A = \sum_{n=1}^{\infty} A_n\, E_n \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (20.41)$$

This process would, however involve a discussion of infinite determinants, which we may avoid by observing that the units E_n may provisionally be defined by means of the equivalence

$$E_n \sim X^{n-1}, \qquad \ldots \quad \ldots \quad \ldots \qquad (20,11)$$

where X is an *arbitrary* complex number. Just as we, in § 3, established the connection between our initial algebra and the theory of D. S., we may now similarly connect our new algebra with Mac-Laurin's series, and speak of it as *an infinitary algebra of Mac-Laurin's type.*

The operations of addition, subtraction and multiplication of this algebra are defined by

$$A \pm B = \sum_{n=1}^{\infty} (A_n \pm B_n)\, E_n\,, \ldots \qquad \ldots \quad \ldots \qquad (20.5)$$

$$A\,.\,B = \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} A_i \cdot B_{n-i+1} \right) E_n \qquad \ldots \quad \ldots \qquad (20\ 6)$$

The process of division is defined by the formula

$$A^{-1} = \frac{1}{A_1}\left[E_1 - \frac{A_2}{A_1} E_2 + \left\{ \left(\frac{A_2}{A_1} \right)^2 - \frac{A_3}{A_1} \right\} E_3 - \left\{ \left(\frac{A_2}{A_1} \right)^3 - 2\,\frac{A_2}{A_1} \cdot \frac{A_3}{A_1} + \frac{A_4}{A_1} \right\} E_4 + \ldots \right], \qquad (20.7)$$

provided the condition

$$A_1 \neq 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20.71)$$

is satisfied.

[As in the case of the algebra of Dirichlet's type, certain anomalies with regard to the operation of division appears in the present algebra, which may be partially eliminated by the admission of units with *negative* and *zero* indices, subject to the multiplication table (20.2), an artifice which will, however, introduce complications in other directions.]

The infinitary algebra of Mac-Laurin's type may now be developed on lines similar to those followed in the case of the infinitary algebra of Dirichlet's type. Although the two algebras are equivalent under linear transformation of units in the field considered, the former appears to be of less interest than the latter, and to furnish less elegant results. It will therefore not be considered further here.

THE GROUP-THEORY ELEMENT OF THE HISTORY OF MATHEMATICS*

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FEW mathematical terms suggest such fundamental human cravings as the term group, and few have been more appropriately chosen. Just as human society has led to perplexities which increased with the advance of civilization, so the mathematical group-theory has given rise to problems which became more and more difficult with the advances in the development of mathematics. In both cases the primitive stages are comparatively simple and their history throws important light on the later developments.

The history of the mathematical group-theory can be conveniently divided into three periods. The first of these extends from the beginning of mathematical history to about 1770 A D, and may be called the *implicit period* since the group concept was then employed without being explicitly stated. The second, or *specialization period*, extends from about 1770 to about 1870. During this period the theory of substitution groups was founded as an autonomous science and the usefulness of this theory in the study of algebraic equations was emphasized. The third, or *generalization period*, extends from about 1870 to the present day, and is characterized by increased generalizations by abstraction and the explicit use of groups in each of the large domains of mathematics.

The most fundamental property of the elements of a group in the common restricted sense of this term is that they satisfy the condition that each group contains one and only one element which satisfies the equation

$$xy = z$$

whenever any two of these symbols are replaced by two equal or unequal elements of the group. This condition is evidently satisfied by the members of the number systems of the ancients, including the Babylonians, the Egyptians and the Greeks. On the other hand, the zero of our common modern number system destroys this group property of the entire system. If the zero is excluded the rest of these numbers (real or complex) constitute a group as regards multiplication.

One of the oldest groups of operations in the history of mathematics is the multiplication group whose elements are all the ordinary rational

* From *The Scientific Monthly*, January 1921.

numbers with the exception of zero. This group is used implicitly in the Ahmes papyrus, written about 1700 B. C., since the linear equation and fractions are found frequently in this work. It is an interesting historical fact that for many centuries after this date the ancients seemed to have considered only numbers which are elements of this group, and one is led to inquire to what extent the guiding influence of the group concept was responsible for the late introduction of zero as a number.

It might at first appear that the introduction of zero as a number tended to show that the group concept was not a fundamental guiding principle in the development of arithmetic since it failed to dominate when opposing forces presented themselves. The impression that the group concept did not dominate when zero was introduced is, however, not quite correct, since this number which was rejected by the multiplication group was destined to become the principal element, or the identity, of the addition group. "The stone which the builders rejected, the same is become the head of the corner."

It is an interesting fact that the addition group was made possible by the introduction of zero and by granting full number citizenship to the negative numbers, and the latter was done about the time when the multiplication group was somewhat impaired by the introduction of zero. These important extensions of our number system were completed during the seventeenth century but some steps in this direction had been taken a thousand years earlier especially by the Hindus. In particular, Brahmagupta had already illustrated negative and positive numbers by debts and credits and he observed that a debt subtracted from zero becomes a credit and a credit subtracted from zero becomes a debt, and if one subtracts a debt from a credit or a credit from a debt one obtains the sum.

The long delay in the general introduction of zero as a number seems to show that the ancient peoples held tenaciously to the view that all of the numbers without exception should constitute a group as regards multiplication. It is doubtless true that the group concept was not clearly observed by them and that they could not have given a satisfactory account of the motives which guided them in their efforts for more knowledge about numbers, but the fact that they were guided by this concept seems to be well established.

H. Poincaré pointed out that the same concept guided the ancients in their efforts to secure a knowledge of geometry and he noted that the absence of any direct reference to groups in Euclid's *Elements* was due to the fact that the group notions were among the oldest mathematical notions to be assimilated and hence they did not seem to require explicit mention even at the time of Euclid, notwithstanding the fact that

the principal foundation of Euclid's demonstrations is really the group and its properties.[1] In view of the fact that the group notions are so fundamental and elementary that they did not seem to require explicit mention at the time of Euclid, it may at first appear strange that during the latter half of the nineteenth century these notions assumed a prominent place in the mathematical literature and that the subject of group-theory began to be regarded as one of the most difficult in the whole range of pure mathematics.

The reason for this change of attitude on the part of the mathematicians is not difficult to discover. As long as only the most general notions of groups were needed the subject was naturally regarded as too elementary to require any special attention. The idea that a set of distinct elements should have the property that any two of them can be combined into one and that this one is also found in the set was illustrated not only by the natural numbers but also by the movements of figures in space, and hence this idea became firmly fixed in the human mind at an early age. It is in accord with the human yearnings for completeness and it is a natural extension of the notion of cyclic changes which were illustrated by the daily and the seasonal apparent movements of the sun.

There is only a short step from this idea of completeness to the idea that a set of distinct elements has the property that: when any two of the symbols in the equation $xy=z$ are replaced by distinct or equal elements of the set, the resulting linear equation has always one and only one root in the set. If we add to these conditions the condition that the associative law shall be satisfied when the elements of the set are combined, we have a complete modern definition of the term group, and it is at once apparent that this definition involves only very fundamental and elementary notions in regard to laws in the world of ideas.

While the general laws of the group are very mild, they required exceptions at an early stage in the growth of mathematics. As was noted above, the entrance of zero into our number system required some modification of these laws as regards the operation of multiplication. The one law of combination imposed by the group notion was too narrow for the full development of our operations with numbers, where two modes of combination, now known as addition and multiplication, were developed even in pre-historic times. Historically the group concept may therefore be said to embody fundamental laws of combination with which human beings became acquainted in pre-historic times but which had to be violated in certain respects in order to secure the most fruitful mathematical developments.

[1] H. Poincaré, "On the Foundations of Geometry," *The Monist*, Vol. 9, (1898) p. 34.

As these laws were not formulated in the abstract until about 1870, their early violators were naturally unconscious of the significance of their steps as regards the group concept. In fact, these laws might never have been formulated if it had not been discovered that without violating any of them it was possible to develop a very useful and extensive body of knowledge. The mathematical world discovered this fact by accident and at a comparatively late date. The discovery seems to have been due to the development of a large body of knowledge relating to a special class of groups now known as *substitution groups*.

The fundamental ideas involved in this body of knowledge are also very elementary. About 1770 J. L. Lagrange and others were much interested in the solution of the general equation in one unknown. More than two centuries had then elapsed since several Italian mathematicians had discovered algebraic solutions of the general cubic and of the general biquadratic equation. All efforts to obtain a solution of the general quintic had failed and the mathematical world was becoming more and more deeply interested in either making further advances along this line, or proving that such advances are impossible.

In the study of the methods which had led to success for the lower degrees it appeared that the number of different formal values which certain unsymmetric rational functions of a number of variables assume when these variables are permuted in every possible manner was of fundamental importance. For instance, the expression $x_1x_2 + x_3x_4$ assumes the following three values

$$x_1x_2 + x_3x_4,\ x_1x_3 + x_2x_4,\ x_1x_4 + x_2x_3$$

when the four variables x_1, x_2, x_3, x_4 are permuted in every possible manner. As there are 24 possible permutations of these four variables eight of them transform such a function into itself. These eight permutations constitute an important substitution group known as the octic group.

In general, all the permutations on n variables which transform into itself a certain rational function of these variables constitute a substitution group. Hence the study of such groups seemed important for the purpose of proving the existence or the non-existence of rational functions of a given number of variables which assume a given number of values when these variables are permuted in every possible manner. The concept of a substitution group on n variables is thus seen to be a very elementary one but the study of such groups led to a large body of theorems. Some of these appeared elegant even if they were supposed to apply only to a rather special field.

[*To be continued.*]

Leaves from a Lecturer's Diary.

The following demonstration for obtaining the recurrence formu'æ of the convergents of a continued fraction is short.

If $\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \ldots$ be the continued fraction it is obvious that the nth convergent

$$\frac{p_n}{q_n} = \frac{Aa_n + Bb_n}{Ca_n + Db_n} \quad \ldots \quad \ldots \quad (\alpha)$$

in lowest terms where A, B, C and D are functions of all the elements up to the $(n-1)$th.

If in $\frac{p_n}{q_n} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots \frac{a_{n-2}}{b_{n-2}} + \frac{a_{n-1}}{b_{n-1}} + \frac{a_n}{b_n}$, $a_n = 0$, this reduces to $\frac{p_{n-1}}{q_{n-1}}$, while if $b_n = 0$, $b_{n-1} + \frac{a_n}{b_n}$ becomes infinity and therefore everything after b_{n-2} becomes zero, thus reducing the convergent to $\frac{p_{n-2}}{q_{n-2}}$.

Now on inserting successively $a_n = 0$ and $b_n = 0$ in (α), we get

$$\frac{p_{n-1}}{q_{n-1}} = \frac{B}{D} \text{ and } \frac{p_{n-2}}{q_{n-2}} = \frac{A}{C}.$$

Hence

$$\frac{p_n}{q_n} = \frac{a_n p_{n-2} + b_n p_{n-1}}{a_n q_{n-2} + b_n q_{n-1}}$$

from which the proof may be completed for equating the numerator and denominator separately as in the usual manner.

*** *** ***

It is fairly well-known that the cissoid $y^2(2a - x) = x^3$ can be employed to solve the Delian problem of inserting two mean proportionals between two given numbers. The following analysis of the proof given in Edwards' *Differential Calculus*, § 438, serves to suggest a curve which can be similarly employed to interpolate n mean proportionals. If two means, say α and β, are to be introduced between a and b (say $a > b$), we are required to have

$$\frac{a}{\alpha} = \frac{\alpha}{\beta} = \frac{\beta}{b}, \text{ i.e., } a^2b = \alpha^3 \ldots \quad \ldots \quad (1)$$

Now describe a circle of radius a with A as centre and let OC be a diameter. Upon AD, which is perpendicular to it, choosing $AB = b$ and $A\alpha = \alpha$, we find for the equations of CB and Oα, respectively

$$ay + bx = 2ab \quad \ldots \quad \ldots \quad (2) \quad \text{and} \quad ay = \alpha x \ldots \quad \ldots \quad (3)$$

Eliminating α and b, we find for the locus of P, the intersection of (2) and (3), the cissoid $x^3 = y^2(2a - x)$.

Conversely when the cissoid is drawn, we have the construction given by Edwards quoted above. Similarly if n mean proportionals are to be inserted, say α, β ... such that $\frac{a}{\alpha} = \frac{\alpha}{\beta} = \ldots \frac{v}{b}$ which gives $\alpha^{n+1} = a^n b$.

Eliminating as above α and b between this and (2) and (3) given above we are led to the curve $x^{n+1} = y^n(2a - x)$.

The converse process of determining α is precisely similar to the above construction when this generalised cissoid is drawn. After α is determined, β, γ, ... are each determined by the ordinary methods from

$$a\beta = \alpha^2; \; \alpha\gamma = \beta^2; \; \ldots$$

SOLUTIONS

Question 661.

(S. RAMANUJAN) :—Solve in integers $x^3 + y^3 + z^3 = u^6$ and deduce the following :—

(i) $6^3 - 5^3 - 3^3 = 2^6$. (ii) $8^3 + 6^3 + 1^3 = 3^6$.

(iii) $12^3 - 10^3 + 1^3 = 3^6$. (iv) $46^3 - 37^3 - 3^3 = 6^6$.

(v) $174^3 + 133^3 - 45^3 = 14^6$. (vi) $1188^3 - 509^3 - 3^3 = 34^6$.

Solution by Prof. N B. Mitra, M.A.

To solve $x^3 + y^3 + z^3 = u^6$ (a)

$= v^3$, where $v = u^2$.

(i) A particular solution may be obtained thus :—

We have $x^3 + y^3 = v^3 - z^3$.

Let $x = p + q$, $y = p - q$, $z = r - s$, $v = r + s$.

Then substituting we get, $p(p^2 + 3q^2) = s(s^2 + 3r^2)$.

Put $p = a(ll + 3m)$, $q = a(l - m)$, $r = bl$, $s = 3bm$,

we get, $4a^3(l + 3m)(l^2 + 3m^2) = 9b^3m(l^2 + 3m^2)$,

so that $4a^3l = 3m(3b^3 - 4a^3)$,

so that we may put $l = 3(3b^3 - 4a^3)$, $m = 4a^3$.

Substituting and writing a for $2a$, we get,

$x = a(9b^3 - a^3)$, $y = a^4$, $z = 3b(3b^3 - a^3)$, $u = 3a^2$.

Putting $a = b$, we get $8^3 + 1^3 + 6^3 = 3^6$.. (ii).

(ii) Another special solution may be obtained from Q. 441 proposed by the late Mr. Ramanujan (J. I. M. S., Vol. V, page 39).

We have $(3a^2 + 5ab - 5b^2)^3 + (4a^2 - 4ab + 6b^2)^3 + (5a^2 - 5ab - 3b^2)^3$
$= (6a^2 - 4ab + 4b^2)^3$.

A solution of the present problem will be obtained if we can make $6a^2 - 4ab + 4b^2$ a perfect square.

Assume therefore $6a^2 - 4ab + 4b^2 = (ka - 2b)^2$,

whence $a(k^2 - 6) = 4b(k - 1)$.

Hence we can take $a = 4(k - 1)$, $b = k^2 - 6$ and we get the following values $x = -5k^4 + 20k^3 + 88k^2 - 216k - 12$

$y = 6k^4 - 16k^3 + 8k^2 - 32k + 184$

$z = -3k^4 - 20k^3 + 136k^2 - 40k - 148$

$u = 2k^2 - 4k + 6$.

Putting

$k=1$, we get $-125^3+150^3-75^3=10^6$ or $-5^3+6^3-3^3=2^6$... (i)

$k=2$, $-12^3+120^3+108^3=12^6$ or $1^3-10^3+12^3=3^6$... (iii)

$k=0$, $-12^3+184^3-148^3=12^6$ or $-3^3+46^3-37^3=6^6$... (iv)

$k=4$, $532^3+696^3-180^3=28^6$ or $133^3+174^3-45^3=14^6$. (v)

(iii) We may proceed also thus :—Mr. S. Narayanan in his solution of Q. 441 shows (J. I. M. S., Vol. VI, page 226) that if

$$(pa^2+mab-mb^2)^3+(na^2-nab+lb^2)^3+(ma^2-mab-pb^2)^3 = (la^2-nab+nb^2)^3 \quad \ldots \quad (1)$$

then $l=\lambda(\lambda^3+1)^2$, $m=2\lambda^3-1$, $n=\lambda(\lambda^3-2)^2$, $p=\lambda^3+1$... (2)

Multiply both sides of (1) by l^3;

then the r . h . s . $=(la^2-lnab+lnb^2)^3$.

Assume $l^2a^2-lnab+lnb^2=(la+ka)^2$, whence $a:b=(ln-k^2):(l2k-n)$, so that we may take $a=ln-k^2$, $b=l(2k-n)$... (3)

Then a solution of the present problem is

$x=l(pa^2+mab-mb^2)\,q^2$,

$y=l(na^2-nab+lb^2)\,q^2$,

$z=l(ma^2-mab-pb^2)q^2$,

$u=l(ln+k^2-kn)\,q$ where q is any quantity.

To get numerical values, we give any value to λ, then (2) gives us the values of l, m, n, p. These together with any value we may give to k determine a and b by (3). Then we choose q so as to make x, y, z integral. If l is a square, say r^2, then we may omit the factor l from the values of x, y, z and substitute r for l in the value of u. Otherwise the values of x, y, z, u will have a common divisor.

This solution is however still not general.

(iv) We may obtain a more general solution in the following way :—

Euler gives the most general solution of $A^3+B^3+C^3=D^3$ as follows :—

$A=(3ac+3bc-ad+3bd)(d^2+3c^2)-(a^2+3b^2)^2$.

$B=(3ac-3bc+ad+3bd)(d^2+3c^2)+(a^2+3b^2)^2$.

$C=(d^2+3c^2)^2-(3ac+3bc-ad+3bd)(a^2+3b^2)$.

$D=(d^2+3c^2)^2+(3ac-3bc+ad+3bd)(a^2+3b^2)$

so that $x=AD$, $y=BD$, $z=CD$, $u=D$ is a solution of (a).

The defect of this method is that the values of x, y, z, u so obtained have a common divisor.

We may attack the problem in another way yet. Give any values we like to a, b, c, d and thus obtain the values of A, B, C, D. If in any case D is a square, we obtain a solution of (a).

The problem of finding the general solution of

$$(a^2 + 3b^2)(3ac - 3bc + ad + 3bd) + (d^2 + 3c^2)^2 = w^2$$

giving all the sets of values of a, b, c, d, w seems to be intractable.

Question 681.

(S. RAMANUJAN) :—Solve in integers $x^3 + y^3 + z^3 = 1$ and deduce the following :—

(i) $6^3 + 8^3 = 9^3 - 1$. (ii) $9^3 + 10^3 = 12^3 + 1$.

(iii) $135^3 + 138^3 = 172^3 - 1$. (iv) $791^3 + 812^3 = 1010^3 - 1$.

(v) $11161^3 + 11468^3 = 14258^3 + 1$. (vi) $65601^3 + 67402^3 = 83802^3 + 1$.

Solution by N. B. Mitra.

A particular solution may be obtained in the following way :—

Put $x = p + q$, $y = p - q$, $z = 2r - 1$.

Then $p(p^3 + 3q^3) = (1 - r)(4r^2 - 2r + 1)$.

Let $p = 3a$, $q = a(4r - 1)$. (This introduces a restriction).

Then we get after reduction and cancellation of the factor $4r^2 - 2r + 1$,

$$r = 1 - 36a^3, \quad q = 3a(1 - 48a^3)$$

whence

$$x = 6a(1 - 24a^3), \quad y = 144a^4,$$

$$z = 1 - 72a^3$$

or writing a for $2a$,

$$x = 3a(1 - 3a^3),$$

$$y = 9a^4,$$

$$z = 1 - 9a^3.$$

Putting $a \pm 1$, we get $-6^3 + 9^3 - 8^3 = 1$ (i)

$a = -1$, $-12^3 + 9^3 + 10^3 = 1$ (ii)

Question 1074.

(V. THIRUVENKATACHARI) :—If $a_1, a_2, a_3, \ldots a_n$ be n unequal positive numbers, show that

(i) $L_n = (G_1 G_3 G_5 \ldots)/(G_2 G_4 \ldots)$,

(ii) $G_n = (L_1 L_3 L_5 \ldots)/(L_2 L_4 \ldots)$,

where G_r denotes the product of the G.C.M.'s of the numbers r at a time and L_r denotes the similar product of the L.C.M.'s.

Solution by P. R. Venkatakrishna Iyer.

Let $a_1 = \frac{\rho_1 PG}{g_1}$, $a_2 = \frac{\rho_2 PG}{g_2}$, ... $a_r = \frac{\rho_r PG}{g_r}$, .. ,

where G is the G.C.M. of $a_1, a_2, \ldots a_n$, and $P = g_1 g_2 \ldots g_n$, (one or more of the numbers $g_1, g_2\ g_n \ldots$ may be equal to unity). Also $\rho_1, \rho_2, \rho_3 \ldots \rho_n$ and $g_1, g_2, g_3 \ldots g_n$ are all prime to each other.

Denoting the product $\rho_1 \rho_2 \ldots \rho_n$ by the letter p,

we have $G_1 = p \cdot P^{n-1} G^n$,

$$G_2 = \frac{P^{{}^nC_2}\ G^{{}^nC_2}}{P^{n-1}} = P^{{}^{n-1}C_2}\ G^{{}^nC_2},$$

similarly $G_3 = P^{{}^{n-1}C_3}\ G^{{}^nC_3}$

...

$$G_r = \frac{P^{{}^nC_r}\ G^{{}^nC_r}}{P^{{}^{n-1}C_{r-1}}} = P^{{}^{n-1}C_r}\ G^{{}^nC_r}$$

$$G_{n-2} = P^{{}^{n-1}C_1}\ G^{{}^nC_2}$$

$$G_{n-1} = P \cdot G^{{}^nC_1}$$

$$G_n = G.$$

When n is even, $\frac{G_1 \cdot G_3 \cdot G_5 \ldots G_{n-1}}{G_2 \cdot G_4 \ldots \ldots G_n}$

$$= p \cdot \frac{P^{{}^{n-1}C_1} + {}^{n-1}C_3 + \ldots + {}^{n-1}C_0}{P^{{}^{n-1}C_2} + {}^{n-1}C_4 + \ldots + {}^{n-1}C_1} \times \frac{G^{{}^nC_1} + {}^nC_3 + \ldots + {}^nC_1}{G^{{}^nC_2} + {}^nC_4 + \ldots + {}^nC_0}$$

$$= p \cdot \frac{P^{2^{n-2}}}{P^{2^{n-2}-1}} \times \frac{G^{2^{n-1}}}{G^{2^{n-1}-1}} = p \cdot P \cdot G = L_n,\ L_n \text{ being the L.C.M. of}$$

$a_1, a_2, \ldots a_n$.

Again, $L_1 = p \cdot P^{n-1} G^n$

$$L_2 = p^{n-1} \cdot (PG)^{{}^nC_2}$$

$$L_3 = p^{{}^{n-1}C_1} \cdot (PG)^{{}^nC_3}$$

...

$$L_r = p \cdot {}^{n-1}C_{r-1} \cdot (PG)^{{}^nC_r}$$

...

$$L_{n-2} = p^{{}^{n-1}C_2}\,(PG)^{{}^nC_2}$$

$$L_{n-1} = p^{n-1}\,(PG)^{{}^nC_1}$$

$$L_n = p\,.\,P\,.\,G.$$

When n is even, $\dfrac{L_1\,.\,L_3\,.\,L_5\,\ldots\,L_{n-1}}{L_1\,.\,L_4\,\ldots\,\ldots\,L_n}$

$$= \frac{p^{{}^{n-1}C_0 + {}^{n-1}C_2 + \ldots + {}^{n-1}C_1}}{p^{{}^{n-1}C_1 + {}^{n-1}C_3 + \ldots + {}^{n-1}C_2 + {}^{n-1}C_0}} \times \frac{p^{{}^{n-1} + {}^nC_3 + {}^nC_5 + \ldots + {}^nC_1}}{p^{{}^nC_2 + {}^nC_4 + \ldots + {}^nC_2 + {}^nC_0}}$$

$$\times \frac{G^{{}^nC_1 + {}^nC_3 + \ldots + {}^nC_1}}{G^{{}^nC_2 + {}^nC_4 + \ldots + {}^nC_2 + {}^nC_0}}$$

$$= \frac{p^{2^{n-2}}}{p^{2^{n-2}}} \times \frac{P^{2^{n-1}-1}}{P^{2^{n-1}-1}} \times \frac{G^{2^{n-1}}}{G^{2^{n-1}-1}} = G = G_n.$$

The results are true when n is odd.

Question 1081.

(K. J. Sanjana, M.A.) :—If a, b, c be real unequal quantities whose sum is zero, prove that the equation

$$4\,(a^2 + b^2 + c^2)\,x^3 + 18\,abc\,x^2 - abc\,(a^2 + b^2 + c^2) = 0$$

has two imaginary or complex roots.

Solution by Hemraj.

The Hessian of the cubic is

$$18\,abc\,x^2 + 2\,(a^2 + b^2 + c^2)^2\,x + 3\,abc\,(a^2 + b^2 + c^2) = 3$$

The roots of the Hessian are real if

$$(a^2 + b^2 + c^2)^3 > 54\,a^2b^2c^2$$

since $a^2 + b^2 + c^2 > 0$

i.e. if $27a^2b^2c^2 + 4\,(bc - a^2)^3 < 0$

i.e. if $15a^2b^2c + 4\,(b^3c^3 + 3a^4bc - a^6) < 0$

i.e. if $3a^2b^2\,(a^2 + b^2) - 12ab\,(a^4 + b^4)$
$- 4\,(a^5 + b^5) + 26\,a^3b^3 < 0$

i.e. if $3a^2b^2 - 12ab\,(a + b)^2 - 4\,(a^2 + ab + b^2)^2 < 0$

since $(a - b)^2 < 0$

i.e. if $4\,(a + b)^4 - 4\,ab\,(a + b)^2 + a^2b^2 > 0$

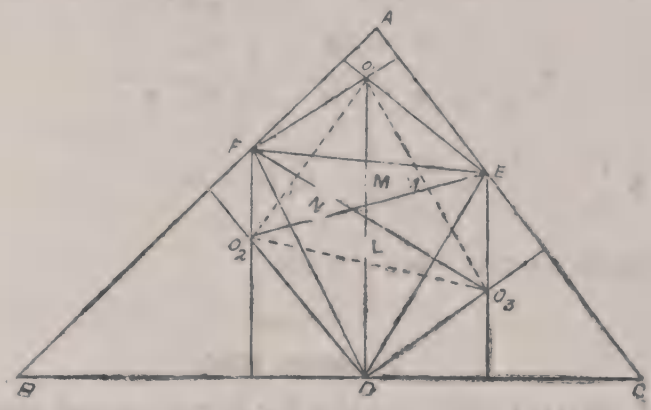
i.e. if $(2c^2 - ab)^2 > 0$ which is so.

Hence the cubic has two imaginary or complex roots.

Question 1087.

(A. A. KRISHNASWAMI AIYANGAR):—ABC is any triangle. DEF is a triangle inscribed within it, D, E, F lying on the sides BC, CA, AB respectively, and O_1, O_2, O_3 are the orthocentres of the triangles AEF, BDF, CED. Prove that (i) the area of the triangle $O_1O_2O_3$ is equal to that of the triangle DEF and (ii) the two triangles $O_1O_2O_3$ and DEF can be inscribed in the same conic.

Solution by Hemraj and Tiruvenkatachari.



Let O_1, O_2, O_3, be the orthocentres.

Let O_1 D, O_2 E, O_3 F form a triangle LMN.

Since $O_1F \parallel O_3D$, $FO_2 \parallel EO_3$, $O_1E \parallel DO_2$.

(i) $\therefore$ $\triangle O_1FO_3 = \triangle O_1FD$

i.e. $\triangle O_1LO_3 = \triangle FLD$

and $\triangle O_1MO_2 = \triangle EMD$

$\triangle O_2NO_3 = \triangle ENF$

$\triangle LMN = \triangle LMN$

Adding we get $\triangle O_1O_2O_3 = DEF$

(ii) $O_1FO_2DO_3E$ is a hexagon whose opposite sides are parallel,

$\therefore$ the points of intersections of the opposite sides are collinear.

Hence O_1, F, O_2, D, O_3, E lie on a conic.

Question 1089.

(S. MAHADEVAN):—Prove that if $|x| < 1$

$$\frac{x}{(1+x)^2} + \frac{x^2}{(1+x^2)^2} + \frac{x^3}{(1+x^3)^2} + \ldots\ldots$$

$$= \frac{x}{1-x} - \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} - \ldots\ldots = \sum_1^\infty \psi(n)\, x^n$$

where $\psi(n)$ denotes the excess of the odd over the even divisors of n, 1 and n being reckoned as divisors.

Solution by Hemraj.

Expand the terms of the 1st series and place them upon a summation diagram.

$\frac{x}{(1+x)^2}$	x	$-2x^2$	$3x^3$	$-4x^4$	$5x^5$	. . .
$\frac{x^2}{(1+x^2)^2}$	x^2	$-2x^4$	$3x^6$	$-4x^8$	$5x^{10}$	. . .
$\frac{x^3}{(1+x^3)^2}$	x^3	$-2x^6$	$3x^9$	$-4x^{12}$	.	. . .
. .	.	.	.	.	.	. . .

Now consider

X	$2X^2$	$3X^3$	$4X^4$	. .
X^2	$2X^4$	$3X^6$	$4X^8$	. .
X^3	$2X^6$	$3X^9$	$4X^{12}$	. .

where $X = |x|$

The second summation-diagram, consisting of positive numbers only is convergent, since if we sum by rows, we get

$$\frac{X}{(1-X)^2} + \frac{X^2}{(1-X^2)^2} + \frac{X}{(1-X^3)^2} + \ldots\ldots$$

the limit of whose convergency-ratio is

$$Lt_{n\to\infty} \frac{X^{n+1}}{(1-X^{n+1})^2} \times \frac{(1-X^n)^2}{X} = X < 1.$$

Hence the original summation diagram is absolutely convergent and may therefore be summed by any convenient system of curves.

If we sum by rows, we obtain $\frac{x}{(1+x)^2} + \frac{x^2}{(1+x^2)^2} + \frac{x^3}{(1+x^3)^2} + \quad (1)$

If we sum by columns, we get $\frac{x}{1-x} - \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} - \ldots\ldots \quad (2)$

Hence (1) and (2) are equal.

If the above double series be arranged according to powers of x, we sum it by columns. Since the double series still converges after every term is made positive

$\therefore$ the sums by rows and columns are certainly equal;

$$\therefore \quad (1) = x - x^2 + 4x^3 - 5x^4 + \ldots\ldots = \sum_1^\infty \psi(n)\, x^n.$$

Question 1090.

(S. MAHADEVAN):—The focus S is joined to any point P_1 on the ellipse, P_1S cuts the ellipse again in Q_1, Q_1S' cuts the ellipse again in P_2, and so on. Show that SP_n ultimately tends to coincide with the major axis.

Solution by Hemraj and Kewalramani.

Let the eccentric angles of P_1, Q_1, P_2, Q_2, Q_{n-1}, P_n be θ_1, ϕ_1, θ_2, ϕ_2 ϕ_{n-1}, θ_n. Then it is easy to show that

$$\tan\frac{\theta_1}{2}\tan\frac{\phi_1}{2} = \cot\frac{\phi_1}{2}\cot\frac{\theta_2}{2} = \tan\frac{\theta_2}{2}\tan\frac{\phi_2}{2} = \ldots\ldots$$

$$= \cot\frac{\phi_{n-1}}{2}\cot\frac{\theta_n}{2} = \lambda \text{ (say)}.$$

$$\therefore \quad \tan\frac{\theta_1}{2}\cot\frac{\theta_n}{2} = \lambda^{2(n-1)}$$

Now $\tan\frac{\theta_1}{2}\tan\frac{\phi_1}{2} = \lambda \ (\neq 1)$.

$\therefore \quad \theta_n \to 0$ or π according as $\lambda \gtrless 1$,

$\therefore$ SP_n tends to coincide with the major axis.

Question 1091.

(ENQUIRER):—If a, b, c be the radii of three circles which touch one another externally, and r_1 and r_2 be the radii of the two circles, that can be drawn to touch these three, prove that

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{2}{a} + \frac{2}{b} + \frac{2}{c}.$$

Solution (1) *by Hemraj and Martyn M. Thomas.*

(2) *by N. Sundaram Aiyar.*

(1) Let A, B, C be the centres of the circles a, b, c. Take A as the origin and AB as the x — axis. Then the equations of the circles are

$$x^2 + y^2 - a^2 = 0$$

$$x^2 + y^2 - 2(a+b)x + a^2 + 2ab = 0$$

$$x^2 + y^2 + 2gx + 2fy + c' = 0$$

where $c' = a^2 + 2ac$,

$$g = -\left(a + \frac{a-b}{a+b}c\right)$$

and $$f^2 = \frac{4abc\,(a+b+c)}{(a+b)^3} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

If they are touched by a fourth circle excluding them all, then let its equation be

$$x^2 + y^2 + 2Gx + 2Fy + C = 0$$

where $C = a^2 + 2ra$

$$G = -\left(a + \frac{a-b}{a+b}\,r\right),$$

$$F^2 = \left\{ ab\,(r+c) - rc\,\frac{(a^2+b^2)}{a+b} \right\}^2 \times \frac{1}{abc\,(a+b+c)}.$$

[These follow from the condition $c_1 + c_2 - 2r_1r_2 - 2g_1g_2 - 2f_1f_2 = 0$].

Also from (1) $F^2 = \dfrac{4abr\,(r+a+b)}{(a+b)^2}$.

Hence equating the values of F^2 we get

$$r^2\left\{ \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - \left(\frac{2}{bc} + \frac{2}{ca} + \frac{2}{ab}\right) \right\}$$
$$- 2r\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + 1 = 0.$$

But if the *first* three circles are touched by a circle enclosing them all, its radius is obtained by changing r into $-r$.

$$\therefore \frac{1}{r_1} = 2\sqrt{\Sigma\frac{1}{bc}} + \Sigma\frac{1}{a} \text{ and } \frac{1}{r_2} = 2\sqrt{\Sigma\frac{1}{bc}} - \Sigma\frac{1}{a}$$

$$\therefore \frac{1}{r_1} - \frac{1}{r_2} = 2\,\Sigma\frac{1}{a}.$$

(2) Let A, B, C be respectively the centres of circles whose radii are a, b and c. Let O be the centre of either of the circles which touch the given circles.

Call the angles BOC, COA, AOB in order θ_1, θ_2 and θ_3 respectively,

Then $\theta_1 + \theta_2 + \theta_3 = 2\pi \quad \ldots \quad \ldots \quad \ldots \quad (1)$

Let OA, OB, and OC be equal to α, β and γ respectively

If r_2 be the radius of the circle which touches the given $\odot^s$ internally, and r_1 that of the other circle, we have

$$r_2 = a + \alpha = b + \beta = c + \gamma \quad \ldots \quad \ldots \quad (2)$$

and $$r_1 = \alpha - a = \beta - b = \gamma - c \quad \ldots \quad \ldots \quad (3)$$

In both cases $(a-b)^2 = (\alpha - \beta)^2$ and two similar relations hold good.

Now $AB^2 = (a+b)^2 = \alpha^2 + \beta^2 - 2\alpha\beta\cos\theta_3$

$= (\alpha-\beta)^2 + 4\alpha\beta\sin^2\frac{\theta_3}{2}$

$\therefore \quad ab = \alpha\beta\sin^2\frac{\theta_3}{2}.$

From this and two similar relations we easily get

$$c = \gamma\,\frac{\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}}{\sin\frac{\theta_3}{2}}$$

or from (1), $\frac{\gamma}{c} = \frac{\sin\frac{\theta_1+\theta_2}{2}}{\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}} = \cot\frac{\theta_1}{2} + \cot\frac{\theta_2}{2}$ (4)

From (4) and two similar results, we have

$$\left(\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}\right)^2 = 4\left(\Sigma\cot\frac{\theta_1}{2}\right)^2$$

$$= 4\Sigma\cot^2\frac{\theta_1}{2} + 8\Sigma\cot\frac{\theta_2}{2}\cot\frac{\theta_3}{2}$$

$$= 2\Sigma\left(\cot\frac{\theta_1}{2}+\cot\frac{\theta_2}{2}\right)^2 + 4\Sigma\cot\frac{\theta_2}{2}\cot\frac{\theta_3}{2}$$

$$= 2\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right) + 4 \quad \text{from (4) and (1).}$$

From this relation, as well as (2) and (3), if r stands for r_1 or r_2, we have $\frac{\alpha}{a} = \frac{r}{a} \pm 1$ and two similar relations.

$\therefore$ We have, since $\Sigma\frac{\alpha^2}{a^2} - 2\Sigma\frac{\beta\gamma}{bc} + 4 = 0,$

$$\Sigma\frac{r^2}{a^2} \pm 2\Sigma\frac{r}{a} + 3 - 2\left\{\Sigma\frac{r^2}{bc} \pm 2\Sigma\frac{r}{a} + 3\right\} + 4 = 0$$

or $$r^2\left(\Sigma\frac{1}{a^2} - 2\Sigma\frac{1}{bc}\right) \mp 2r\Sigma\frac{1}{a} + 1 = 0$$

Hence r_1 takes the negative and r_2 the positive sign

$$\therefore \frac{1}{r_1^2} - \frac{2}{r_1}\Sigma\frac{1}{a} = -\left(\Sigma\frac{1}{a^2} - 2\Sigma\frac{1}{bc}\right) = \frac{1}{r_2^2} + \frac{2}{r_2}\Sigma\frac{1}{a}$$

$$\therefore \frac{1}{r_1^2} - \frac{1}{r_2^2} = 2\left(\Sigma\frac{1}{a}\right)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)$$

$$\therefore \frac{1}{r_1} - \frac{1}{r_2} = \frac{2}{a} + \frac{2}{b} + \frac{2}{c}.$$

Question 1092.

(ENQUIRER) :—Find the relation existing between α, β, γ, δ, if

$$\int \frac{\alpha + \beta x + \gamma x^2 + \delta x^3}{\sqrt{x(1-x)}}\, dx$$

does not depend on circular functions.

Solution by Hemraj and S. R. Ranganathan.

$$\int \frac{dx}{\sqrt{x - x^2}} = \cos^{-1}(1 - 2x) = \text{vers}^{-1}\, 2x$$

$$\int \frac{\alpha + \beta x + \gamma x^2 + \delta x^3}{\sqrt{x - x^2}}\, dx = \alpha \int \frac{dx}{\sqrt{x - x^2}} + \frac{\beta}{2} \int \frac{1 - (1 - 2x)}{\sqrt{x - x^2}}\, dx$$

$$+ \gamma \int \frac{x - x(1-x)}{\sqrt{x(1-x)}}\, dx + \delta \int \frac{x - x(1-x) - x^2(1-x)}{\sqrt{x(1-x)}}\, dx$$

$$= \alpha\, \text{vers}^{-1}\, 2x + \frac{\beta}{2}\, \text{vers}^{-1}\, 2x - \beta \sqrt{x - x^2} + \frac{\gamma}{2}\, \text{vers}^{-1}\, 2x - \gamma \sqrt{x - x^2}$$

$$+ \gamma \left\{ \frac{(1-2x)\sqrt{x-x^2}}{4} + \tfrac{1}{8} \sin^{-1}(1 - 2x) \right\} + \frac{\delta}{2}\, \text{vers}^{-1}\, 2x - \delta \sqrt{x - x^2}$$

$$+ \frac{3\delta}{2} \left\{ \frac{(1 - 2x)\sqrt{x-x^2}}{4} + \tfrac{1}{8} \sin^{-1}(1 - 2x) \right\} + \frac{\delta}{2} (x - x^2)^{\frac{3}{2}}$$

since $\sin^{-1}\theta + \cos^{-1}\theta = \frac{\pi}{2}$.

$\therefore$ if this integral is independent of circular functions, then

$$\alpha + \frac{\beta}{2} + \frac{\gamma}{2} - \frac{\gamma}{8} + \frac{\delta}{2} - \frac{3}{16}\delta = 0,$$

i.e.

$$16\alpha + 8\beta + 6\gamma + 5\delta = 0.$$

Question 1093.

(MARTYN M. THOMAS, M.A.) :—A fixed ray of light falls on a plane mirror revolving about an axis in its plane. Show that the locus, in the plane, of the point of incidence is a conic, and that the reflected rays generate a ruled conicoid.

Solution by N. Sundaram, N. B. Mitra and N. B. Pendse.

Let AO be the axis, OP the shortest line between the axis and the fixed ray, and PQ the fixed ray, inclined to the plane AOP at an angle λ, say :

In the fist place, let OA be taken as the y-axis, and the straight line through O, in the revolving plane perpendicular to OA as the x-axis. Let the revolving plane cut the ray at Q in a certain position. Let QR be

perpendicular to plane AOP, and RS to OA. Then the co-ordinates of Q, the point of incidence are (SQ, OS).

Now $SQ^2 = SR^2 + RQ^2$ and $\frac{RQ}{RP} = \tan \lambda$.

$\therefore$ the locus of Q in the mirror is

$$x^2 = \alpha^2 + y^2 \tan^2 \lambda, \text{ where } \alpha \text{ is SR or OP,}$$

i.e. $x^2 - y^2 \tan^2 \lambda = \alpha^2$ which is a conic.

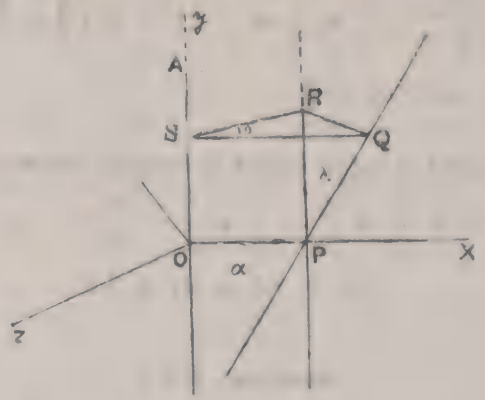
To find the locus of the reflected rays, let OP be the x-axis, OA the y-axis, and the line through O perpendicular to plane AOP, the z-axis Q being the point of incidence, if $QSR = \theta$, the equation to the revolving plane is $z = x \tan \theta$.

The equations to PQ are $x = \alpha$, $z = y \tan \lambda$.

$\therefore$ the co-ordinates of Q are $(\alpha, \alpha \tan \theta \cot \lambda, \alpha \tan \theta)$.

The co-ordinates of the image of P are obviously,

$(\alpha \cos 2\theta, 0, \alpha \sin 2\theta)$.



The equations to the reflected ray which is the join of these two points are therefore

$$\frac{x - \alpha}{2 \sin^2 \theta} = \frac{y - \alpha \tan \theta \cot \lambda}{\tan \theta \cot \lambda} = \frac{z - \alpha \tan \theta}{\left(\tan \theta - \frac{2 \tan \theta}{1 + \tan^2 \theta} \right)}$$

or $$\frac{x - \alpha}{\frac{2t^2}{1 + t^2}} = \frac{y \tan \lambda - \alpha t}{t} = \frac{z - \alpha t}{\frac{t(t^2 - 1)}{t^2 + 1}} \text{ where } t = \tan \theta$$

or $$\frac{x - \alpha}{2t} = \frac{y \tan \lambda - \alpha t}{1 + t^2} = \frac{z - \alpha t}{t^2 - 1}$$

Each is equal to $\frac{y \tan \lambda - z}{2}$ or $\frac{y \tan \lambda + z}{2t^2} - \frac{2a}{2t}$.

$$\therefore \quad \frac{x - a}{t} = y \tan \lambda - z$$

and $$(x + a)\, t = y \tan \lambda + z.$$

∴ Locus of the reflected rays is the ruled conicoid

$$x^2 - a^2 = y^2 \tan^2 \lambda - z^2$$

Question 1095.

(R. S. NARASIMHAM):—Seven thieves A, B, C, ... secure a sum of rupees, but are obliged to conceal it without counting it. A returns alone, divides the sum into 7 parts, finds that there are 6 rupeees over, takes the 6 rupees and one-seventh part and departs. B returns alone and does the same with the diminished sum. He divides it into 7 parts, finds that there are 5 rupees over, takes the 5 rupees and one-seventh part and departs. Each does the same, the successive remainders being 6, 5, 4, 3, 2, 1, 0. Finally, they all come together and divide the remainder which is a multiple of 7, into 7 equal parts and each takes one part. Find the least number of rupees stolen and the amount that each gets.

Solution (by the author).

Let x be the no. of rupees stolen. Then since it is divided into 7 parts and Rs. 6 are left over $x = 7t + 6$ where t is a +ve integer.

Now A takes Rs. $t + 6$ and departs. ∴ The remainder is $6t$ which is divided into 7 parts and Rs. 5 are left over.

$$\therefore \quad 6t = 7p + 5.$$

Similarly
$$6p = 7q + 4$$
$$6q = 7r + 3$$
$$6r = 7s + 2$$
$$6s = 7u + 1$$
$$6u = 7v$$
$$6v = 7y,$$ where $t, p, q, \ldots\ldots y$ are +ve integers.

By eliminating $t, p, \ldots v$, we get $279936x - 5764801y = 7812540$.

By writing down the successive convergents of $\frac{5764801}{279936}$, we find

$279936x - 5764801y = 7812540(123073 \times 5764801 - 2534477 \times 279936)$.

so that

$279936\,(x + 7812540 \times 2534477) = 5764801\,(y + 7812540 \times 123073)$,

i.e. $\dfrac{x + 7812540 \times 2534477}{5764801} = \dfrac{y + 7812540 \times 123073}{279936} = k$ (say).

$$\therefore \quad \left.\begin{aligned} x &= 5764801\,k - (\,)\,(\,) \\ y &= 279936\,k - (\,)\,(\,) \end{aligned}\right\}.$$

Solving these two equations we find that the least positive integral value of k in order that both x and y may be positive integers is 3434780.

The corresponding value of x, *i.e.*, the least no. of rupees that must have been stolen = **4941180.**

The amount that each got is :

$$\left.\begin{aligned} A = 945828\,;\ B = 844986\,;\ C = 758550\,;\ D = 684462\,; \\ E = 620958\,;\ F = 566526\,;\ G = 519870. \end{aligned}\right\} \textit{Ans.}$$

Similar solutions from N. B. Mitra and several others.

Question 1096.

(T. P. Trivedi, M.A., LL.B.) :—Show that if p is prime and n any positive integer, $(np\,!) - n\,!\,(p\,!)^n$ is a multiple of p^{n+3}.

Solution by N. B. Mitra and N. B. Pendse.

Let the notation $M(p)$ denote a multiple of p.

Now $np\,!$ can be written in the form

$$\begin{array}{ccccccc} 1. & 2. & 3. & \ldots & \ldots & (p-1)\,.\,p. \\ (p+1). & (p+2). & (p+3). & & \ldots & (p+p-1)\,.\,2p. \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \{(n-1)\,p+1\}\,.\,\{(n-1)\,p+2\} & \ldots & \{(n-1)(p+p-1\} & . & np. \end{array}$$

Now the product of the factors in the last column is $n\,!\ p^n$.

Hence $np\,! = n\,!\ p^n \prod_{r=0}^{r=n-1} f(rp)$,

where $f(rp) = (rp+1)\,(rp+2) \quad \ldots \quad \ldots \quad (rp+p-1)$.

Now, $f(rp) = (rp)^{p-1} + A_1\,(rp)^{p-2} + A_2\,(rp)^{p-3} + \ldots$

$$+ A_{p-2}\,rp + A_{p-1} \quad \ldots \quad (1),$$

where it is known that all the odd A's except A_1 are $M(p^2)$ and all the remaining A's except A_{p-1} are $M(p)$ and $A_{p-1} = (p-1)\,!$ Hence, if $p > 3$, $p-2$ is odd since p is prime. Therefore $A_{p-2} = M(p^2)$.

$$\therefore \quad f(rp) = M(p^3) + (p-1)\,!$$

Hence $\prod_{n-1} f(rp) = \mathrm{M}(p^3) + \{(p-1)!\}^n$.

Therefore from (1), $(np)! = n!\, p^n \{\mathrm{M}(p^3) + (p-1)!\}^n$

$= \mathrm{M}(p^{n+3}) + n!\,(p!)^n$.

$\therefore \quad (np)! - n!\,(p!)^n = \mathrm{M}(p^{n+3})$.

Question 1097.

(T. P. Trivedi, M.A., LL.B.) :— Prove that

(i) $\left[\int_0^\infty e^{-x^4}\,dx\right]^2 = \int_0^\infty e^{-x^2}\,dx \times \int_0^{\pi/2} \frac{d\theta}{(4 - 2\sin^2\theta)^{\frac{1}{2}}}$,

(ii) $\left[\int_0^\infty e^{-x^6}\,dx\right]^2 = \int_0^\infty e^{-x^3}\,dx \times \int_0^{\pi/2} \frac{d\theta}{(8 - 6\sin^2\theta)^{\frac{1}{3}}}$.

Solution by K. J. Sanjana, M.A.

Putting $x^{2n} = y$, it is readily seen that

$$\int_0^\infty e^{-x^{2n}}\,dx \text{ or } \mathrm{I}_{2n} \text{ is } \frac{1}{2n}\,\Gamma\left(\frac{1}{2n}\right).$$

Hence $$\frac{\mathrm{I}_{2n}^2}{\mathrm{I}_n} = \frac{1}{4n}\,\frac{\left\{\Gamma\left(\frac{1}{2n}\right)\right\}^2}{\Gamma\left(\frac{1}{n}\right)}.$$

When $n = 2$, we get

$$\frac{\mathrm{I}_4^2}{\mathrm{I}_2} = \frac{1}{8}\,\frac{\{\Gamma(\frac{1}{4})\}^2}{\sqrt{\pi}} = \frac{1}{8\sqrt{\pi}} \cdot 4\sqrt{\pi}\,\mathrm{K}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{d\theta}{(1 - \frac{1}{2}\sin^2\theta)^{\frac{1}{2}}}.$$

This gives (i); see *Roberts*, § 134.

Similarly, we have

$$\frac{\mathrm{I}_6^2}{\mathrm{I}_3} = \frac{1}{12}\,\frac{\{\Gamma(\frac{1}{6})\}^2}{\Gamma(\frac{1}{3})} = \frac{1}{12} \cdot 3\sqrt{2}\,\mathrm{K}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{d\theta}{(1 - \frac{3}{4}\sin^2\theta)^{\frac{1}{2}}}, \text{ which gives (ii).}$$

The latter result may be obtained as in Ex. 56, Ch. VI of *Roberts*.

The result for $\mathrm{I}_8^2 \div \mathrm{I}_4$ could be obtained without much difficulty.

Question 1098.

(K. J. SANJANA, M.A.):—(i) If the circle of curvature at a point of the parabola $y^2 = 4ax$ touches the directrix, the point lies at the intersection of the parabola with the straight line $4x - 5a = 0$.

(ii) If the circle of curvature at a point of the hyperbola $4xy = a^2 + b^2$ touches one of the asymptotes, the point lies at the intersection of the hyperbola with one of the curves

$$3x^4 - 2(a^2 - b^2)x^2 + 6x^2y^2 - y^4 = 0.$$
$$3y^4 - 2(a^2 - b^2)y^2 + 6y^2x^2 - x^4 = 0.$$

Solution by Martyn M. Thomas, C. N. Sreenivasa Iyengar, S. V. Venkatroya Sastri, C. Ranganathan, V. Ramachandran, N. Sundaram and N. B. Mitra.

First Part.—At the point $(at^2, 2at)$ on the parabola, the x-co-ordinate of the centre of curvature is $2a + 3at^2$; and $\rho = 2a(1 + t^2)^{3/2}$,

$\therefore$ If the directrix is to touch the circle of curvature,

$$(2a + 3at^2) + a = \rho = 2a(1 + t^2)^{3/2}$$

$\therefore$ $3(1 + t^2) = 2(1 + t^2)^{3/2}$ $\therefore$ $(1 + t^2)^{\frac{1}{2}} = \frac{3}{2}$. $\therefore$ $t^2 = \frac{5}{4}$.

$\therefore$ The point $(at^2, 2at)$ lies on the line $x = \frac{5a}{4}$.

Second Part.—Let $c^2 = \frac{a^2 + b^2}{4}$, so that $\left(ct, \frac{c}{t} \right)$ is a point on the hyperbola $xy = c^2$, the equation being referred to its asymptotes as axes of oblique co-ordinates.

If ω be the angle between the asymptotes

$$\cos\omega = \frac{1 - \tan^2\frac{\omega}{2}}{1 + \tan^2\frac{\omega}{2}} = \frac{1 - \frac{b^2}{a^2}}{1 + \frac{b^2}{a^2}} = \frac{a^2 - b^2}{a^2 + b^2}$$

Normal at $\left(ct, \frac{c}{t} \right)$ is $y(1 - t^2\cos\omega) - x(t^2 - \cos\omega) = \frac{c}{t}(1 - t^4)$.

Normal at $\left(ct_1, \frac{c}{t_1} \right)$ is $y(1 - t_1^2\cos\omega) - x(t_1^2 - \cos\omega) = \frac{c}{t_1}(1 - t_1^4)$.

Eliminating x,

$$y(t + t_1)\sin^2\omega = \frac{c}{tt_1}\{t^3 + tt_1 + t_1^3 - \cos\omega + t^3t_1^3 - \cos\omega\, tt_1(t_1^2 + tt_1 + t^2)\}.$$

Putting $t_1 = t$, the y co-ordinate of the centre of curvature at $\left(ct, \frac{c}{t} \right)$ is given by $\bar{y} \cdot 2t \sin^3 \omega = \frac{c}{t^3} (3t^2 - \cos \omega + t^6 - 3t^4 \cos \omega)$.

Also $\rho = \frac{(CD)^3}{ab} = \frac{(CP^2 - a^2 + b^2)^{3/2}}{ab}$

$$= \{ c^2t^2 + \frac{c^2}{t^2} - 2(ct)\left(\frac{c}{t}\right) \cos(\pi - \omega) - a^2 + b^2 \}^{3/2} / ab$$

$$= \{ c^2(t^4 + 1) + t^2(2c^2 \cos \omega - a^2 + b^2) \}^{3/2} / abt^3$$

$$= \{ (t^4 + 1)(a^2 + b^2) - 2t^2(a^2 - b^2) \}^{3/2} / 8abt^3.$$

If the circle of curvature touches the x-axis, we have $\bar{y} \sin \omega = \rho$

$$\therefore \frac{c\{3t^2 + t^6 - \cos \omega (1 + 3t^4)\}}{2t^3 \sin \omega}$$

$$= \frac{\{(t^4 + 1)(a^2 + b^2) - 2t^2(a^2 - b^2)\}^{3/2}}{8abt^3}.$$

Since $\sin \omega = \frac{2ab}{a^2 + b^2}$ and $\cos \omega = \frac{a^2 - b^2}{a^2 + b^2}$,

$$(a^2 + b^2)\{(3t^2 + t^6)(a^2 + b^2) - (1 + 3t^4)(a^2 - b^2)\}^2$$
$$= \{(t^4 + 1)(a^2 + b^2) - 2t^2(a^2 - b^2)\}^3.$$

Simplifying, and removing the common factor $4a^2b^2$,

$$t^6 \cdot 3(a^2 + b^2) - 8t^6(a^2 - b^2) + 6t^4(a^2 + b^2) - (a^2 + b^2) = 0.$$

$\therefore$ Dividing by t^6 and putting $a^2 + b^2 = 4c^2$,

$$3t^4 \cdot 4c^2 - 8t^2(a^2 - b^2) + 6 \cdot 4c^2 - \frac{4c^2}{t^4} = 0.$$

Multiplying by $\frac{c^2}{4}$ and putting x for ct and y for $\frac{c}{t}$,

$$3x^4 - 2(a^2 - b^2)x^2 + 6x^2y^2 - y^4 = 0.$$

Interchanging x and y, we get the equation corresponding to the y-axis.

Question 1098.

(K. J. Sanjana, M.A.):—(i) If n and r are positive integers, prove that

$${}^nC_{r-1} - \tfrac{1}{2} \cdot {}^nC_{r-2} + \tfrac{1}{3} \cdot {}^nC_{r-3} - \ldots + (-1)^{r-1} \frac{1}{r} {}^nC_0$$

$$= {}^nC_r \left\{ \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \ldots + \frac{1}{n-r+1} \right\}$$

(ii) If r is a positive integer, and n any rational positive number, prove that

$$ {}_nP_{r-1} + \tfrac{1}{2} \cdot {}_nP_{r-2} + \tfrac{1}{3} \cdot {}_nP_{r-3} + \ldots\ldots + \frac{1}{r} \cdot {}_nP_0 $$

$$ = {}_nP_r \left\{ \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{n+r-1} \right\}, $$

where ${}_nP_r = \dfrac{n(n+1) \ldots\ldots (n+r-1)}{\lfloor r}$ and ${}_nP_0 = 1$.

Solution by Martyn M. Thomas, N. Sundaram, V. M. Gaitonde and V. Ramachandran.

First Part.

The left-hand series = coefft. of x^r in $(1+x)^n \times \log (1+x)$

$$ = \frac{1}{r} \text{ coefft. of } x^{r-1} \text{ in } \frac{d}{dx} \left\{ (1+x)^n \cdot \log (1+x) \right\} $$

$$ = \frac{1}{r} \text{ coefft. of } x^{r-1} \text{ in } \left\{ (1+x)^{n-1} + n (1+x)^{n-1} \log (1+x) \right\} $$

$$ = \frac{1}{r} \cdot {}_{n-1}C_{r-1} + \frac{n}{r} \text{ . coefft. of } x^{r-1} \text{ in } (1+x)^{n-1} \log (1+x) $$

$$ = \frac{1}{r} \cdot {}_{n-1}C_{r-1} + \frac{n}{r} \cdot \frac{1}{r-1} \text{ coefft. of } x^{r-2} \text{ in} $$

$$ \left\{ (1+x)^{n-2} + (n-1) (1+x)^{n-2} \log (1+x) \right\} $$

$$ = \frac{1}{r} \cdot {}_{n-1}C_{r-1} + \frac{n}{r} \cdot \frac{1}{r-1} \cdot {}_{n-2}C_{r-2} + \frac{n(n-1)}{r(r-1)} \text{ coefft. of } x^{r-2} $$

$$ \text{in } (1+x)^{n-2} \log (1+x) $$

$$ = \frac{1}{n} \cdot {}_nC_r + \frac{1}{n-1} \cdot {}_nC_r \ldots\ldots + \frac{n(n-1) \ldots\ldots (n-r+2)}{r(r-1) \ldots\ldots 2} \text{ coefft. of } x \text{ in} $$

$$ (1+x)^{n-r+1} \log (1+x) $$

$$ = \frac{1}{n} \cdot {}_nC_r + \frac{1}{n-1} \cdot {}_nC_r + \ldots\ldots + \frac{1}{n-r+1} \cdot {}_nC_r. $$

Second Part.

The left-hand series = coefft. of x^r in $-\log (1-x) \times (1-x)^{-n}$

$$ = \frac{1}{r} \text{ coefft. of } x^{r-1} \text{ in } \frac{d}{dx} \left\{ -\log (1-x) \cdot (1-x)^{-n} \right\} $$

$$= \frac{1}{r} \text{coefft. of } x^{r-1} \text{ in} \left\{ (1-x)^{-n-1} - n(1-x)^{-n-1} \log(1-x) \right\}$$

$$= \frac{1}{r} \cdot {}_{n+1}P_{r-1} + \frac{n}{r} \text{ coefft. of } x^{r-1} \text{ in}$$

$$\left\{ -\log(1-x) \cdot (1-x)^{-n-1} \right\}$$

$$= \frac{1}{r}\, {}_{n+1}P_{r-1} + \frac{n}{r} \cdot \frac{1}{r-1} \text{ coefft. of } x^{r-2} \text{ in}$$

$$\left\{ (1-x)^{-n-2} - (n+1)(1-x)^{-n-2} \cdot \log(1-x) \right\}$$

$$= \frac{1}{r} \cdot {}_{n+1}P_{r-1} + \frac{n}{r} \cdot \frac{1}{r-1} \cdot {}_{n+2}P_{r-2} + \frac{n}{r} \frac{(n+1)}{(r-1)} \text{ coefft. of } x^{r-2}$$

$$\text{in} \left\{ -(1-x)^{-n-2} \log(1-x) \right\}$$

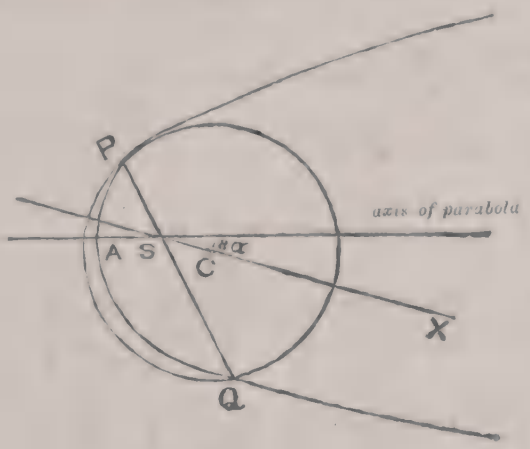
$$= \frac{1}{n} \cdot {}_{n}P_{r} + \frac{1}{n+1} \cdot {}_{n}P_{r} + \ldots\ldots + \frac{n(n+1)\ldots(n+r-2)}{r(r-1)\ldots\ldots 2} \text{coefft. of } x \text{ in}$$

$$\left\{ -(1-x)^{-(n+r-1)} \cdot \log(1-x) \right\}$$

$$= \frac{1}{n} \cdot {}_{n}P_{r} + \frac{1}{n+1} \cdot {}_{n}P_{r} + \ldots\ldots + \frac{1}{n+r-1} \cdot {}_{n}P_{r}.$$

Question 1105.

(N. P. Pandya):—S is a fiixed point in the plane of a given circle. A variable parabola is described with S as focus to cut the circle in P and Q so that PSQ is always a straight line. Find the locus of the vertex o the parabola.



Solution by Giridhari Lal Gupta and C. Ranganatham.

Let S be taken for the pole and SC for the initial line where C is the centre of the given circle. Let the axis of the parabola make an angle α with the initial line. Let SC $= b$ and radius of the circle $= a$.

The equation of the parabola is $\frac{l}{r} = 1 - \cos(\theta - \alpha)$(i)

And that of the circle is $r^2 - 2b\cos\theta . r + b^2 - a^2 = 0$.....(2)

Let the equation of SP be $\theta = \beta$(3)

and let SP and SQ taken positively be denoted by r_1 and r_2 respectively.

$\therefore$ From (2) and (3), r_1 and $-r_2$ are the roots of

$$r^2 - 2b\cos\beta . r + b^2 - a^2 = 0.$$

Whence $r_1 r_2 = a^2 - b^2$ and $r_1 - r_2 = 2b\cos\beta$.....(A)

Again from (1) and (3) $\frac{l}{r_1} = 1 - \cos(\beta - \alpha) = 2\sin^2\frac{\beta-\alpha}{2}$

and $\frac{l}{r_2} = 1 - \cos(\pi + \beta - \alpha) = 1 + \cos(\beta - \alpha) = 2\cos^2\frac{\beta-\alpha}{2}$

$$\left.\begin{aligned} \therefore r_1 r_2 &= \frac{l^2}{4\sin^2\frac{\beta-\alpha}{2}\cos^2\frac{\beta-\alpha}{2}} = \frac{l^2}{\sin^2(\beta-\alpha)} \\ \text{and } r_1 - r_2 &= \frac{l}{2}\left(\frac{1}{\sin^2\frac{\beta-\alpha}{2}} - \frac{1}{\cos^2\frac{\beta-\alpha}{2}}\right) = \frac{2l\cos(\beta-\alpha)}{\sin^2(\beta-\alpha)} \end{aligned}\right\} \text{..... (B)}$$

Thus from relations (A) and (B) we have the following equations :—

$$\frac{l^2}{\sin^2(\beta-\alpha)} = a^2 - b^2 = c^2 \text{ (suppose)} \text{(4)}$$

and $\frac{l\cos(\beta-\alpha)}{\sin^2(\beta-\alpha)} = 2b\cos\beta$(5)

From (4), $\sin(\beta-\alpha) = \frac{l}{c}$(6)

and $\therefore$ from (5) $\cos(\beta-\alpha) = \frac{bl}{c^2}\cos\beta$(7)

Squaring and adding the two sides of (6) and (7) we get

$$1 = \frac{l^2}{c^2} + \frac{b^2l^2}{c^4}\cos^2\beta$$

$$\therefore \cos^2\beta = \frac{(c^2-l^2)c^2}{b^2l^2} \text{ and also } \sin^2\beta = \frac{a^2l^2-c^4}{b^2l^2}.$$

Now (4) may be written as $\frac{l^2}{c^2} = \{\sin\beta\cos\alpha - \cos\beta\sin\alpha\}^2$

$$= \frac{(c^2-l^2)c^2}{b^2l^2}\sin^2\alpha + \frac{a^2l^2-c^4}{b^2l^2}\cos^2\alpha - \sin 2\alpha \cdot c\,\frac{\sqrt{c^2-l^2}\,\sqrt{a^2l^2-c^4}}{b^2l^2}.$$

or $c\sqrt{c^2-l^2}\sqrt{a^2l^2-c^4}\sin 2\alpha = (a^2l^2-c^4)\cos^2\alpha$

$$+ c^2(c^2-l^2)\sin^2\alpha - \frac{b^2l^4}{c^2} \quad \ldots \quad \ldots \quad (8)$$

Now the co-ordinates of the vertex are given by

$$\gamma = \frac{l}{2} \text{ and } \theta = \alpha + \pi.$$

Thus eliminating l and α we have the following equation as the locus :—

$$c\sqrt{c^2-4\gamma^2}\sqrt{4a^2\gamma^2-c^4}\sin^2\theta = (4a^2\gamma^2-c^4)\cos^2\theta + c^2(c^2-4\gamma^2)\sin^2\theta - 16\frac{b^2}{c^2}\gamma^4.$$

Question 1107.

(CHARLES SALDANHA) :—If x be any odd integer not divisible by 3, prove that the integral part of $4^x - (2+\sqrt{2})^x$ is a multiple of 112.

Solution by K. J. Sanjana and N. Sundaram Iyer.

Lemma. Let $a+b+c=0$, and put $bc+ca+ab=s$ and $abc=t$.

Then $\left(1-\frac{a}{x}\right)\left(1-\frac{b}{x}\right)\left(1-\frac{c}{x}\right) = 1+\frac{s}{x^2}-\frac{t}{x^3}$

$$= 1-\left(\frac{t}{x^3}-\frac{s}{x^2}\right).$$

Take logarithms and equate the co-efficients of $1/x^n$; we get

$$\frac{1}{n}(a^n+b^n+c^n) = \text{co-efficient of } \frac{1}{x^n} \text{ in } \sum\frac{1}{r}\left(\frac{t}{x^3}-\frac{s}{x^2}\right)^r.$$

Let $n = 6m+1$; the co-efficient of $1/x^n$ on the right side will occur in the terms from $\frac{1}{2m+1}\left(\frac{t}{x^3}-\frac{s}{x^2}\right)^{2m+1}$ to $\frac{1}{3m}\left(\frac{t}{x^3}-\frac{s}{x^2}\right)^{3m}$. It will be found to contain $t^{2m-1}.s^2$, $t^{2m-3}.s^5$, ... $t.s^{3m-1}$, which shows at once that it is divisible by s^2t, *i.e.*, by $abc(bc+ca+ab)^2$.

When $n = 6m - 1$, the co-efficient will contain factors ranging from $t^{2m-1}.s$ to $t.s^{3m-2}$, and is thus always divisible by st, *i.e.*, by

$$abc(bc + ca + ab).$$

Thus we deduce the theorem (which goes by the name of *Cauchy*) that when n is an odd number not divisible by 3, $a^n + b^n + c^n$ is divisible by $abc(bc + ca + ab)$ or $abc(bc + ca + ab)^2$ according as n is of the form $6m - 1$ or $6m + 1$, provided that $a + b + c$ vanishes.

As $(2 + \sqrt{2})^x + (2 - \sqrt{2})^x$ is an integer and $(2 - \sqrt{2})^x$ is a proper fraction, the integral part of $4^x - (2 + \sqrt{2})^x$ is $4^x - (2 + \sqrt{2})^x - (2 - \sqrt{2})^x$, which is seen to be $4^x + (-2 - \sqrt{2})^x + (-2 + \sqrt{2})^x$, as x is an odd integer. Now $4 + (-2 - \sqrt{2}) + (-2 + \sqrt{2}) = 0$ and x is not divisible by 3; hence by the Lemma the integral part is divisible by

$$(2 - 8 + 4\sqrt{2} - 8 - 4\sqrt{2}) \cdot 4(2 - \sqrt{2})(-2 + \sqrt{2})$$

or by $\quad (2 - 8 + 4\sqrt{2} - 8 - 4\sqrt{2})^2 \cdot 4(-2 - \sqrt{2})(-2 + \sqrt{2})$

according as x is of the form $6m - 1$ or $6m + 1$; *i.e.*, by 112 or 112×14 numerically, and hence by 112 in the case proposed.

[*Note*:—The above is a solution with the aid of Cauchy's theorem. A solution without the aid of the above theorem is wanted. The question appears to be solvable without reference to Cauchy's theorem. It was taken from a Cambridge paper and set at the last B.A. Hon. Exam. of the Bombay University.]

Question 1108.

(Communicated):—Prove that

$$\int_0^{\pi/2} \text{arc tan}\,(a\sqrt{\tan x})\,dx = \pi \,\text{arc tan}\,(a\sqrt{2} + 1) - \frac{\pi^2}{4}.$$

Hence, or otherwise, prove that

$$\int_0^{\pi/2} \text{arc tan}\left[\frac{\sqrt{2\tan x}}{1 + \tan x}\right] dx = \frac{\pi^2}{12}$$

and

$$\int_0^{\pi/2} \text{arc tan}\left[\frac{2\sqrt{\tan x}}{1 + \tan x}\right] dx = \pi \,\text{arc sin}\,\tfrac{1}{3}.$$

Solution by Martyn M. Thomas and several others.

Let $\quad I = \int_0^{\pi/2} \arctan\,(a\,\sqrt{\tan x})\,dx.$

Then $\quad \dfrac{dI}{da} = \int_0^{\pi/2} \dfrac{\sqrt{\tan x}}{1 + a^2 \tan x}\,dx$

$$= \int_0^{\infty} \frac{2t^2\,dt}{(1 + a^2t^2)(1 + t^4)}, \text{ where } t^2 = \tan x.$$

$$= \int_0^{\infty} \left[\frac{2}{a^4 + 1}\left(\frac{-a^3}{1 + a^2t^2} + \frac{t^3 + a^2}{1 + t^4}\right)\right] dt,$$

resolving into partial fractions

$$= -\frac{2a}{a^4 + 1}\int_0^{\infty} \frac{d(at)}{1 + a^2t^2} + \frac{1}{a^4 + 1}\int_0^{\infty} \frac{2(t^3 + 1)}{t^4 + 1}\,dt$$

$$+ \frac{2(a^2 - 1)}{a^4 + 1}\int_0^{\infty} \frac{dt}{t^4 + 1}$$

$$= -\frac{2a}{a^4 + 1}\left[\tan^{-1}(at)\right]_0^{\infty} + \frac{1}{a^4 + 1}\left[\sqrt{2}\tan^{-1}(t\sqrt{2} + 1)\right.$$

$$\left. + \sqrt{2}\tan^{-1}(t\sqrt{2} - 1)\right]$$

$$+ \frac{2(a^2 - 1)}{a^4 + 1}\left[\frac{1}{4\sqrt{2}}\log\left(\frac{t^2 + t\sqrt{2} + 1}{t^2 - t\sqrt{2} + 1}\right)\right.$$

$$\left. + \frac{1}{2\sqrt{2}}\tan^{-1}(t\sqrt{2} + 1) + \frac{1}{2\sqrt{2}}\tan^{-1}(t\sqrt{2} - 1)\right]$$

$$= -\frac{2a}{a^4 + 1}\left[\frac{\pi}{2}\right] + \frac{1}{a^4 + 1}\left[\sqrt{2}\cdot\frac{\pi}{2} + \sqrt{2}\cdot\frac{\pi}{2}\right]$$

$$+ \frac{2(a^2 - 1)}{a^4 + 1}\left[\frac{1}{2\sqrt{2}}\cdot\frac{\pi}{2} + \frac{1}{2\sqrt{2}}\cdot\frac{\pi}{2}\right]$$

$$= \frac{\pi}{a^4 + 1}\left\{-a + \sqrt{2} + \frac{a^2 - 1}{\sqrt{2}}\right\} = \frac{\pi}{\sqrt{2}}\cdot\frac{a^2 - \sqrt{2}a + 1}{a^4 + 1}$$

$$= \frac{\pi}{\sqrt{2}}\cdot\frac{1}{a^2 + \sqrt{2}a + 1}.$$

$\therefore$ Integrating with respect to a,

$$I = \frac{\pi}{\sqrt{2}} \int \frac{da}{a^2 + \sqrt{2}a + 1} = \pi \tan^{-1} (a\sqrt{2} + 1) + C.$$

When $a = 0$, $I = 0$. $\therefore 0 = \pi \cdot \frac{\pi}{4} + C$. $\therefore C = -\frac{\pi^2}{4}$.

Hence $I = \pi \text{ arc tan } (a\sqrt{2} + 1) - \frac{\pi^2}{4}$ 1)

Second Part.

Now, $I = \int_0^{\pi/2} \tan^{-1} (a\sqrt{\tan x})\, dx.$

Also $I = \int_0^{\pi/2} \tan^{-1} [a\sqrt{\tan (\pi/2 - x)}]\, dx$

$$= \int_0^{\pi/2} \tan^{-1} (a\sqrt{\cot x})\, dx.$$

$$\therefore 2I = \int_0^{\pi/2} \left[\tan^{-1} (a\sqrt{\tan x}) + \tan^{-1} (a\sqrt{\cot x})\right] dx$$

$$= \int_0^{\pi/2} \tan^{-1} \left[\frac{a(\sqrt{\tan x} + \sqrt{\cot x})}{1 - a^2}\right] dx$$

$$= \int_0^{\pi/2} \tan^{-1} \left[\frac{a}{1 - a^2} \cdot \frac{\sin x + \cos x}{\sqrt{\cos x \sin x}}\right] dx$$

$$= \int_0^{\pi/2} \tan^{-1} \left[\frac{a}{1 - a^2} \cdot \frac{1 + \tan x}{\sqrt{\tan x}}\right] dx$$

$$= \int_0^{\pi/2} \left[\frac{\pi}{2} - \cot^{-1} \left(\frac{a}{1 - a^2} \cdot \frac{1 + \tan x}{\sqrt{\tan x}}\right)\right] dx$$

$$= \int_0^{\pi/2} \left[\frac{\pi}{2} - \tan^{-1} \left(\frac{1 - a^2}{a} \cdot \frac{1 + \tan x}{\sqrt{\tan x}}\right)\right] dx.$$

$$\therefore \int_0^{\pi/2} \tan^{-1} \left(\frac{1 - a^2}{a} \cdot \frac{1 + \tan x}{\sqrt{\tan x}}\right) dx = \frac{\pi}{2} \cdot \frac{\pi}{2} - 2I$$

$$= \frac{3\pi^2}{4} - 2\pi \tan^{-1}(a\sqrt{2} + 1), \text{ from (1)} \quad \ldots \quad \ldots \quad (2)$$

Putting $\frac{1 - a^2}{a} = \sqrt{2}$ in the relation (2) obtained above, that is, $a = \frac{\sqrt{3} - 1}{\sqrt{2}}$,

$$\int_0^{\pi/2} \tan^{-1}\left(\frac{\sqrt{2\tan x}}{1 + \tan x}\right) dx = \frac{3\pi^2}{4} - 2\pi \tan^{-1}(\sqrt{3})$$

$$= \frac{3\pi^2}{4} - 2\pi\left(\frac{\pi}{3}\right) = \frac{\pi^2}{12}.$$

Again, putting $\frac{1 - a^2}{a} = 2$ in the relation (2), that is, $a = \sqrt{2} - 1$, we have

$$\int_0^{\pi/2} \tan^{-1}\left(\frac{2\sqrt{\tan x}}{1 + \tan x}\right) dx = \frac{3\pi^2}{4} - 2\pi \tan^{-1}(3 - \sqrt{2})$$

$$= \frac{3\pi^2}{4} - \pi \tan^{-1}\left(\frac{6 - 2\sqrt{2}}{6\sqrt{2} - 10}\right)$$

$$= \pi\left[\tan^{-1}(-1) - \tan^{-1}\left(\frac{3 - \sqrt{2}}{3\sqrt{2} - 5}\right)\right]$$

$$= \pi \tan^{-1}\left[\frac{2 - 2\sqrt{2}}{4\sqrt{2} - 8}\right]$$

$$= \pi \tan^{-1}\left(\frac{1}{2\sqrt{2}}\right) = \pi \sin^{-1}\left(\frac{1}{3}\right).$$

QUESTIONS FOR SOLUTIONS.

1151. (Selected) :—In any closed oval curve without singularities a normal chord PQ will have its maximum or minimum values, either when Q is the centre of curvature at P, or when PQ is normal at Q as well as at P, which must always be the case for two positions at least of PQ.

1152. (Selected) :—Find the complete primitive of the differential equation

$$9xy^2 y_2 - 2 = 0.$$ (Forsyth : *Diff. Eqns.*)

1153. (R. Vaidyanathaswami) :—Find three collinear points on a cusped cubic, the tangents at which are concurrent. Prove that the line containing the three points passes through a fixed point and the point of concurrence of the tangents traces a fixed line.

1154. (R. Vaidyanathaswami) :—Find all the conics which reciprocate a cusped cubic into itself.

1155. (R. Vaidyanathaswami) :—A B C E are four points on a conic S. The polar line of ABC *w. r. t.* S, interests S in P Q and the polar line of E *w. r. t.* the triangle ABC intersects S in X, Y. Prove that the conic which is inscribed to ABC and XYP passes through E.

1156. (R. Vaidyanathaswami) :—Shew that there are four conics inscribed to a Δ ABC which pass through a given point E, and touch a given conic S passing through A B C E. Shew also that the four points of contact are such that the polar line of the Δ formed by any three *w. r. t.* S passes through the fourth.

1157. (K. J. Sanjana and K. B. Madhava) :—In all integral right-angled triangles in which the sides are prime to each other, the sums and the differences of the sides are (1) either primes every one of which has 2 for a quadratic residue or (2) the products of such primes and their powers.

1158. (N. Dorai Rajan) :—If the sides of a triangle are in arithmetical progression prove that, with the usual notation, the straight line IG is parallel to one of the sides.

1159. (N. Dorai Rajan) :—Is the following theorem true? : "The three cusped hypocycloid is the only quartic curve, (excluding a pair of concentric circles) in which all tangent chords are equal."

1160. (N. Dorai Rajan) :—A Number N consists of p digits. It has the property that, when squared, it ends with the same digits in the same order as in itself. Find the number when (1) $p = 4$ and (2) $p = 5$.

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SOME APPLICATIONS OF HEAWOOD'S THEOREM*

This note deals with some applications of Heawood's Theorem, *viz.* f $f(z) = 0$ is a cubic representing the vertices of a triangle, then $z) = 0$ gives the foci of the maximum inscribed ellipse." The theorem s been discussed in this Journal by Prof. Naraniengar in Vol. IV, p. 96, d Vol. V, p. 14.

Notation. α, β, γ are the vectors to the angular points of a Δ ABC; hs vector to any point P is $(x + iy)$; and l, m, n are real qnantities.

The equation $\frac{l}{z-\alpha} + \frac{m}{z-\beta} + \frac{n}{z-\gamma} = 0$ has roots z_1, z_2 which e geometrically related to the Δ ABC and are denoted by the points and H.

P is the point ζ, given by $\zeta = \frac{l\alpha + m\beta + n\gamma}{l + m + n}$.

1. The points S and H are isogonal conjugates, (Vide: Prof. Naraengar's Note in Vol. V, p. 14 of the Journal),

The equation on reduction is

$$z^2(l + m + n) - z\{(l + m + n)\Sigma\alpha - \Sigma l\alpha\} + \Sigma l \cdot \beta\gamma = 0.$$

Taking the origin at A, $z_1 z_2 \cdot \Sigma l = l\beta\gamma$.

Hence AS, AH, have the same bisectors as AB, AC. Similarly for S, BH. The product of the moduli are equal. Hence S, H are ogonal.

* *Quarterly Journal of Mathematics*, Vol. 33, p. 84.

$$|z_1| \cdot |z_2| \cdot \Sigma l = l, |\beta| \cdot |\gamma|.$$

$$\therefore \quad AS \cdot AH = \frac{l}{l+m+n} \cdot AB \cdot AC.$$

$$\therefore \quad \frac{AS \cdot AH}{AB \cdot AC} + \ldots + . = 1 \text{ [(Q. 254. (2)]};$$

$$\text{also} \quad \frac{AS \cdot BS \cdot CS \cdot AH \cdot BA \cdot CH}{a^2 - b^2 - c^2} = \frac{l - m - n}{(\Sigma l)^3} = \frac{1}{27}.$$

When $l = m = n$, S, H are the foci of the maximum inscribed ellips and we have Question 511.

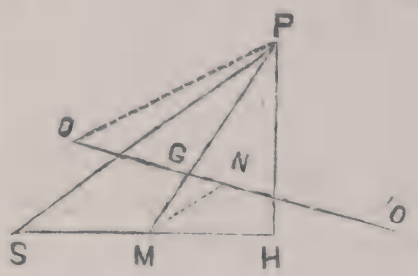
2. Δs ABC, PSH, have the same centroid, for

$$z_1 + z_2 = \frac{\Sigma l \cdot \Sigma\alpha - \Sigma l\alpha}{\Sigma l} = \Sigma\alpha - \zeta.$$

$$\therefore \quad z_1 + z_2 + \zeta = \alpha + \beta + \gamma.$$

3. If M be the mid-pt. of SH, and N the centre of the nine-pt circle, O the circumcentre, $MN = \frac{1}{2}OP$.

This follows from the similarity of the Δs OGP, MGN.



4. $OS \cdot OH = R \cdot OP$, where R is the circumradius.

Take the origin at O, the circumcentre. Then

$$\therefore \quad |\alpha| = |\beta| = |\gamma| = R.$$

$$z_1 \cdot z_2 = \frac{\alpha\beta\gamma}{l+m+n} \cdot \left(\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma}\right).$$

Put $\alpha = R \cdot e^{i\theta_1}$; $\beta = R \cdot e^{i\theta_2}$; $\gamma = R \cdot e^{i\theta_3}$.

$$\therefore \quad z_1 \cdot z_2 = \frac{1}{R^2} \cdot \frac{\alpha\beta\gamma}{l+m+n} \cdot (l \cdot R e^{-i\theta_1} + \ldots + \ldots)$$

The modulus of $\frac{\Sigma l \cdot R e^{-i\theta_1}}{\Sigma l}$ is the same as that of $\frac{\Sigma l \cdot R e^{i\theta_1}}{\Sigma l}$, which latter is OP (the two points being reflections about x-axis).

$$\therefore \quad |z_1| \cdot |z_2| = \frac{1}{R^3} \cdot |\alpha| \cdot |\beta| \cdot |\gamma| \cdot OP.$$

$\therefore$ $OS \cdot OH = R \cdot OP,$

d $OP = 2 \cdot MN = 2d.$

here d is the distance of the centre of the inconic from the nine-point ntre.

Hence $OS \cdot OH = 2R \cdot d.$ (See Vol. III, p. 68).

Similarly we can establish the result

$$+ m + n) \frac{KS \cdot KH}{KA \cdot KB \cdot KC} = \left[\Sigma \frac{l^2}{KA^2} + \Sigma \frac{mn(KB^2 + KC^2 - BC^2)}{KB^2 \cdot KC^2} \right],$$

being any point in the plane.

5. If AG, BG, CG are produced to meet the circumcircle in A'B'C', ow that G is a focus of the maximum inscribed ellipse of the Δ A'B'C'. Question 351 of Prof. Naraniengar).

The foci of the max. inscribed ellipse of the Δ A'B'C' are given by

$$\frac{1}{z - \alpha'} + \frac{1}{z - \beta'} + \frac{1}{z - \gamma'} = 0, \ (l = m = n).$$

here α', β', γ' are the vectors to the points A', B', C'.

Choose the origin at G. If G should be a focus, $z = 0$ is a root of he above ; and $\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'} = 0$, which is obviously true since G is the entroid of ABC.

6. If BCP, BCQ are equilateral triangles on the base BC of a triangle BC, show that the bisectors of the angle PAQ are parallel to the axes of he max. inscribed ellipse. (Question 444 of Prof. Swaminarayan).

Take A as origin the points P, Q as δ, δ' and S, H the foci of the ax. inscribed ellipse.

The equation giving S, H is

$$3z^2 - 2z(\beta + \gamma) + \beta\gamma = 0. \quad \ldots \quad \ldots \quad (i)$$

If θ, β, γ form an equilateral triangle,

$$\theta^2 + \beta^2 + \gamma^2 - \theta\beta - \theta\gamma - \beta\gamma = 0.$$

(See Hardy's *Pure Mathematics*).

Hence δ, δ' are the roots of the above equation.

$\therefore \ \delta + \delta' = \beta + \gamma,$

$\delta\delta' = \beta^2 + \gamma^2 - \beta\gamma.$

The direction SH is that of the vector $(z_1 - z_2)$, z_1 z_2 being roots of (i)

$\therefore \ z_1 - z_2 = \frac{1}{3} \sqrt{\beta^2 - \beta\gamma + \gamma^2}$... from (1).

$\therefore \ \delta\delta' = 9 (z_1 - z_2)^2.$

Hence one of the bisectors of the angle PAQ, is parallel to SH.

That is, the bisectors of $\angle$ PAQ are parallel to the axes of the ellipse

Also the product of the moduli being equal,

$$AP \cdot AQ = 9 \cdot SH^2.$$

7. $\overline{AS} \cdot \overline{BS} \cdot \overline{CH} + \overline{AH} \cdot \overline{BH} \cdot \overline{CH} = 2\overline{AG} \cdot \overline{BG} \cdot \overline{CG}$.

(Question 478 of Mr. M. T. Naraniengar

S, H are the roots of

$$3z^2 - 2z(\alpha + \beta + \gamma) + \alpha\beta + \beta\gamma + \gamma\alpha = 0.$$

Take the origin at the centroid G, then $\alpha + \beta + \gamma = 0$.

$\therefore$ S, H, are the points $+ z$, $- z$ given by

$$3z^2 + \alpha\beta + \beta\gamma + \gamma\alpha = 0.$$

Now $\overline{\alpha + z} \cdot \overline{\beta + z} \cdot \overline{\gamma + z} + \overline{\alpha - z} \cdot \overline{\beta - z} \cdot \overline{\gamma - z}$

$= (\alpha\beta\gamma + z \cdot \Sigma\beta\gamma + z^2 \cdot \Sigma\alpha + z^3) + (\alpha\beta\gamma - z \cdot \Sigma\beta\gamma + z^2 \cdot \Sigma\alpha - z^3)$

$= 2 \cdot \alpha \cdot \beta \cdot \gamma$., on simplification.

Hence the result.

N. Dubai Rajan.

THE THEORY OF RATIONAL TRANSFORMATION.

By R. Vythyanathaswami.

1. The general Rational Transformation is of the form

$$x' = \mathrm{R}(x) \equiv \frac{f(x)}{\phi(x)},$$

$$f(x) = \sum_{r=0}^{r=n} a_r x^{n-r}, \quad \phi(x) = \sum_{r=0}^{r=n} b_r x^{n-r},$$

n being the *order* of the transformation. The roots of $x = \mathrm{R}(x)$, $(n+1)$ in number, are the *fixed points*. The transformation R involves $(2n + 1)$ effective constants ; that is, n constants other than the fixed points.

The pencil $[f(x) - \lambda \phi(x)]$ where λ is a parameter is called the pencil associated with R or briefly the pencil of R. Two transformations having the same pencil will be said to be *congruent*. It is obvious that any transformation congruent to R is of the form SR where S is homographic.

The Rational Transformation R effects an (1, 1) correspondence between the number x' and the members of the pencil of R, the x' corresponding to any assigned member being the transform by R of any of the roots of the member, — this transform being the value of the parameter λ corresponding to the member.

2. If $\alpha_0\ \alpha_1 \ldots \alpha_n$ are the fixed points (supposed distinct) of $\mathrm{R}(x)$, the equation $x' = \mathrm{R}(x)$ can be written in the form

$$\sum_{r=0}^{r=n} k_r \frac{x' - \alpha_r}{x - \alpha_r} = 0. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

The numbers k_r will be called the *parameters* of the transformation.

Note.—If all the parameters are finite and their sum is zero, the transformation is the identical transformation $x' = x$.

The value of $\frac{dx'}{dx}$ at a fixed point is called a *multiplier*. When the fixed points are all distinct, the multiplier at α_r is seen to be

$$m_r = 1 - \frac{\Sigma(k)}{k_r}.$$

The multipliers $m_0\ m_1 \ldots m_n$ thus satisfy the identical relation

$$\Sigma \frac{1}{1 - m_r} = 1.$$

NOTE.—When the transformation is homographic, the identical relation becomes $m_0 m_1 = 1$, which is obviously true.

When the fixed points $\alpha_0\ \alpha_1 \ldots \alpha_{p-1}$ coalesce at α, equation (1) becomes

$$k\,\frac{x'-\alpha}{x-\alpha} + (x'-x)\sum_{1}^{p-1}\frac{k_r}{(x-\alpha)^{r+1}} + \sum_{p}^{n}\frac{k_r\,(x'-\alpha_r)}{(x-\alpha_r)} = 0.$$

k may be called the principal parameter at α and $k_1, k_2 \ldots k_{p-1}$, the secondary parameters. The multiplier at α is unity and at α_p is

$$-\frac{(k + k_{p+1} + k_{p+2}\ldots)}{k_p}.$$

The multipliers are thus functions of the principal parameters only.

In this case there is no identical relation between the multipliers.

Note (1).—If R and R′ have a common fixed point α with multipliers m, m' respectively, then α is a fixed point of RR′ (and also R′R) with the multiplier mm'.

NOTE (2).—The multipliers of R are the same as those of $S^{-1}RS$ where S is homographic. For, let α be a fixed point and m the corresponding multiplier of R. Let $S(\alpha + p) = \alpha + \lambda p$ where p is infinitesimal. Then $S^{-1}(\alpha)$ is a fixed point of $S^{-1}RS$ and the corresponding multiplier is m, because

$$S^{-1}RS\left(\alpha + \frac{\rho}{\lambda}\right) = S^{-1}R(\alpha+\rho) = S^{-1}(\alpha + m\rho) = \alpha + m\frac{\rho}{\lambda}.$$

NOTE (3).—*The multiplier at an infinite fixed point.*

Let R have a fixed point at infinity so that

$$R(x) = \frac{a_0 x^n + a_1 x^{n-1} + \ldots}{b_1 x^{n-1} + b_2 x^{n-2} + \ldots}.$$

Let $S(x) = \frac{1}{x}$. Then $S^{-1}RS(x) = \frac{b_1 x + b_2 x^2 + \ldots}{a_0 + a_1 x + \ldots}$.

The multiplier of R at ∞ = the multiplier of $S^{-1}RS$ at $0 = \frac{b_1}{a_0}$.

NOTE (4).—The multiplier at a finite fixed point α of

$$R(x) \equiv \frac{f(x)}{\phi(x)} \text{ is } \{f'(\alpha) - \alpha\phi'(\alpha)\}/\phi(\alpha).$$

For the polynomial transformation $R(x) \equiv f(x)$, the multiplier at the fixed point α is $f'(\alpha)$ and at the fixed point ∞ is zero.

3. *The Derived or Polar Transformation.*

There is one type of Rational Transformation—which we term the Derived or Polar Transformation—which is completely determined by its fixed points. The derived transformation with respect to a polynomial $\theta(x)$ of the $(n+1)$th degree is

$$x' = \mathrm{R}(x) = -\frac{d\theta}{dt} \Big/ \frac{d\theta}{dx}$$

= the root of the nth polar of $\theta(y)$ with respect to x ; t being the usual unit variable. The fixed points of the transformation are given by $\theta(x) = 0$.

NOTE.—All the parameters of the derived transformation are equal ; and all the multipliers are also equal each being $-n$, where n is the order of the transformation.

For, the equation $x' \frac{d\theta}{dx} + \frac{d\theta}{dt} = 0$ reduces to

$$\Sigma \frac{x' - \alpha_r}{x - \alpha_r} = 0, \ \alpha\text{'s being the roots of } \theta.$$

In the case when θ has multiple roots, the order of the derived transformation is equal to the number of distinct roots.

For, if α is an m-ple root of θ, $(x - \alpha)^{m-1}$ is a common factor of $\frac{d\theta}{dx}$ and $\frac{d\theta}{dt}$. It will be noticed, that when this common factor is cancelled, α is only a simple fixed point of the reduced form of the transformation. Further, the parameter of the transformation at an m-ple fixed point is seen to be proportional to m and the corresponding multiplier is

$$\left(1 - \frac{n+1}{m}\right).$$

NOTE (1).—A rational transformation with distinct fixed points and parameters proportional to positive integers is a derived transformation.

NOTE (2).—The derived transformation of order 1 is the involution.

Those of orders 2 and 3 may be reduced to

$$x' = \frac{1}{x^2} \text{ and } x' = \frac{\lambda x^2 - 1}{x(x^2 - \lambda)}.$$

The Focal Rational Transformation.

When $\theta(x)$ possesses an apolar quadratic $\gamma(x)$, the derived transformation is *focal*, the roots of $\gamma(x)$ being termed the foci. The focal transformation transforms each focus into the other and transforms no point

not a focus into a focus. Taking the foci as 0 and ∞, the transformation reduces to the form $x' = \frac{k}{x^n}$.

4. *Certain Families of Rational Transformation* $(n > 2)$.

When the pencil of R is given, *three* of the fixed points of R may be chosen arbitrarily, but the rest are determinate ; so that the co-efficients of $\theta(x)$ are linear functions of *four* parameters. Conversely the pencil of a transformation with given fixed points can not be chosen arbitrarily.

Theorem : If $\theta(x) = 0$ gives the fixed points of R, then $\theta(x)$ is apolar to F_{2n-2} which is the unique form of the $(2n-2)$th degree apolar to every member of the pencil of R. We have seen that $\theta(x)$ belongs to a four-parameter system of polynomials. To determine this system, suppose $f(x)$ to be a member of the pencil of R and two of its roots α, β (say) to be fixed points of R. Then $R(\alpha) = \alpha$ and $R(\beta) = \beta$. But $R(\alpha) = R(\beta)$ since α, β are the roots of $f(x)$. Hence α, β are each carried into more than one point by R, so that $(x - \alpha)(x - \beta)$ is a common factor of the numerator and denominator of R. Hence since the numerator and denominator are members of the pencil of R, they must contain the common factor $f(x)$, so that $R(x)$ reduces to the singular form $k\frac{f(x)}{f(x)}$.

Hence whenever *two* of the roots of $\theta(x)$ are roots of $f(x)$ a member of the pencil of R, $\theta(x)$ is of the form $f(x).(x-k)$ where k may be arbitrary. In other words the four-parameter system of polynomials to which $\theta(x)$ belongs, is that four-parameter system which has the pencil of R for a singular pencil.

Now the four-parameter system of $(n+1)$-ics apolar to F_{2n-2} is also a system having the pencil of R for a singular pencil.

Hence $\theta(x)$ is apolar to F_{2n-2}.

Theorem : If $R_1(x)$ is a particular transformation whose fixed points are given by $\theta(x) = 0$ and the pencil of R_1 be the pencil determined by two n-ics which are written in the forms

$$\theta(x) \sum \frac{b_r}{x - \alpha_r} \text{ and } \theta(x) \sum \frac{b'_r}{x - \alpha_r},$$

where α's are the roots of θ, then the pencil of any other transformation whose fixed points are α's is the pencil determined by two n-ics of the forms

$$\theta(x) \sum \frac{k_r b_r}{x - \alpha_r} \text{ and } \theta(x) \sum \frac{k_r b'_r}{x - \alpha_r}.$$

Dem.—Since the pencil $\theta(x) \Sigma \frac{b_r + \lambda b'_r}{x - \alpha_r}$ is the pencil of R_n, its apolar, $(2n-2)$ -ic viz. F_{2n-2} is apolar to $\theta(x)$.

Thus F_{2n-2} is of the form $\Sigma A_r (x - \alpha_r)^{2n-2}$.

The condition that this is apolar to $\theta(x) \Sigma \frac{b_r + \lambda b'_r}{x - \alpha_r}$ for all values of λ is

$$\Sigma A_r (b_r + \lambda b'_r) (x - \alpha_r)^{n-2} \prod (\alpha_r - \alpha_s) \equiv 0.$$

This equation is unaltered if we write $k_r b_r$ for b_r, $k_r b'_r$ for b'_r and A_r/k_r for A_r.

Thus the pencil $\theta(x) \Sigma k_r \frac{b_r + \lambda b'_r}{x - \alpha_r}$ is the apolar pencil of some F_{2n-2} apolar to $\theta(x)$; that is, it is the pencil of some rational transformation whose fixed points are given by $\theta(x)$.

Cor. Considering R_1 as the derived transformation defined by $\theta(x)$. Its pencil is the pencil of first polars of $\theta(x)$ and is determined by $\theta(x) \Sigma \frac{1}{(x - \alpha_r)}$ and $\theta(x) \Sigma \frac{\alpha_r}{x - \alpha_r}$. Hence the pencil of any transformation whose fixed points are roots of $\theta(x)$ is the pencil determined by two n-ics of the form $\theta(x) \Sigma \frac{k_r}{x - \alpha_r}$ and $\theta(x) \Sigma \frac{k_r \alpha_r}{x - \alpha_r}$ where the k_r's are arbitrary.

Geometrical Interpretation.

Represent every n-ic as a point in n dimensions. The n-ics which are perfect nth powers will then correspond to points on a twisted n-ic curve Γ. An n-ic $(x - \alpha_1)(x - \alpha_2) \ldots\ldots (x - \alpha_n)$ corresponds to the point of intersection of the osculating $(n-1)$ -dimensional regions at $P_1 P_2 \ldots\ldots P_n$ where P_r is the point on Γ corresponding to $(x - \alpha_r)^n$. By Para (1), the general Rational transformation is an (1, 1) correspondence between Γ and an arbitrary straight line. The fixed points are the points (α_r) on Γ the osculating planes at which intersect the straight line in the corresponding point. If the points (α_r) are given, the possible straight lines form an n-ply infinite family such that any line of the family is the transform of a fixed line L of the family by some collineation whose fixed $(n+1)$ -hedron Δ' is the one formed by the osculating planes at (α_r). This complex of ∞^n straight lines is evidently the generalisation of the tetrahedral complex in three dimensions.

As in the corollory above the line L may be taken to be the line which represents the pencil of first polars.

Theorem: The line L which represents the pencil of first polars is ths locus of poles with respect to Δ' of osculating planes of Γ.

The equation in plane co-ordinates of the twisted n-tic with respect Δ' may be taken to be

$$L_r = \frac{\lambda_r}{t - \alpha_r} \quad (r = 0, 1 \ldots\ldots n)$$

t representing the parameter of the current point on the curve. The equation to L which is the locus of poles with respect to Δ' of osculating planes of Γ, is then

$$X_r = \frac{t - \alpha_r}{\lambda_r}.$$

If the osculating plane at t_1 on the curve passes through the point t on the line

$$\Sigma \frac{\lambda_r}{t_1 - \alpha_r} \cdot \frac{t - \alpha_r}{\lambda_r} = \Sigma \frac{t - \alpha_r}{t_1 - \alpha_r} = 0.$$

Thus the transformation whose pencil is represented by the line is a derived transformation, so that the line represents the pencil of first polars.

The line L may be called the polar line of Δ with the respect to Γ.

5. *Argand Representations.*

(1) Transformations with real multipliers (and therefore real parameters).

Let $k_0\ k_1 \ldots k_n$ be the (real) parameters corresponding to the fixed points $\alpha_0\ \alpha_1 \ldots \alpha_n$ respectively of the transformation R. With the point z as centre describe a circle in the Argand diagram and let α'_r be the inverse of α_r with respect to this circle.

The inverse with respect to the same circle of the mean centre of $\alpha'_0\ \alpha'_1 \ldots \alpha'_n$ for multiples $k_0\ k_1 \ldots k_n$ is the point R (z).

Dem. If T represent the operation of reflection in the real axis, then

$$(z - \alpha'_r) = T\left(\frac{1}{z - \alpha_r}\right).$$

If M is the mean centre

$$(z - M) = \frac{\Sigma k_r (z - \alpha'_r)}{\Sigma k_r}.$$

Therefore, if z' is the inverse of M,

$$(z - z') = \mathrm{T}\left(\frac{1}{z - \mathrm{M}}\right) = \Sigma k_r \,.\, \mathrm{T}\,.\left[\Sigma k_r \frac{1}{(z - \alpha'_r)}\right]$$

$$= \Sigma k_r \,.\, \mathrm{T}\left[\frac{1}{\Sigma k_r\, \mathrm{T}(3' - \alpha_r)}\right] = \frac{\Sigma k_r}{\Sigma \dfrac{k_r}{z - \alpha_r}},$$

so that $$\Sigma k_r \frac{z' - \alpha_r}{z - \alpha_r} = 0.$$

This construction is suitable for all derived transformations.

(2) The above construction may be extended to any transformation whatever, provided we define the mean centre for multiples $m_0\, m_1 \ldots m_r$ of points $z_0 \ldots z_r$ in the Argand plane to be the point $z = \dfrac{\Sigma m_r z_r}{\Sigma m_r}$, even in case the m's are complex.

6. *Powers of a Rational Transformation.*

The rth power of a Rational Transformation R of order n is a rational transformation of order n^r, which may be denoted by R^r. The $n^r + 1$ fixed points of R^r include obviously the fixed points of R and the fixed points of R^d where d is any divisor of R. Those fixed points of R^r which are not fixed points of R^d are the *special* fixed points of R^r. The special fixed points are $n^{(r)}$ in number, where

$$n^{(r)} = n^r - \Sigma n^{r/p_1} + \Sigma n^{r/p_1 p_2} - \text{etc.}$$

$p_1, p_2, \ldots$ being the prime factors of r. (For proof: see my 'Note in Combinatory Analysis,' J. I. M. S.)

$n^{(r)}$ is divisible by r. (Extension of Fermat's Theorem, *Ibid*).

Hence the $n^{(r)}$ special fixed points can be divided into sets of r points each set being carried into itself cyclically and therefore *primitively* by R. Such a set may be called a *primitive* set of R and the corresponding polynomial a *primitive* polynomial. Any polynomial carried into itself by R is either primitive or the product of primitive polynomials.

The number of primitive r-ics of R is $\dfrac{n^{(r)}}{r}$.

Theorem: The multipliers of R^r corresponding to the points of a primitive r-set, are equal to the same rational function of the co-efficients of the corresponding primitive r-ic.

Let $\alpha_1\ \alpha_2 \ldots \alpha_r$ be a primitive r-set so that $\alpha_2 = R(\alpha_1), \ldots \alpha_k = R(\alpha_{k-1})$ and $\alpha_1 = R(\alpha_r)$.

Then $R(\alpha_1 + d\alpha_1) = \alpha_2 + d\alpha_2, \ldots R(\alpha_k + d\alpha_k) = \alpha_{k+1} + d(\alpha_{k+1})$ $[k = 2 \ldots (r-1)]$ and $R(\alpha_r + d\alpha_r) = \alpha_1 + d\alpha'_1$.

The multiplier of R^r at α_1

$$= \frac{d\alpha'_1}{d\alpha_1} = \frac{d\alpha'_1}{d\alpha_r} \cdot \frac{d\alpha_r}{d\alpha_{r-1}} \cdots \frac{d\alpha_2}{d\alpha_1}$$

$$= \alpha_1 \cdot \left(\frac{f'(\alpha_r)}{f(\alpha_r)} - \frac{\phi'(\alpha_r)}{\phi(\alpha_r)} \right) \times \alpha_r \left(\frac{f'(\alpha_{r-1})}{f(\alpha_{r-1})} - \frac{\phi'(\alpha_{r-1})}{\phi(\alpha_{r-1})} \right) \times \ldots$$

$$\left(\text{where } R(x) \equiv \frac{f(x)}{\phi(x)} \right)$$

$$= \alpha_1\ \alpha_2 \ldots \alpha_r \prod \left(\frac{f'(\alpha_k)}{f(\alpha_k)} - \frac{\phi'(\alpha_k)}{\phi(\alpha_k)} \right)$$

$$=$$ a symmetric function of $\alpha_1\ \alpha_2 \ldots \alpha_r$,

which proves the theorem.

The multipliers of R^r can be divided into the following sets :—

(1) The multipliers at the special fixed points of R^r. There are $\frac{n^{(r)}}{r}$ such multipliers, each multiplier corresponding to a primitive r-ic in virtue of the theorem proved. Call these $M_{r_1}, M_{r_2} \ldots$

Each fixed point, not a special fixed point of R^r, must be a special fixed point of one and only one power of R, say R^d, where d is a divisor of r.

Hence the other multipliers can be classified into—

(2) The multipliers of R^r at the special fixed points of R^d. These multipliers are the $\left(\frac{r}{d} \right)^{\text{th}}$ powers of the multipliers of R^d at the same points. That is, in our previous notation $\left(M^d_k \right)^{r/d} \left(k = 1, 2, 3 \ldots \frac{n^{(d)}}{d} \right)$ each multiplier corresponding to a primitive d-set.

(3) The multipliers at the fixed points of R. These are the rth powers of the multipliers of R.

Thus the problem of finding all the multipliers of R^r reduces to that of finding the multipliers at the special fixed points.

Ex. (1) Deduce from simple considerations a proof of $\sum n^{(d)} = n^r$, where the values of d are the divisors of r (unity and r included).

Ex. (2) To find $R^2(x)$, where

$$a_1 \frac{R(x) - \alpha_1}{x - \alpha_1} + a_2 \frac{R(x) - \alpha_2}{x - \alpha_2} + a_3 \frac{R(x) - \alpha_3}{x - \alpha_3} = 0.$$

Let m_1 m_2 m_3 be the multipliers of R and $(\beta\gamma)$ the primitive pair.

The multipliers of R^2 at β, γ are equal to the same number m and at α_1 α_2 α_3 are m_1^2 m_2^2 m_3^2, m is thus determined by the equation

$$\frac{1}{1 - m_1^2} + \frac{1}{1 - m_2^2} + \frac{1}{1 - m_3^2} + \frac{2}{1 - m} = 1.$$

This determines m as a function of m_1 m_2 m_3, where

$$m_1 = \frac{-(a_2 + a_3)}{a_1}, \text{ etc.}$$

Hence $R^2(x)$ is given by

$$\sum_{r=1}^{3} \frac{1}{1 - m_r^2} \cdot \frac{(R^2(x) - \alpha_r)}{x - \alpha_r} + \frac{1}{1 - m} \left[\frac{R^2(x) - \beta}{x - \beta} + \frac{R^2(x) - \gamma}{x - \gamma} \right] = 0.$$

7. *Powers of the Derived Transformation.*

If $\theta(x) = 0$, represent the fixed points of a derived transformation, then any primitive r-ic is a covariant of $\theta(x)$ but not a rational covariant. However *the product of all primitive r-ics is a rational covariant of* $\theta(x)$ (which we may call the rth primitive covariant). To prove this assume this to be true for all values of r up to a given one r. The fixed points of the rth power of the derived transformation is a rational covariant of $\theta(x)$, say L_x. If d is a divisor of r, the dth primitive covariant (which by hypothesis is rational) is a factor of L_x. On removing all such factors, the part which remains in L_x is a rational covariant and is the rth primitive covariant. Thus the rth primitive covariant is a rational covariant—shewing that the theorem is true for r. Since obviously the theorem is true for $r = 2$, the induction is complete.

Ex. (1) The primitive quadratic of a derived quadratic transformation is the Hessian of $\theta(x)$.

For since a cubic is apolar to its Hessian, either root of the Hessian must be the second polar of the other.

Ex. (2) The primitive quadratics of the derived cubic transformation are the apolar quadratic factors of the sextic covariant of $\theta(x)$.

Let $\alpha\beta\gamma\delta$ be the roots of $\theta(x)$ and (pq) the pair harmonic to $(\alpha\beta)$ such that $(\alpha\beta pq)$ is apolar to $(\alpha\beta\gamma\delta)$, (pq) is thus a covariant of $(\alpha\beta)$ and

$(\gamma\delta)$ and must therefore be the common harmonic pair of $(\alpha\beta)$ and $(\gamma\delta)$. Thus $(pq\alpha\beta)$ and $(pq\gamma\delta)$ are both apolar to $(\alpha\beta\gamma\delta)$ and therefere $(pppq)$ and $(pqqq)$ are both apolar to $(\alpha\beta\gamma\delta)$. Thus either of p, q is the third polar of the other, shewing that (pq) is a primitive pair.

The following are examples of this theorem.

Ex. (3) To find pairs of points on the rational space-quartic, such that the osculating plane at each passes through the other.

Let $f(t) = 0$, represent parametrically the super-osculation points. Then if the osculating plane at the point t cuts the curve again in the point t', t' is the derived cubic transformation of t determined by $f(t)$. Thus the pairs sought are given by the quadratic factors of the sextic covariant of $f(t)$.

Ex, (4) To find pairs of points on an ellipse the osculating circle at each of which passes through the other.

Let $\theta(t)$ represent parametrically the axial extremities of the ellipse. If $f(t) = 0$ represent four concyclic points on the curve, then obviously f is apolar to θ. Thus if the osculating circle at t cuts the ellipse in t', t' is the derived transformation of t determined by $\theta(t)$. The pairs sought are therefore the quadratic factors of the sextic covariant of $\theta(t)$, *i.e.*, the points at ∞ on the ellipse and the extremities of the equiconjugate diameter.

8. . *Reducible Pencils.*

A pencil Γ_n of n-ics is said to be *reducibls* over a pencil Γ_p of p-ics (where p is a factor of n other than 1 or n) if every member of Γ_n is the product of members of Γ_p.

The Jacobian of Γ_n is then the product of the Jacobian of Γ_p and of $\left(\frac{2n}{p} - 2\right)$ members of Γ_p. Conversley it is easy to see that Γ_n is reducible over Γ_p whenever the Jacobian of Γ_n is such a product.

From the definition of reducibility it follows that Γ_n is certainly irreducible if n ia a prime.

Ex. (1) A pencil of quartics is reducible if it contains two perfect squares.

For if Γ_4 is reducible over Γ_2, the Jacobian of Γ_4 is the product of the Jacobian of Γ_2 and ot two members P_1, P_2 of Γ_2. Obviously P_1^{22}, P_2 are members of Γ_4.

Ex. (2) A pencil Γ_{pq} is reducible if it contains two perfect qth powers.

Let P_1^q, P_2^q be members of Γ_{pq}, P_1, P_2 being of order p. $P_1^q - \lambda P_2^q$ is resoluble into q factors of the form $P_1 - \mu P_2$ and therefore Γ_{pq} is reducible over the pencil (P_1, P_2).

Ex. (3) An example of a pencil of n-ics which is reducible over a pencil of order p *for every divisor* p *of* n, is the pencil of vertices of n-gons inscribed in one conic, circumscribed to another.

Ex. (4) A pencil of quartics if reducible over more than one pencil of quadratics, is reducible over *three* such pencils and is a standard pencil.

For if Γ_4 is reducible over two distinct pencils, it must contain three perfect squares P^2, Q^2, R^2 (since it cannot contain four). Thus Γ_4 is reducible over each of the three pencils (P, Q), (Q, R), (R, P). Obviously Γ_4 is a standard pencil.

9. *Reducible Transformations.*

A rational transformation (or function) $R(x)$ of order n is said to be reducible if $R(x) \equiv R_p\{R_q(x)\}$ where R_p, R_q are rational functions of orders p, q respectively ($p, q \neq 1$). If R is reducible, any transformation congruent to R is also reducible; for if $R = R_p R_q$ then $SR = SR_p . R_q$ where S is a homography. Thus the reducibility or irreducibility of R is determined purely by the pencil of R.

Theorem: The necessary and sufficient condition for the reducibility of R is the reducibility of the pencil of R.

Firstly, let $R = \frac{f}{\phi}$ and the pencil $f + \lambda\phi$ be reducible over a pencil of order p ($n = pq$). Then

$$f \equiv a_0 P_1^q + a_1 P_1^{q-1} P_2 + \ldots\ldots + a_q P_2^q$$

$$\phi \equiv b_0 P_1^q + b_1 P_1^{q-1} P_2 + \ldots\ldots + b_q P_2^q$$

where P_1, P_2 are p-ics.

Hence $R \equiv R_1 R_2$,

where $R_2(x) = \frac{P_1(x)}{P_2(x)}$ $R_1(x) = \frac{a_0 x^q + a_1 x^{q-1} \ldots + a_q}{b_0 x^q + b_1 x^{q-1} \ldots + b_q}$

Secondly, let $R \equiv R_1 R_2$, $R = \frac{f}{\phi}$, $R_1 = \frac{f_1}{\phi_1}$, $R_2 = \frac{f_2}{\phi_2}$.

Then the equation $f + \lambda\phi = 0$ is equivalent to $f_1(R_2) + \lambda\phi_1(R_2) = 0$ and therefore the pencil of R is reducible over the pencil of R_2.

Note.—A rational function of prime order is certainly irreducible. When a rational function R is reducible, the reduction can be performed in a triply infinite number of ways: For if $R = R_1 R_2$, then R is also equal to $R_1 S \,.\, S^{-1} R_2$ where S is homographic.

If we call R_2 the pre-factor and R_1 the post-factor of R, then all the possible pre-factors are congruent to one another and the pencils of the post-factors are linearly transformable into one another.

Ex. (1) A reducible quartic transformation can be expressed in infinity ways as the product of two congruent transformations.

For S can be chosen in infinity ways so as to make the pencil of $R_1 S$ identical with that of R_2.

Hence there is an infinity of quadratic transformations R_1 such that R_1^2 is congruent to a given reducible quartic transformation.

[This is an attempt to study the general Rational Transformation of one variable in some of its most characteristic aspects. More questions have been raised than solved; and the positive results obtained are few. In spite of the fragmentary character of the article, it is believed that it will be of some value to readers interested in the subject.]

THE GROUP-THEORY ELEMENT OF THE HISTORY OF MATHEMATICS.

By Professor G. A. Miller.

(*Continued from p.* 12.)

FOR about a century mathematicians studied these special groups with only occasional glimpses into their deeper meanings and wider applications. E. Galois, A. L. Cauchy, A. Cayley, and W. R. Hamilton made references to these deeper meanings, especially as regards an abstract theory, but none of these men formulated the abstract laws governing this theory. About 1870, an eminent triumvirate of mathematicians, C. Jordan, S. Lie and F. Klein, began to exhibit the applications of the group concept to new fields. In his "Traité des Substitutions" (1870) and in an article on the groups of movements (1868), C. Jordan made fundamental geometric applications, which were greatly extended by F. Klein. About the same time S. Lie founded a new theory of continuous groups of transformations and made extensive applications of these groups in the theory of differential equations and in other mathematical subjects.

It may be of interest to note that during the first, or implicit period, of the development of our subject, groups involving an infinite number of elements exercised the greatest influence. During the second, or specialization period, the attention was centered on groups of a finite number of elements, while during the third, or generalization period, groups involving an infinite number of elements again moved to the foreground, but groups of finite order continued to receive considerable attention. Two types of groups of infinite order were studied during this period, *viz.*, those in which the transformations were continuous and those in which the transformations were discontinuous.

The fundamental abstract notions involved in group theory are so elementary that they can be easily understood by those who are not professional mathematicians. Hence it is the more interesting that these notions were not explicitly formulated before 1870. In formulating these for the special case where the elements obey the commutative law when they are combined, L. Kronecker expressed himself as follows: "The extremely simple principles upon which the method of Gauss is founded, find applications not only in the place named but also in others, and, indeed, already in the elementary parts of the theory of numbers. This circumstance points to the fact, about which it is easy to convince oneself, that the said principles belong to a more general and more abstract sphere of

ideas. Hence it appears appropriate to free their development from all non-essential limitations so that one will be spared the trouble of repeating the same method of reaching a conclusion in the different instances of its use. The advantage of this appears even in the development itself, and the presentation gains at the same time in simplicity, and, by the clear exhibition of the essentials only, also in distinctness when it is given in the most general permissible way."[1]

The student of the history of science may be especially interested in the fact that the formulation of a definition of an abstract group came so late in the development of this subject. For a full century mathematicians were dealing with special substitution groups before making a serious effort to develop an abstract theory embodying the fundamental principles of these groups as a special case. It was not until such an abstract theory was being developed that mathematicians began to see that the group concept had been a dominant factor in some of the most important early mathematical work and hence it became an important means not only for suggesting further advances but also for securing an insight into the large body of earlier mathematical developments.

A few statements found in well-known text-books may serve to illustrate the attitude of leading mathematicians at the beginning of the present century as regards the theory of groups. In the preface of his "Géometrie," 1905, E. Borel says :

The new foundation (of elementary geometry) has been laid in the nineteenth century by the works of leading mathematicians. It consists of the recognition that elementary geometry is equivalent to the investigation of the group of movements. Such a view is in accord with the characteristic tendency of modern scientists to replace static investigations of the phenomena by dynamic ; or, to speak in more general terms, the thought of development penetrates more and more our observations.

In his "Lehrbuch der Algebra" (kleine Ausgabe), 1912, page 180, H. Weber notes that :

There are chiefly two large general concepts which dominate modern algebra. The existence and importance of these concepts could be observed only after algebra was completed to a certain extent, and had become the property of the mathematicians. Only then could be observed the combining and guiding principles. These are the concepts of groups and of domains (koerper) which we now proceed to explain. The more general of these is the concept of group.

In his "Berührungstransformationen" 1914, page 11, H. Liebmann makes the following statement :

The rules and concept development of group theory may be compared with the organizing laws of nature according to which crystals arise. If it is allowed to

[1] L. Kronecker, Berlin *Monatsberichte*, 1881, p. 882.

ntinue the figure of speech it may be added that the remaining mother liquor is a h fostering soil on which luxuriant organized life unfolds itself.

These quotations may suffice to indicate in a general way to what extent oup-theory influenced the trend of mathematical progress since the ginning of the third period of its development. The infinite number of ite groups, each of which exhibits special laws of operations, which had en discovered during the second period of the development of this sub- ct, showed that this theory can never be completely mastered in its tails. There are, however, large categories of groups which have many operties in common and whose common operational laws throw light on her mathematical developments.

Comparatively little progress has been made in the study of those stract properties which all groups have in common, yet it is just ese common properties which were popularized by the mathematical erature of the last quarter of the nineteenth century. While they are simple that the ancients did not consider it necessary to mention them plicitly it was found that they furnish a point of view which offers any advantages. For instance, few mathematical terms are more use- l than the term equivalent, and one of the services which group- eory has rendered is to give this term a flexible yet perfectly definite eaning by noting that the equivalence of two objects implies that ne can be transformed into the other by the operations of a certain oup.

Hence the term equivalent is relative to the group under consi- eration. For instance, in Euclidean geometry two figures are equiv- ent if they can be made to coincide by operations of the group com- osed of displacements and symmetries. The distance between any two oints is an absolute invariant under this group. On the other hand, elementary geometry two figures are equivalent when they can be ansformed into each other by the operators of the group composed f the similarity transformations which includes the preceding group s an invariant sub-group. In elementary geometry all circles are quivalent, and all squares are equivalent, but this is not true in uclidean geometry.

Euclid's "Elements" could have been enriched not only by the xplicit use of groups of infinite order but also by the introduction f groups of finite order. In particular, the five regular solids which lay an important rôle in Greek mathematics and in Greek philosophy epresent three interesting groups of finite order. In the words of E. Picard:

A regular polyhedron, say an icosahedron, is on the one hand the solid that all the world knows; it is also, for the analyst, a group of finite order, corresponding to the divers ways of making the polyhedron coincide with itself. The investigation of all the types of groups of motion of finite order interests not only the geometers, but also the crystallographers; it goes back essentially to the study of groups of ternary linear substitution of determinant unity, and leads to the thirty-two systems of symmetry of the crystallography for the particular complex.

While it seems impossible to establish the reasons why Euclid did not make explicit use of groups of finite and of infinite order in his "Elements," the fact that Aristotle frequently expressed the view that mathematics has to do with the *immovable* objects except such as relate to astronomy, is suggestive. While movements were used to illustrate the demonstrations of theorems the Greek philosophers seemed to hold the view that geometry itself was essentially a static subject. It is difficult to overestimate the great influence which this view had on the later history of mathematics.

If Euclid had emphasized in his "Elements" the dynamic rather than the static elements of mathematics, it is likely that his work would have exerted a more vigorous influence. The cube of Euclid, for instance, is of great interest but it is not so inspiring as the cube composed of the twenty-four movements of space which leave Euclid's cube invariant. These movements affect all space and convey big and far-reaching notions. Moreover, they suggest many questions as regards sub-groups and abstract laws of operation. In particular, this group of order 24 is completely defined by the fact that it contains two operators of orders 2 and 3 respectively whose product is of order 4.

While a group-theory of the third century B. C. is conceivable, it could not have been the group-theory of the nineteenth century since the latter century had a much richer mathematical heritage. The rapid strides of group-theory during the last century were largely due to the utilization of old results, as is always the case in generalizations by abstraction. The soil had been prepared by the labors of earlier centuries and it was only necessary to sow on it the new seed to secure the bountiful harvest with which the labors of many workers in this field were rewarded, especially during the last decades of the nineteenth century.

When group-theory appeared explicitly, it naturally took a form which was in accord with the spirit of the times. Substitution groups constitute a type of combinatory analysis and arose about the time when the Combinatorial School flourished in Germany under the leadership of C. F. Hindenburg (1741-1808). Abstract group-theory is a type of postulational

mathematics and its early development during the middle of the preceding century was in the van of the postulational activity which was so prominent during the second half of the nineteenth century. Continuous and geometric group-theory are mainly applied group-theory and their rapid development during the last quarter of the preceding century is in accord with the spirit of this age when the fear of mathematical isolation through over-specialization tended to make the study of applications especially popular.

SOLUTIONS.

Question 1102.

(M. Bhimasena Rao):—P is the inverse of the incentre of a triangle ABC with respect to the circum-circle of ABC. Show that the isogonal conjugate of P with respect to ABC lies on the common diameter of the in-circle and the nine-point circle of ABC.

Solution by N. Sundaram Aiyar and C. N. Sreenivasa Iyengar.

Let O be the circumcentre, I the incentre and N the nine-point centre of $\triangle$ ABC. Then $OI \cdot OP = R^2$ (R being the circum-radius). The trilinear co-ordinates of O and I are (R cos A, R cos B, R cos C) and (r, r, r) respectively. Let α, β, γ be the co-ordinates of P.

Then obviously

$$\frac{r - R\cos A}{OI} = \frac{\alpha - R\cos A}{OP} = \frac{\sqrt{(r - R\cos A)(\alpha - R\cos A)}}{R}.$$

$$\therefore \quad \frac{\alpha - R\cos A}{r - R\cos A} = \frac{R^2}{OI^2} = \frac{R}{R - 2r}.$$

$$\therefore \quad \alpha = \frac{R^2\cos A - 2R\gamma\cos A + R\gamma - R^2\cos A}{R - 2r}$$

$$= \frac{R\gamma}{R - 2\gamma}\,.\,(1 - 2\cos A).$$

The co-ordinates of P are thus proportional to $(1-2\cos A)$, $(1-2\cos B)$ and $(1-2\cos C)$.

The co-ordinates of its isogonal conjugate are proportional to

$$\frac{1}{1 - 2\cos A}, \quad \frac{1}{1 - 2\cos B}, \quad \frac{1}{1 - 2\cos C}.$$

To prove that this point is on a line with I and N whose co-ordinates are proportional to (1, 1, 1) and $[\cos(B - C), \cos(C - A), \cos(A - B)]$, respectively, we are to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ \cos(B - C) & \cos(C - A) & \cos(A - B) \\ \dfrac{1}{1 - 2\cos A} & \dfrac{1}{1 - 2\cos B} & \dfrac{1}{1 - 2\cos C} \end{vmatrix} = 0.$$

Now $\Sigma(1 - 2\cos A)\,[\cos \overline{C - A}\,(1 - 2\cos C)$

$= \Sigma(1 - 2\cos A)\,[\cos \overline{C - A} - \cos \overline{A - B} - (\cos \overline{2C - A}\,(\cos \overline{A - 2B})]$

$= \Sigma(\cos \overline{C - A} - \cos \overline{A - B}) - \Sigma(\cos C - \cos B) - \Sigma[\cos \overline{C - 2A}$

$- \cos (2A - B) - \Sigma(\cos \overline{2C - A} - \cos \overline{A - 2B}$

$+ \Sigma(\cos 2C - \cos 2B) + \Sigma(\cos \overline{2C - 2A} - \cos \overline{2A - 2B}) = 0.$

Hence the result.

Similar solution by M. K. Kewalramani.

Question 1104.

(M. Bhimasena Rao):—A circle cuts the sides of the triangle of reference at angles α, β, γ. Show that it cuts the nine-point circle at the angle θ given by the equation

$$\cos\theta(a\cos\alpha + b\cos\beta + c\cos\gamma) = \Sigma a\cos A\sin^2\alpha + \Sigma a\cos\beta\cos\gamma + \ldots .$$

If $\vartheta = \alpha + \beta + \gamma$, show that either

(i) $\sin A\sin\alpha + \sin B\sin\beta + \sin C\sin\gamma = 0$,

or (ii) $\cos A\sin\alpha + \cos B\sin\beta + \cos C\sin\gamma + \sin(\alpha+\beta+\gamma) = 0$.

Interpret these results geometrically.

Solution by N. Sundaram Aiyar.

The trilinear co-ordinates of the centre of the circle of radius r, which cuts the sides of the Δ of reference at angles, α, β, γ are

$$r\cos\alpha,\ r\cos\beta,\ r\cos\gamma,$$

where $$r(a\cos\alpha + b\cos\beta + c\cos\gamma) = 2\Delta.$$

If d be the distance of this centre from the nine-point centre, we have

$$d^2 = r^2 + \frac{R^2}{4} - 2r\cdot\frac{R}{2}\cos\theta. \qquad \ldots \qquad \ldots \quad (1)$$

But the nine-point centre is the point

$$\frac{R}{2}\cos(B - C),\ \frac{R}{2}\cos(C - A),\ \frac{R}{2}\cos(A - B).$$

$$\therefore\ d^2 = \frac{abc}{4\Delta^2}\,\Sigma a\cos A\left(r\cos\alpha - \frac{R}{2}\cos\overline{B - C}\right)^2$$

$$= \frac{-abc}{4\Delta^2}\cdot\Sigma a\left(r\cos\beta - \frac{R}{2}\cos\overline{C - A}\right)\left(r\cos\gamma - \frac{R}{2}\cos\overline{A - B}\right)$$

$$= \tfrac{1}{2}\cdot\frac{abc}{4\Delta^2}\left[\Sigma a\cos A\cdot r^2(1 - \sin^2\alpha) - rR\Sigma a\cos\alpha\cos A\cos\overline{B - C}\right.$$

$$+ \frac{R^2}{4} \Sigma a \cos A \cos^2 \overline{B - C} - r^2 \Sigma a \cos \beta \cos \gamma$$

$$- \frac{R^2}{4} \Sigma a \cos \overline{C - A} \cos \overline{A - B} + \frac{rR}{4} \Sigma (a \cos \beta \cos \overline{A - B} + a \cos \gamma \cos \overline{C - A})]$$

$$= \frac{R}{2\Delta} [r^2 . \frac{2\Delta}{R} - r^2 (\Sigma a \cos A \sin^2 \alpha + \Sigma a \cos \beta \cos \gamma)$$

$$+ \frac{rR}{2} \Sigma \cos \alpha (c \cos \overline{C - A} + b \cos \overline{A - B} - 2a \cos A \cos \overline{B - C})$$

$$+ \frac{R^2}{4} \Sigma (a \cos A \cos^2 \overline{B - C} - a \cos \overline{C - A} \cos \overline{A - B})].$$

But $c \cos \overline{C - A} + b \cos \overline{A - B} - 2a \cos A \cos \overline{B - C}$

$= R (\sin \overline{2C - A} + \sin A + \sin A + \sin \overline{2B - A} - \sin \overline{2A + B - C}$

$- \sin (2A - B + C) = 2R \sin A = a.$

Also $\Sigma \{ a \cos A \cos^2 \overline{B - C} - a \cos \overline{C - A} \cos (A - B) \}$

$$= \Sigma \frac{a \cos A}{2} + \frac{\Sigma a \cos A \cos \overline{2B - 2C}}{2} - \Sigma R \sin A (\cos \overline{C - B} + \cos \overline{C + B - 2A})$$

$$= \frac{\Delta}{R} + \frac{R}{4} \Sigma (\sin \overline{2A + 2B - 2C} + \sin \overline{2A - 2B + 2C})$$

$$- \frac{R}{2} \Sigma (\sin \overline{A + C - B} + \sin \overline{A - C + B} + \sin \overline{C + B - A} + \sin \overline{3A - B - C})$$

$$= \frac{\Delta}{R} + 2R \sin 2A \sin 2B \sin 2C - 2R \sin A \sin B \sin C - 2R \cdot \pi \sin 2A$$

$= 0.$

$$\therefore \quad d^2 = r^2 - \frac{Rr^2}{2\Delta} (\Sigma a \cos A \sin^2 \alpha + \Sigma a \cos \beta \cos \gamma)$$

$$+ \frac{rR^2}{4\Delta} \Sigma a \cos \alpha = r^2 - rR \cos \theta + \frac{R^2}{4}.$$

$$\therefore \quad \cos \theta = \frac{r}{2\Delta} (\Sigma a \cos A \sin^2 \alpha + \Sigma a \cos \beta \cos \gamma)$$

or $$\cos \theta \Sigma (a \cos \alpha) = \Sigma a \cos A \sin^2 \alpha + \Sigma a \cos \beta \cos \gamma. \qquad (2)$$

If $\theta = \alpha + \beta + \gamma$, this result becomes

$$(\Sigma a \sin \alpha) (\Sigma \cos A \sin \alpha) + \Sigma a \sin^2 \alpha \cos (\beta + \gamma)$$

$$+ \Sigma 2 \sin \alpha \cos \alpha \sin (\beta + \gamma) = 0,$$

or $$(\Sigma a \sin \alpha) (\Sigma \cos A \sin \alpha + \sin \overline{\alpha + \beta + \gamma}) = 0.$$

Hence either

(i) $\sin A \sin \alpha + \sin B \sin \beta + \sin C \sin \gamma = 0,$

or (ii) $\cos A \sin \alpha + \cos B \sin \beta + \cos C \sin \gamma + \sin (\alpha + \beta + \gamma) = 0.$

Question 1109.

(SALDHANA AND MADHAVA).—Prove that

$$e^{\lambda_1}\left[f(x)\right] = \left(1 + \frac{1}{x}\right) f(x+1) \qquad \text{(Bombay, B.A.)}$$

where $\lambda_1 = \left(\frac{d}{dx} + \frac{1}{x}\right)$;

and extend the result as follows :—

$$e^{\lambda_2}\left[\tfrac{1}{2} f(x)\right] = \left(\frac{1}{2} + \frac{1}{x} + \frac{1}{2x^2}\right) f(x+1)$$

$$e^{\lambda_3}\left[\tfrac{1}{3} f(x)\right] = \left(\frac{1}{3} + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{3x^3}\right) f(x+1)$$

where $\lambda_2 = \frac{d}{dx} + \frac{2}{x}$ and $\lambda_3 = \frac{d}{dx} + \frac{3}{x}$.

Solution by A. V. Subba Rau; K. J. Sanjana and several others.

The three results to be proved are particular cases of

$$e^{\lambda r}\left[f(x)\right] = \left(1 + \frac{1}{x}\right)^r f(x+1) \text{ where } \lambda_r = \frac{d}{dx} + \frac{r}{x}$$

r being any positive integer.

Let f, f_1, f_2, f_3 etc., be $f(x)$ and its derived functions with regard to x.

Then $\lambda_r f = f_1 + \frac{r}{x} f$,

$$\lambda_r^2 f = f_2 + \frac{r}{x} f_1 + \frac{r}{x} f_1 - \frac{r}{x^2} f + \frac{r^2}{x^2} f$$

$$= f_2 + 2 \cdot \frac{r}{x} f_1 + \frac{r(r-1)}{x^2} f,$$

$$\lambda_r^3 f = \lambda_r f_2 + 2 \cdot \lambda_r \left[\frac{r}{x} f_1\right] + r(r-1)\, r_r \left[\frac{1}{x^2} f\right]$$

$$= f_3 + \frac{r}{x} f_2 + \frac{2r}{x} f_2 - \frac{2r}{x^2} f_1 + \frac{2r^2}{x^2} f_1$$

$$+ \frac{r(r-1)}{x^2} f_1 - \frac{2r(r-1)}{x^3} f + \frac{r^2(r-1)}{x^3} f$$

$$= f_3 + 3 \cdot \frac{r}{x} f_2 + 3 \cdot \frac{r(r-1)}{x^2} f_1 + \frac{r(r-1)(r-2)}{x^3} f.$$

In general, we get

$$\lambda_r^n f = f_n + {}_nC_1 \frac{r}{x} f_{n-1} + {}_nC_2 \frac{r(r-1)}{x^2} f_{n-2} + \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

the last term being

(i) ${}_nC_n \dfrac{r(r-1)(r-2)\ldots(r-n+1)}{x^n} f$, if $n < r$;

(ii) $\dfrac{r!}{x^r} f$, if $n = r$;

(iii) ${}_nC_r \dfrac{r!}{x^r} f_{n-r}$, if $n > r$.

To prove this expansion of $\lambda_r^n f$ in terms of f_n, f_{n-1}, f_{n-2}, etc., we assume it in the case of n and deduce it in the case of $n+1$ as follows :—

Now $\lambda_r^{n+1} f = \left(\dfrac{d}{dx} + \dfrac{r}{x}\right) \lambda_r^n f.$

and the general term in $\lambda_r^{n+1} f$

$$= {}_nC_{s-1}\, r(r-1)(r-2)\ldots(r-s+2) f_{n-s+1} \left(\frac{d}{dx} + \frac{r}{x}\right) \frac{1}{x^{r-1}}$$

$$+ {}_nC_s \frac{r(r-1)(r-2)\ldots(r-s+1)}{x^s} f_{n-s+1}$$

$$= {}_nC_{s-1}\, r(r-1)(r-2)\ldots(r-s+2)(r-s-1) f_{n-s+1}$$

$$+ {}_nC_s \frac{r(r-1)(r-2)\ldots(r-s+1)}{x^s} f_{n-s+1}$$

$$= ({}_nC_{s-1} + {}_nC_s) \frac{r(r-1)(r-2)\ldots(r-s+1)}{x^s} f_{n-s+1}$$

$$= (n+1)\, C_s \frac{r(r-1)\ldots(r-s+1)}{x^s} f_{n+1-s}.$$

Hence by Mathematical Induction the expansion of $\lambda_r^n f$ as given in (1) above, is true.

Now $$e^{\lambda_r} f = \sum \frac{\lambda_r^n}{n!} f,$$

and $\sum \dfrac{f_n}{n!}$, $\sum \dfrac{f_{n-1}}{n-1}$, $\sum \dfrac{f_{n-2}}{n-2!}$ etc. are all equal and each is equal to $f(x+1)$, by Taylor's Theorem.

$$\therefore \ e^{\lambda r}\left[f(x)\right] = \left(1 + \frac{{}^{r}C_1}{x} + \frac{{}^{r}C_2}{x^2} + \frac{{}^{r}C_3}{x^3} + \ \ldots \ \frac{{}^{r}C_r}{x^r}\right) f(x+1)$$

$$= \left(1 + \frac{1}{x}\right)^{r} f(x+1).$$

By putting $r = 1, 2$ and 3 the three results required to be proved are got.

Similar solution by C. N. Sreenivasa Aiyangar, V. M. Gaitonde, N. Sundaram and others.

Question 1110.

(A. Narasinga Rao) :—If for a curve there exists a functional relation connecting the area of every escribed triangle, and that of the triangle formed by the points of contact, such a curve must be a straight line or a parabola.

Solution by N. Sundaram.

Since the functional relation is true for all inscribed Δs, let us take the inscribed triangle formed by three consecutive points on the curve. Let P, Q, R be the points forming the inscribed Δ of area Δ_1. Let C be the corresponding centre of curvature, and ρ the radius of curvature. Then the area of the triangle LNM formed by the tangents at P, Q, R to the circle of curvature will differ from that formed by tangents at P, Q, R to the curve by an infinitesimal of lower order than Δ_1. Let Δ_2 be the area of the escribed triangle.

It is obvious from a figure that

$$\Delta_1 = \tfrac{1}{2}\cdot 2\rho \sin\frac{\theta}{2}\cdot 2\rho \sin\frac{\phi}{2}\sin\frac{\theta+\phi}{2} = 2\rho^2 \sin\frac{\theta}{2}\cdot\sin\frac{\phi}{2}\sin\frac{\theta+\phi}{2},$$

and

$$\Delta_2 = \rho^2 \tan\frac{\theta}{2}\cdot\tan\frac{\phi}{2}\cdot\tan\frac{\theta+\phi}{2},$$

where θ, ϕ stand for the angles PCQ, QCR.

Thus since θ, ϕ are small, we have as the functional relation

$$\Delta_1 = 2\Delta_2.$$

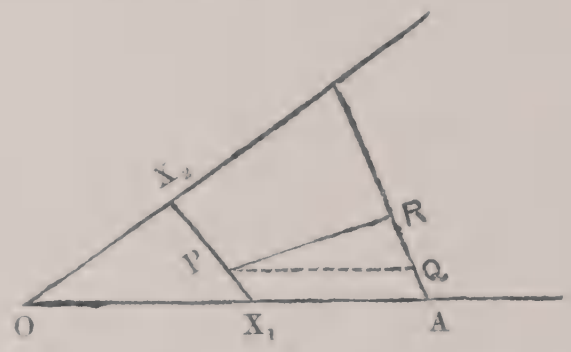
Hence either $\Delta_1 = \Delta_2 = 0$, which is obviously the case when the curve is a straight line; or the curve is a parabola which satisfies the relation $\Delta_1 = 2\ \Delta_2$.

For if $(at_1^2,\ 2at_1)$, $(at_2^2,\ 2at_2)$, $(at_3^2,\ 2at_3)$ be the vertices of the inscribed Δ, $[at_1t_2,\ a(t_1+t_2)\ ;\ at_1t_3,\ a(t_1+t_3)\ ;\ at_2t_3,\ a(t_2+t_3)]$ are the vertices of the corresponding escribed triangle.

$$\therefore\ \Delta_1 = \tfrac{1}{2}a \,.\, 2a \begin{vmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{vmatrix} = a^2 (t_1 - t_2)(t_1 - t_3)(t_2 - t_3)$$

$$\text{and}\ \Delta_2 = \tfrac{1}{2}a^2 \begin{vmatrix} t_1t_2 & t_1+t_2 & 1 \\ t_2t_3 & t_2+t_3 & 1 \\ t_3t_1 & t_3+t_1 & 1 \end{vmatrix} = \tfrac{1}{2}a^2 (t_1-t_2)(t_2-t_3) \times (t_3-t_1)$$

$$\therefore \qquad \Delta_1 = 2\,\Delta_2, \text{ in a parabola.}$$



[We can prove that the only form of the curve other than a straight line is a parabola, as under :

Let OA and OB be two fixed tangents to the curve at A and B. Let X_1X_2 be a variable tangent to the curve at P. It will be shown that P can trace only a parabola.

Let OA and OB be the axes of co-ordinates, and the equations to AB and X_1X_2 be $\frac{X}{a}+\frac{Y}{b} = 1$ and $\frac{X}{x_1} + \frac{Y}{x_2} = 1$, respectively. Let P be (x,y).

Now Δ APB $= 2\,\Delta$ OX_1X_2 as above.

$\therefore$ $\frac{1}{2}$ AB . $p = x_1x_2$ sin $A\hat{O}B$, where p is the perpendicular PR on AB.

PQ $= p_1$ (say) is drawn parallel to OA to cut AB at Q.

$\therefore$ $\frac{1}{2}$ AB . p_1 sin A $= x_1x_2$ sin $A\hat{O}B$.

But $$\frac{\text{AB}}{\sin A\hat{O}B} = \frac{b}{\sin \text{A}},$$

$$\therefore \qquad bp_1 = 2x_1x_2. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1)$$

And length PQ or p_1 is $a\left(1 - \frac{y}{b}\right) - x$.

Writing $\frac{X}{x_1} + \frac{Y}{x_2} = 1$ in the form $Y - y = \frac{dy}{dx}(X - x)$

and remembering that $\frac{x}{x_1} + \frac{y}{x_2} = 1$, we have

$$x_1 = x - \frac{y}{y'}, \text{ and } x_2 = -y'\left(x - \frac{y}{y'}\right).$$

Substituting in (1) for x_1, x_2 and p_1, we have

$$-2\left(x - \frac{y}{y'}\right)^2 y' = b\left\{a\left(1 - \frac{y}{b}\right) - x\right\},$$

or
$$2\left(x\frac{dy}{dx} - y\right)^2 = \frac{dy}{dx}\{ay + b.(x - a)\}. \quad \ldots \quad (2)$$

To solve this, put $x = au^2$, $y = bv^2$, so that

$$\frac{dy}{dx} = \frac{bv}{au} \cdot \frac{dv}{du}.$$

Equation (2) becomes

$$\left(buv\frac{dv}{du} - bv^2\right)^2 = \frac{bv}{au} \cdot \frac{dv}{du} \cdot ab(u^2 + v^2 - 1);$$

or
$$2uv\left(u\frac{dv}{du} - v\right)^2 = \frac{dv}{du}(u^2 + v^2 - 1), \quad \ldots \quad \ldots \quad (2')$$

which is satisfied by $v + u = 1$.

Hence the original equation is satisfied by

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1. \quad \ldots \quad \ldots \quad (3)$$

Since the differential equation is of the first order, (3) should be a solution got for a particular value of the single arbitrary constant which the complete primitive can have. But since our curve is to touch the axes at distances a and b respectively and (3) does so, it is the locus of P.]

Question 1112.

(C. KRISHNAMACHARI, M.A.) :—If V is the normal velocity drawn outwards to a closed surface S in a liquid, show that

$$\iint \rho V dS + \iiint \frac{d\rho}{dt}\, dx\, dy\, dz = 0,$$

where the volume integral extends throughout the volume enclosed by S, and the surface integral over the surface S.

Hence establish the equation of continuity.

Solution by Martyn M. Thomas and S. R. Ranganatham.

Let ρ be the density at any point (x, y, z) of the fluid inside the given closed surface S, so that ρ is a function of x, y, z.

Then $\iiint \rho\, dx\, dy\, dz$ being the mass of the fluid within S we have

$\frac{\partial}{\partial t}\iiint \rho\, dx\, dy\, dz.\ \delta t$ = increase in mass inside the surface in time δt

= excess of flow in, over flow out, across the surface S in time δt

$$= \Sigma\left[x\, dy\, dz\ .\ \rho - \left(x + \frac{\partial x}{\partial t}\right)\delta t\ .\ dy\, dz\, \rho\right]$$

+ two similar terms, the summation being extended to points all over the bounding surface S

$= -\iint [u\rho dydz + v\rho dzdx + w\rho dxdy]\ \delta t$, where u, v, w are the components of the normal velocity V on the surface parallel to coordinate axes

$= -\iint (u\rho\ .\ ldS + v\rho\ .\ mdS + w\rho\ .\ ndS)\ \delta t$, where l, m, n are the direction cosines of the normal to S

$$= -\iint (l\rho u + m\rho v + n\rho w)\ dS.\delta t \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$= -\iint [\rho\,(l^2\mathrm{V}) + \rho(m^2\mathrm{V}) + \rho(n^2\mathrm{V})]\, dS\ .\ \delta t = -\iint (\rho\mathrm{V})\ dS.\delta t.$$

$$\therefore \left\{\frac{\partial}{\partial t}\iiint \rho\, dxdydz + \iint \rho\mathrm{V}dS\right\}\delta t = 0.$$

Hence $$\iiint \frac{\partial \rho}{\partial t}\, dxdydz + \iint \rho\mathrm{V}dS = 0.$$

To deduce the equation of continuity :

Green's Theorem states that

$$\iint (lf + mg + nh)\, dS = \iiint \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dxdy\, dz,$$

where f, g, h are functions of (x, y, z), which, with their first derivatives, are finite and continuous through a region bounded by a closed surface S.

Hence from the equality (1) obtained above, we have

$$\frac{\partial}{\partial t}\left[\iiint \rho dxdydz\right]\delta t = -\iint [l(\rho u) + m(\rho v) + n(\rho w)]\ dS.\delta t$$

$$= -\iiint \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] dx dy dz \,.\, \delta t,$$

using Green's Theorem.

$$\therefore \iiint \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] dx dy dz = 0. \quad \ldots (2)$$

Since equation (2) is true for all ranges of integration within the fluid, we have $\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$, at every point of the fluid ; and this is the equation of continuity, in Cartesian co-ordinates.

Question 1113.

(Selected):—An ellipse, of eccentricity $\sin 2\alpha$ passes through the focus of a parabola of latus-rectum $4a$ and has its foci on the curve. Show that the major axis envelopes the parabola

$$y^2 = 4a(1 - \tan^2\alpha)(x - a)\tan^2\alpha);$$

and that the minor axis envelopes the semi-cubical parabola

$$27a \sec^2\alpha \; y^2 = 4(x - a - a\sec^2\alpha)^3.$$

Solution by A. A. Krishnaswami Aiyangar and Martyn M. Thomas.

Let S be the focus of the parabola and P, P′ the foci of the ellipse. Denote P, P′ by the parameters t, t' respectively.

Since $$\frac{SP + SP'}{PP'} = \frac{1}{e} = \frac{1}{\sin 2\alpha},$$

$$\frac{2 + t^2 + t'^2}{(t - t')\sqrt{[(t + t')^2 + 4]}} = \frac{1}{\sin 2\alpha} = \frac{1 + \tan^2\alpha}{2\tan\alpha}$$

$$\therefore \quad \tan\alpha = \frac{t - t'}{\sqrt{(t + t')^2 + 4}} \text{ or } \frac{\sqrt{(t + t')^2 + 4}}{t - t'}.$$

$$\therefore \quad tt' = \left(\frac{t + t'}{2}\right)^2 (1 - \tan^2\alpha) - \tan^2\alpha,$$

taking only the first value. (i)

The equation of PP′ is

$$x + att' = y \cdot \frac{t + t'}{2},$$

$$i.e., \quad x + a(1 - \tan^2\alpha)\left(\frac{t + t'}{2}\right)^2 - a\tan^2\alpha = y\frac{t + t'}{2},$$

after substituting from (i) ;

$$i.e., \quad x - a\tan^2\alpha - y.\frac{t+t'}{2} + a(1-\tan^2\alpha)\left(\frac{t+t'}{2}\right)^2 = 0,$$

whose envelope is easily seen to be

$$y^2 = 4a(1-\tan^2\alpha)(x - a\tan^2\alpha).$$

If we take the second value of $\tan\alpha$, we can get similarly the envelope $y^2 = 4a(1-\cot^2\alpha)(x - a\cot^2\alpha)$, which is not given in the question.

Again, to find the envelope of the minor-axis, we may write its equation in the form

$$y + (x-2a)\frac{t+t'}{2} = \frac{a}{2}.\frac{t+t'}{2}.(t^2+t'^2)$$

$$= a\sec^2\left(\frac{t+t'}{2}\right)^3 + a\tan^2\alpha.\frac{t+t'}{2} \text{ on substitution from (i) ;}$$

$$i.e., \quad a\sec^2\alpha\left(\frac{t+t'}{2}\right)^3 - (x-2a-a\tan^2\alpha)\frac{t+t'}{2} - y = 0.$$

Considering this relation as an equation in the variable $\frac{t+t'}{2}$, we easily find its envelope to be

$$27a\sec^2\alpha\, y^2 = 4(x-2a-a\tan^2\alpha)^3$$
$$= 4(x-a-a\sec^2\alpha)^3.$$

Similary, we can get another envelope of the form

$$27a\,\mathrm{cosec}^2\alpha y^2 = 4(x-a-a\,\mathrm{cosec}^2\alpha)^3.$$

It can be easily verified that the locus of the centre of the ellipse is the parabola $y^2 = 4a\cos^2\alpha(x - a\tan^2\alpha)$.

Similar Solutions by N. B. Mitra, K. B. Madhava, Martyn M. Thomas and several others.

Question 1114.

(T. P. Trivedi, M.A., LL.B.) :—If p, q are any two integers selected at random ; prove that that the probability that the fraction (p/q) is in its lowest terms is $6/\pi^2$.

Solution by N. B. Mitra and A. Narasinga Rao.

The probability that p/q is in its lowest terms is the same as the probability that p and q may be prime to each other.

Let a be any prime number, If any integer p be taken at random, the probability that p may have a factor a may be found thus :

At first suppose $p \not> m$. Two cases arise.

(1) Let m be a multiple of a say $= np$; there are n integers $\not> m$ which contain a as a factor, *viz.*, the integers $a, 2a, \ldots\ldots na$: also the total number of integers $\not> m$ is m. Hence the probability that p may have a factor a is $n/m = 1/a$.

(2) Next, let m lie between na and $(n-1)a$, where n is an integer. In this case the number of integers $\not> m$ which have a factor a is $n-1$. Hence the probability that p may contain a as a factor is $(n-1)/m$ which lies between $1/a$ and $1/a - 1/m$. But as m is taken larger and larger $1/a$ and $1/a - 1/m$ tend to equality.

Hence the probability that an integer p taken at random may contain a as a factor is ultimately $1/a$.

Similarly the probability that another integer q taken at random may contain a as a factor is $1/a$.

Hence the probability that the two integers p, q taken at random may both contain a as a factor is $1/a^2$.

Therefore the probability that p, q may not have a common factor a is $1 - 1/a^2$.

Similarly the probability that p, q may not have a common factor b (where b is another prime) is $1 - 1/b^2$.

Hence the probability that p and q may not have a common factor which is either a or b is $(1 - 1/a^2)(1 - 1/b^2)$.

And so on.

Hence the probability that p and q may be prime to each other is $P = (1 - 1/a^2)(1 - 1/b^2)(1 - 1/c^2) \ldots\ldots$ where $a, b, c \ldots\ldots$ are the succession of natural primes.

To evaluate P.

Let S denote the convergent series $1 + 1/2^2 + 1/3^2 + \ldots\ldots$ multiply by $1 - 1/2^2$. The effect will be to remove all terms from the series whose denominators contain 2 as a factor. Thus

$$S(1 - 1/2^2) = 1 + 1/3^2 + 1/5^2 + \ldots\ldots$$

Next multiply this by $1 - 1/3^2$. This will remove from the right side all terms whose denominators are multiples of 3. Thus

$$S(1 - 1/2^2)(1 - 1/3^2) = 1 + 1/5^2 + 1/7^2 + 1/11^2 + 1/13^2 + \ldots\ldots$$

Proceeding in this way we shall arrive at

$$S(1 - 1/2^2)(1 - 1/3^2) \ldots\ldots (1 - 1/k^2) = 1 + 1/l^2 + \ldots\ldots$$

where 2, 3, ... k are the successive natural primes up to k and l is the prime next to k. We may make l as large as we please and then $1/l^2 + \ldots$ will tend to 0, since it is less than the residue after $(l - 1)$ term of the convergent series S.

Hence finally $SP = 1$. But $S = \pi^2/6$. Therefore $P = 6/\pi^2$.

Note by G. V. Krishnaswami.

This problem is due to the Russian Mathematician Tchebycheff.

(i) The probability that *none* of the primes from 3 onwards is a common factor of p/q is

$$P = \left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\left(1 - \frac{1}{11^2}\right) \ldots \textit{ad inf.}$$

$$= \frac{6}{\pi^2} \div \left(1 - \frac{1}{2^2}\right).$$

(ii) Suppose furthermore, we were assured that none of the two primes 2, 3 was a common factor of both p and q. The probability that the fraction might not be reduced by division by one or more of the other primes is, from the above,

$$\frac{6}{\pi^2} \div \left[\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\right].$$

(iii) Generally if we know that none of the n primes $m_1, m_2, \ldots m_n$ was a common factor of both p and q, the probability that the fraction might not be reduced by division by one or more of the other primes is

$$\frac{6}{\pi^2} \div \left[\left(1 - \frac{1}{m_1^2}\right)\left(1 - \frac{1}{m_2^2}\right)\left(1 - \frac{1}{m_3^2}\right) \ldots\ldots \left(1 - \frac{1}{m_n^2}\right)\right].$$

Question 1115.

(T. P. Trivedi, M.A., LL.B.) :- Prove that

(i) $$\frac{2^2 - 1}{2^2 + 1}\,\frac{3^2 - 1}{3^2 + 1}\,\frac{5^2 - 1}{5^2 + 1} \ldots \textit{ad inf.} = \frac{2}{5}.$$

(ii) $$\frac{2^4 - 1}{2^4 + 1}\,\frac{3^4 - 1}{3^4 + 1}\,\frac{5^4 - 1}{5^4 + 1} \ldots \textit{ad inf.} = \frac{6}{7}.$$

Solution by N. B. Mitra, H. R. Kapadia and
C. N. Sreenivasa Iyengar and others.

[N.B. : *Master S. Chowla, aged 13, has sent a solution on similar lines.*—Ed.]

Let us consider the series $\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} \ldots$ *ad inf.* ... (1)

We know that it is convergent if $n > 1$. Let us denote it by S_n.

Multiply the series by $1 - \frac{1}{2^n}$. The result will be to deprive the series of all terms whose denominators are multiples of 2.

Thus, $$S_n\left(1 - \frac{1}{2^n}\right) = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \ldots$$

Let us multiply this by $\left(1 - \frac{1}{3^n}\right)$, 3 being the prime next to 2. The result will be to deprive the series on the right of all terms whose denominators are multiples of 3.

Proceeding in this way we get

$$S_n\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{5^n}\right) \ldots \left(1 - \frac{1}{p^n}\right) = 1 + \frac{1}{q^n} + \ldots,$$

where 2, 3, 5, ... p are the primes up to p and q is the prime next to p Also $1/q^n + \ldots$ is less than the residue after $q - 1$ terms of the convergent series (1) and hence can be made as small as we please by making q as large as we please.

Hence we get finally,

$$\left(1 - \frac{1}{2^n}\right)^{-1}\left(1 - \frac{1}{3^n}\right)^{-1}\left(1 - \frac{1}{5^n}\right)^{-1} \ldots \text{ to } \infty = S_n \quad \ldots \quad (a)$$

Similarly

$$\left(1 - \frac{1}{2^{2n}}\right)^{-1}\left(1 - \frac{1}{3^{2n}}\right)^{-1}\left(1 - \frac{1}{5^{2n}}\right)^{-1} \ldots \text{ to } \infty = S_{2n} \quad \ldots \quad (b)$$

Dividing (*b*) by (*a*) we get

$$\left(1 + \frac{1}{2^n}\right)^{-1}\left(1 + \frac{1}{3^n}\right)^{-1}\left(1 + \frac{1}{5^n}\right)^{-1} \ldots \text{ to } \infty = \frac{S_{2n}}{S_n} \quad \ldots \quad (c)$$

Dividing (*c*) by (*a*), wet get, after clearing

$$\frac{2^n - 1}{2^n + 1} \cdot \frac{3^n - 1}{3^n + 1} \cdot \frac{5^n - 1}{5^n + 1} \cdot \ldots \textit{ad inf.} = \frac{S_{2n}}{S^2_n} \quad \ldots \quad \ldots \quad (A)$$

The questions proposed are particular cases of this for $n = 2$ and 4.

We know $S_2 = \frac{\pi^2}{6}$, $S_4 = \frac{\pi^4}{90}$, $S_8 = \frac{\pi^8}{9450}$.

These can be successively calculated from the well-known formula

$$S_2 \frac{\pi^{2n-2}}{(2n-1)!} - S_4 \frac{\pi^{2n-4}}{(2n-3)!} + \ldots + (-1)^n S_{2n} = \frac{n\pi^{2n}}{(2n+1)!}.$$

Hence putting $n = 2, 4$ in (A) and substituting the above values, we get the results :—

(i) $\frac{2^2-1}{2^2+1} \cdot \frac{3^2-1}{3^2+1} \cdot \ldots\ldots = \frac{6^2}{90} = \frac{2}{5}$,

(ii) $\frac{2^4-1}{2^4+1} \cdot \frac{3^4-1}{3^4+1} \cdot \ldots\ldots = \frac{90^2}{9450} = \frac{6}{7}$.

Question 1117.

(Prof. K. J. Sanjana) :—If a given function of x, y, z be transformed by the substitutions

$$r = \tfrac{1}{2} \log (x^2 + y^2 + z^2), \quad \phi = \tan^{-1} \frac{y}{x}, \quad \theta = \tan^{-1} \frac{(x^2+y^2)^{\frac{1}{2}}}{z},$$

prove that the operation

$$(x^2 + y^2 + z^2) \left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2} \right)$$

is transformed into

$$\left(\frac{\delta^2}{\delta r^2} + \frac{\delta^2}{\delta \theta^2} + \operatorname{cosec}^2 \theta \frac{\delta^2}{\delta \phi^2} + \frac{\delta}{\delta r} + \cot \theta \frac{\delta}{\delta \theta} \right).$$

Solution by Martyn M. Thomas and several others.

Let $x^2 + y^2 + z^2 = R^2$,

Then the given relations are equivalent to :—

$R = e^r$; $\quad x = R \sin \theta \cos \phi$

$y = R \sin \theta \sin \phi$

$z = R \cos \theta$.

It has been proved in Edward's *Diff. Calculus*, § 532, that

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = \frac{d^2V}{dR^2} + \frac{2}{R}\frac{dV}{dR} + \frac{1}{R^2}\frac{d^2V}{d\theta^2} + \frac{\cot \theta}{R^2}\frac{dV}{d} + \frac{1}{R^2} \operatorname{cosec}^2 \theta \frac{d^2V}{d\phi^2}.$$

$$\therefore \quad R^2\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}\right)$$

$$= R^2\frac{d^2V}{dR^2}+2R\frac{dV}{dR}+\frac{d^2V}{d\theta^2}+\cot\theta\frac{dV}{d\theta}+\operatorname{cosec}^2\theta\frac{d^2V}{d\phi^2}$$

$$= \frac{d}{dR}\left[R^2\frac{dV}{dR}\right]+ \quad ,, \quad ,, \quad ,,$$

$$= \frac{d}{e^r dr}\left[e^{2r}\frac{dV}{e^r dr}\right]+ \quad ,, \quad ,, \quad ,,$$

$$= \frac{1}{e^r}\cdot\frac{d}{dr}\left[e^r\frac{dV}{dr}\right]+ \quad ,, \quad ,, \quad ,,$$

$$= \frac{d^2V}{dr^2}+\frac{dV}{dr}+ \quad ,, \quad ,, \quad ,,$$

Thus $$(x^2+y^2+z^2)\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)V$$

$$= \left(\frac{d^2}{dr^2}+\frac{d}{dr}+\frac{d^2}{d\theta^2}+\cot\theta\frac{d}{d\theta}+\operatorname{cosec}^2\theta\frac{d^2}{d\phi^2}\right)V.$$

Question 1118.

(N. Dorai Rajan):—Show that in the cubical parabola $x^3 = a^2y$, all tangent chords are trisected by the y-axis.

Solution by V. M. Gaitonde and several others.

Tangent at any point (x, y) on the curve $X^3 = a^2Y$ is

$$X(3x^2) + Y(-a^2) + (-2a^2y) = 0.$$

Eliminating Y from these two equations, the abscissae of the points of intersection of the tangent chord with the curve are given by the cubic equation

$$X^3 - 3x^2X + 2a^2y = 0.$$

Let x correspond to the two coincident points at the point of contact P of the tangent, and x' to the other extremity Q of the tangent chord. Then

$$x + x + x' = \text{co-efficient of } X^2 = 0.$$

Let the tangent chord PQ intersect the y-axis in T.

Then, by similar triangles, $\frac{PT}{TQ} = \frac{x}{x'} = \frac{1}{2}$ numerically.

Hence T is a point of trisection of PQ.

Question 1120.

(N. Dorai Rajan):—Show that the curve whose (p, r) equation is $r = (a + p)$ is an involute of an involute of a circle. Find its polar equation and trace the curve.

Solution by N. B. Mitra, C. N. Sreenivasa Iyengar and several others.

If (r', p') be the pedal co-ordinates of a point on the evolute corresponding to the point (r, p) on the curve, we have

$$p'^2 = r^2 - p^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

and

$$r'^2 = r^2 + \rho^2 - 2\rho p, \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where ρ refers to the given curve and $= r\,dr/dp$.

Here the given curve is

$$r = a + p. \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$\therefore \quad \rho = r,$$

and from (2), $r'^2 = 2r(r - p) = 2ra$.

Also from (1), $p'^2 = a(r + p) = a(2r - a)$.

$$\therefore \quad p'^2 + a^2 = r'^2.$$

Hence the pedal equation of the evolute of (3) is $r^2 = a^2 + p^2$, which is the involute of the circle $r = p$.

In other words, the evolute of $r = a + p$ is the involute of the circle $r = p$.

Thus $r = (a + p)$ is the involute of the involute of the circle $r = p$.

To obtain the polar equation of (3):

We know that

$$\frac{ds}{d\theta} = \frac{r^2}{p} - \frac{r^2}{r - a}.$$

$$\therefore \quad \left(\frac{dr}{d\theta}\right)^2 + r^2 = \frac{r^4}{(r - a)^2},$$

$$\therefore \quad \frac{dr}{d\theta} = \frac{r}{r - a}\sqrt{(a\,2r - a)}.$$

$$\therefore \quad \sqrt{a}\,d\theta = ar.(r - a) / \{r\sqrt{(2r - a)}\}.$$

Integrating,

$$\pm\,\theta = \text{vers}^{-1}\frac{a}{r} + \frac{\sqrt{2r - a}}{a}, \text{ which may be put into the form}$$

$$r = \frac{a}{2}\,\text{cosec}^2\phi\,; \quad \theta = 2\phi + \cot\phi,$$

where ϕ is an arbitrary parameter.

QUESTIONS FOR SOLUTION

1161. (K. J. SANJANA) :—Prove that

$$\frac{n(n+1)\ldots(2n-1)}{n!} + \frac{n(n+1)\ldots 2n}{n+1!} \cdot \frac{1}{2} + \frac{n(n+1)\ldots(2n+1)}{n+2!} \cdot \frac{1}{2^2} + \ldots \text{ad. inf.} = 2^{2n-1}.$$

1162. (K. J. SANJANA) :—Solve the following differential equations and explain their geometrical significance :—

(i) $x^2 + y^2 - \frac{2xy_1(1+y_1^2)}{y_2} + \frac{2y(1+y_1^2)}{y_2} = k^2$;

(ii) $x^2 + y^2 - \frac{2xy_1(1+y_1^2)}{y_2} + \frac{2y(1+y_1^2)}{y_2} + \frac{(1+y_1^2)^2}{y_2^2} = k^2$;

(iii) $y - mx + \frac{1+y_1^2}{y_2} + \frac{my_1(1+y_1^2)}{y_2} = 0.$

Here y_1, y_2 stand for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and k and m are given constants.

1163. (V. RAMASWAMI AIYAR) :—If a rectangular hyperbola passes through the incentre of a triangle and the feet of the perpendiculars drawn therefrom to the sides, prove that it cuts the inscribed circle again at the point which is diametrically opposite the Feuerbach point.

1164. [G. T. V.] :—O is the circumcentre of a triangle ABC. Points X, X'; Y, Y'; Z, Z' are taken on BC, CA, AB respectively, such that O is the common incentre of the triangles AXX', BYY', CZZ'. Show that (1) their circumcircles intersect at the Euler point E of the triangle ABC; (ii) if these circles intersect again in points A', B', C', then AA', BB', CC' concur at the centre of the N.P. circle of the triangle ABC; (iii) E is the twin-point of O with respect to the triangle A'B'C'.

1165. (N. P. PANDYA) :—Find the lowest prime of 17 digits.

1166. (HEMRAJ) :—If n be prime and

$$\prod_{k=1}^{n-1}(x+k) = \sum_{k=0}^{n-1} A_k\, x^{n-k-1},$$

prove that

$$2A_k \equiv \left\{ \sum_{r=1}^{\Sigma} (-1)^{r-1}\, n^r\, {}^{n-k+r-1}C_r\; A_{k-r} \right\}, \left(\text{mod. } n^{r+3}\right),$$

where k and r are both odd and $k > (r+2)$.

1167. (HEMRAJ) :—If $I=(ae-4bd+3c^2)$ is a negative number ; prove that a necessary and sufficient condition that the roots of the biquadratic

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$$

with real co efficients may be concyclic in the Argand Diagram is that a root of its reducing cubic should be proportional to $\sqrt{I}$.

1168. (S. KRISHŇASWAMI AIYANGAR) :—If $(x\ y)$ stands for $\text{sn}\,(x+y)\,\text{sn}\,(x-y)$, then

$$\frac{1-k\,(\alpha\beta)}{1+k\,(\alpha\beta)} \times \frac{1-k\,(\beta\gamma)}{1+k\,(\beta\gamma)} = \frac{1+k\,(\gamma\delta)}{1-k\,(\gamma\delta)} \times \frac{1+k\,(\delta\alpha)}{1+k\,(\delta\alpha)}.$$

1169. (S. KRISHNASWAMI AIYANGAR) :—

If $(x\ y) \equiv \dfrac{\text{dn}^2 x - \text{dn}^2 y}{1+k^2\text{dn}^2\,x\,\text{dn}^2 y}$, show that

$$(\alpha\beta) + (\beta\gamma) + (\gamma\alpha) = k^2\,(\alpha\beta)\,(\beta\gamma)\,(\gamma\alpha).$$

1170. (P. V. SESHU AIYAR) :—If y_r is the rth term in the expansion of $(p+q)^n$, show that

$$\sum_{r=1}^{n+1} (y_r \,.\, r^s) = \frac{d}{dq}\left(q\frac{d}{dq}\right)\left(q\frac{d}{dq}\right) \ldots\ldots q(p+q)^n,$$

where $\dfrac{d}{dq}$ is repeated s times.

1171. (R. VAIDYANATHASWAMI) :—If $S + \lambda S'$ is a pair of lines for the values a, $\dfrac{1}{a}$, $-\dfrac{1}{a}$ of λ, prove that quadrilaterals can be inscribed in $S - aS'$, with their four sides touching respectively the conics

$$S + bS',\ S - bS',\ S + \frac{1}{b}S',\ S - \frac{1}{b}S'.$$

1172. (R. VAIDYANATHASWAMI) :—E is a conic of a given four-point system and S_1, S_2 are the two conics of the system which are inscribed to a self-conjugate $\triangle$ of E. $\triangle$s are inscribed in E with two of their sides touching S_1, S_2 respectively. Prove that the envelope of the third side is composed of the two conics of the system each of which has $\triangle$s inscribed in itself, circumscribed to the other.

INDIAN MATHEMATICAL SOCIETY.

3RD. CONFERENCE, MARCH 1921. LAHORE.



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The Indian Mathematical Society.

THIRD CONFERENCE—MARCH 1921, LAHORE.

First Row (*standing*) *from Left to Right.*

Rulia Ram, Mulraj, S. B. Belekar, K. B. Madhawa, N. Raghunatha Aiyangar, K. S. Patrachari, Mehrchand Suri, Bhagwan Das, Dhanpat Rai, Daulat Ram, Ishar Das, Mulk Raj, Sandhya Lal, M. A. Majid, Hukum Chand.

Second (*middle*) *Row* (*standing*).

Pandit Hemraj, A. B. Chandrachud, S. P. Shah, Ram Behari, M. K. Kewalramani, L. S. Vaidyanathan, M. V. Arunachala Sastri, R. Vythianatha Swami, T. V. Mone, G. G. Pendse, Sardari Lal, Ghulam Abbas, Mukund Lal, Moh'd Arshad Khan.

Third Row (*sitting*).

C. V. H. Rao, S. Narayana Aiyar, M. T. Naraniengar, P. V. Seshu Aiyar, Balak Ram, Sarvadaman Chowla, Devi Dayal, G. S. Chowla, T. P. Bhaskara Sastri, D. D. Kapadia, T. S. Narayana, D. K. Hardikar, S. N. Das Gupta.

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THE THIRD CONFERENCE OF THE INDIAN MATHEMATICAL SOCIETY.

The Third Conference of the Indian Mathematical Society was held at Lahore under the kind invitation of the Punjab University Mathematical Association on the 26th, 27th and 28th March 1921.

The opening ceremony on the first day was held at the Town Hall, and the meetings on the second and third days took place in the Law College Buildings, near the University Library, Lahore.

The Punjab University Mathematical Association, under the able guidance of their Patron, Sir H. J. Maynard, Vice-Chancellor of the Punjab University, their President, Prof. Devi Dayal, and other members of the Committee had done all that was possible to make the Conference a success. The arrangements for the comforts of guests were highly satisfactory.

The Conference commenced its proceedings at 11 A M. on the first day in the Town Hall, in the presence of a large gathering consisting of members of the Society, members of the Punjab University Mathematical Association and a number of visitors—both official and non-official—interested in the objects of the Society. Some ladies also graced the occasion by their presence.

Punctually at the appointed time, His Excellency Sir Edward Maclagan, K.C.S.I., K.C.I.E., Governor of the Punjab, accompanied by his Military A. D. C., arrived, and was received on the steps of the Town Hall by the Hon'ble Sardar Bahadur Sunder Singh Majithia (Chairman of the Reception Committee), Mr. Balak Ram (President of the Society), Mr. Devi

Dayal (President of the Punjab University Mathematical Association) and members of the Managing Committee of the Indian Mathematical Society. After the formal introduction of the vsrious officials of the Society, His Excellency was conducted to the Conference Hall where the whole gathering received him standing. After His Excellency had taken his seat on the dais, the proceedings commenced.

The Chairman of the Reception Committee, the Hon'ble Sardar Bahadur Sunder Singh Majithia, Member, Executive Council, Punjab, first read his Address of Welcome.[1]

This was followed by the Opening Address[2] of His Excellency Sir Edward Maclagan, Governor of the Punjab.

The Hon. Joint Secretary (Prof. D. D. Kapadia) then read the Report[3] of the Society.

Mr. Balakram, M.A., I.C.S., afterwards delivered the Presidential Address.[4]

The Hon'ble Sir H. J. Maynard, K.C.I.E., I.C.S., Vice-Chancellor of the Punjab University, next proposed a hearty vote of thanks to His Excellency the Governor for the trouble taken by him to preside at the opening of the Conference and thereby contributing to the success of the Conference.

The proposal was received with acclamation, and the formal proceedings of the Conference for the day terminated.

After the departure of His Excellency and visitors, the members of the Indian Mathematical Society and of the Punjab University Mathematical Association met under the presidentship of Mr. Balakram and the following business was transacted :—

(1) It was proposed by Prof. P. V. Seshu Aiyar and seconded by Prof. G. S. Chowla—that the Society records with extreme regret its deep sense of the loss sustained by India in the sad and untimely death of the late Mr. S. Ramanujan, B.A., F.R.S., at the very early age of 32 years.

The proposal was received most solemnly, all the members standing.

[1] Sardar Bahadur Sunder Singh Majithia's Address of Welcome is given on p. 90.

[2] The Opening Address of His Excellency is given on p. 92.

[3] The Hon. Secretary's Report is given on p. 93.

[4] Mr. Balakram's Address is given on p. 97.

(2) It was next proposed—that a befitting memorial, in a form to be determined by a Committee to be appointed hereafter, be instituted to perpetuate the memory of the late Mr. Ramanujan.

The proposal was moved by Prof. P. V. Seshu Aiyar and seconded and supported by Professors Samuel Lal and Mukundlal, and was carried unanimously.

During the discussion that followed the proposal the general consensus of opinion of the members present seemed to be in favour of founding a prize to be awarded every year, or once in two years according to the funds available, in open competition throughout the whole of India, for original mathematical work.

(3) It was then proposed by Prof. P. V. Seshu Aiyar and seconded by Prof. Mehr Chand Suri—that a Committee, consisting of the following gentlemen, with power to add, be formed to give effect to the formulation of such a scheme to found a Ramanujan Memorial Prize :—

Messrs. Balakram, M. T. Naraniengar, P. V. Seshu Aiyar, R. Ramachandra Rao, G. S. Chowla, Hemraj and D. D. Kapadia.

The proposal was accepted unanimously.

(4) The discussion afterwards turned on the subject-matter of Mr. Balakram's letter for the improvement of the Journal. The main points of Mr. Balakram's letter were to issue our Journal in two parts —one containing mathematical notes, short articles, problems, &c., forming the elementary section and the other containing original advanced contributions.

In opening this subject for discussion, Mr. Balakram informed the audience, that the members of the Committee were generally sympathetic with his views but some had indicated the practical difficulties in the execution of such a scheme, and he wished to know the opinions of those present on these points. An interesting discussion then followed in which Messrs. Hemraj, P. V. Seshu Aiyar, M. T. Naraniengar, C. V. H. Rao, Samuel Lal and others took part.

In winding up the discussion Mr. Balakram suggested that as the matter had been sufficiently well discussed, it might now be left to the Managing Committee to go into the details and put forward definite proposals in the matter.

(5) Mr. P. V. Seshu Aiyar had issued a circular letter to the Committee for the more efficient working of the Library—his principal suggestion being to establish branch circulating libraries at various Provincial centres. In connection with this, the various replies which had been received were put before the meeting, and the long letter containing nearly all the details of our Library from our Assistant Librarian Prof. V. B. Naik was also read in part. General unanimity was not to be found and there was nearly equality of opinions for and against such branch Libraries.

The question of the efficient working of the Library has been a moot point ever since the foundation of the Society, and it has been found hard to arrive at a generally satisfactory solution of this problem.

After some further general discussion, the meeting terminated at 2 P.M.

The Hon'ble Sardar Bahadur Sunder Singh Majithia, Chairman of the Reception Committee, was "At Home" to members of the Indian Mathematical Society and of the Punjab University Mathematical Association in the evening on the same day at 5 P.M., at his residence, and the function was a grand success. The Hon'ble Sardar Bahadur spared no pains to look to the comforts of the guests and the cool and shady recesses of the spacious garden lawns added greatly to the enjoyment. The visitors regaled themselves till dusk and parted with happy memories of the Sardar's hospitality. This terminated the first day's programme.

On the second day, at about 8 A.M., the regular work of the Conference commenced in the Law College Buildings, near the University Library under the Chairmanship of Mr. Balakram, the President of the Society. The following papers were read, and the Conference adjourned for the next day at about 10 A.M. [The sitting had to be suspended for a few minutes at 9 A.M. for a group photo of the members of the Conference] :—

1. Mutually Locked Tetrahedra—by Prof. C. V. H. Rao.
2. Normals to an Algebraic Curve—by Prof. Hemraj.

3. Generating Regions of a Quadric in space of n-Dimensions—by Mr. R. Vythynathaswami.

4. An Elementary Exposition of the allied properties of the Simson Line, the Sternier Envelope, and the Orthopole—by Mr. A. A. Krishnaswami Aiyangar. [Read by Mr. R. Vythynathaswami, in the absence of the author.]

5. Note on the area of a Triangle circumscribed about a Parabola—by Mr. M. K. Kewalramani. [Read by Mr. R. Vythynathaswami, in the absence of the author.]

6. Convergents to $\sqrt{2}$ in the solution of some Pythagorean Equations—by Messrs. K. B. Madhava and Charles Saldhana.

In the afternoon, at about 4 P.M., the members were taken to Shahdara in carriages specially provided for the purpose, and had the splendid opportunity of seeing the historic monuments of the Mausoleums of Jehangir and Nur Jahan. After light refreshments and a stroll in the gardens, the party returned to Lahore at 6 P.M.

In the evening at 7 P.M., a popular lecture under the auspices of the Conference was delivered by Mr. T. P. Bhaskara Sastri of Nizamiah Observatory, Hyderabad (Deccan), in the Hall of the Society for the Promotion of Scientic Knowledge. The lecture was on "Star-Motion" and was illustrated by a number of lantern slides. Mr. Balakram, President of the Society, presided on the occasion, and the lecture was listened to with great attention by a crowded audience. After a hearty vote of thanks to the lecturer, the audience dispersed at about 8 P.M.

On the third and last day, the Conference met again at 8 A.M. and the following papers were read :—

7. Dr. Stewart's Theorems in Non-Euclidean Geometry—by Mr. M. K. Kewalramani. [Read by Mr. R. Vythynathaswami.]

8. Theory of the Rational Transformation—by Mr. R. Vythynathaswami.

[This paper is an attempt to form an integral conception of the general rational transformation in one variable in its most characteristic features. The positive results established are not many ; in fact, more problems are raised than solved. In spite of its fragmentary character, the paper is believed to be of some value as it fixes certain general lines of enquiry in the subject of rational transformation.]

9. An Extension of Feuerbach's Theorem (Supplementary)—by Mr. M. Bhimasena Rao. [Read by Mr. K. B. Madhava.]

[Theorem : "Of the system of pedal-contact circles, each circle touches four circles, which latter are also touched by three more circles of the system."

The present paper contains the proof of the above theorem on the following lines :—

(1) Of the system of pedal circles of isogonal conjugate points whose join passes through a fixed point, each circle is touched by four and only four circles of the system.

(2) The common tangents at the points of contact touch a conic which is considered in some detail. This conic reduces in particular cases to the *ortho-polar conic* of a point.

(3) Each circle c_1 of the system is the pedal circle of a pair of isogonal conjugate points, say, (P_1, P'_1) and the four circles $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ which touch c_1 are the pedal circles of isogonal conjugate points, say (Q_r, Q'_r), $(r = 1, 2, 3, 4)$.

When the join of the isogonal conjugate points passes through the ortho-centre of the anti-medial triangle, the circles belong to the *pedal-contact* system. In this case, it is shown that the 8 Q-points form a pair of Steinerian quadruples.

It follows that the three remaining circles besides c_1 which touch a γ-circle are the pedal circles of points (P_r, P'_r), $(r = 2, 3, 4)$ forming with (P_1, P'_1) a pair of Steinerian quadruples.

(4) The three sides of a triangle and the nine-point circle may be taken to be c-circles when the γ-circles will be the in-and the ex-circles. This is the Feuerbach's theorem.

The relation between the P and Q points are under investigation.]

10. Mathematics in India, Then and Now—by Mr. V. Sankaran. [Extracts were read by Mr. D. D. Kapadia].

11. Some Diophantine Equations—by Mr. Balakram.

[Denoting by $f(a)$ the diophantine equation $x^2 - ny^2 = a$, where n is an integer without non-square factors, it is well-known that $f(1)$ can always be solved, and that a *necessary* (but not always *sufficient*) condition for $f(a)$ to have solutions is that $(-a)$ should be a quadratic residue of n. This note is written with the object of enquiring if the conditions *sufficient* to ensure the solution of $f(-1)$, $f(-2)$ and $f(+2)$ were known. It can be shown that if $n \neq 2$, not more than one of these equations has solutions ; and that if n is an odd prime or twice an odd prime (not 2), at least one has a solution. The question "which one ?" can be completely answered by elementary considerations if n is odd ; or is even and of the

forms $2(8k + 3)$, $2(8k + 5)$, $2(8k + 7)$. If $n = 2(8k + 1)$, the following results are obtained :—

Put, as can always be done, $\frac{n}{2} = \alpha^2 + 16\beta^2$, α being odd. Then

if β is odd, and $\alpha = 8p \pm 1$, $f(+2)$ has solutions ;
if β is odd, and $\alpha = 8p \pm 3$, $f(-1)$ has solutions ;
if β is even, and $\alpha = 8p \pm 3$, $f(-2)$ has solutions.

The remaining case (β, even, $\alpha = 8p \pm 1$ remains unsolved. Calculation shows that out of 28 values of n of this type up to 7522, $f(-1)$ has solutions in 10 cases, $f(+2)$ in 10 cases, and $f(-2)$ in 8 cases.]

12. On a Special Cubic—by Mr. Hemraj.

13. On the Summation of Series of the Type

$$1 + \frac{1}{3^2} - \frac{1}{5^2} - \frac{1}{7^2} + \ldots\ldots$$

by Mr. C. Krishnamachari. [Read by Mr. G. S. Chowla.]

14. The Herpolhode Theorem—by Mr. S. R. Ranganathan. [Read by Mr. Balakram.]

[It is well-known that the motion of a rigid body fixed at a point O and acted on by no forces or by forces whose resultant pass through O, is equivalent to the rolling of the Ellipsoid of Inertia of the body at O on a plane known as the Invariable Plane.

The locus, on the plane of the point of contact of the ellipsoid is called the herpolhode.

Now, if P is the point of contact at any instant, it is also the point of intersection with the plane of the instantaneous axis of rotation, so that OP is the instantaneous axis of rotation. It is, further, easily shown that OP represents in magnitude and direction the instantaneous angular velocity of the body. Let us represent this vector by ω.

Then, P is a point on the herpolhode and the tangent at P to the herpolhode must be co-directional with $\dot{\omega}$. For, both the directions coincide with the limiting position of PP', &c. ...

Further, if P is to be a point of inflection of the herpolhode, the direction of the tangent at P to the herpolhode must be stationary, *i.e.*, the change in the tangential vector $\dot{\omega}$ must be in its own direction, *i.e.*, ω and $\ddot{\omega}$ must be co-directional or in vectorial language $V\,\omega\,\ddot{\omega}$ must be zero. That this is impossible is easily derived from Euler's equations, as follows :

Now, the vectorial equivalent of Euler's equations

$$K\dot{\omega} = L - V\omega q$$

where L is the moment of the impressed forces about the fixed point O, q is the vector representing the moment of momentum of the system with reference to the point O, K is the linear vector operator or the inertia operator of the system with reference to O, representing the operations on ω indicated by the formula

$$K\omega = \omega\Sigma mr^2 - \Sigma m(\omega r)r$$

where the summation is extended over the whole of the rigid body and r is the vector connecting O to the element m.

K being independent of time and in the motion considered, L being zero, we have

$$K\dot{\omega} = -V\omega q = Vq\omega ;$$

so that

$$K\ddot{\omega} = \frac{d}{dt} Vq\omega$$

$$= Vq\dot{\omega}$$

since q is constant in the motion under consideration. Thus, it follows that $\ddot{\omega}$ is at right angles to the plane containing the vectors q and $\dot{\omega}$, so that it is at right angles to the vectors ω. Hence $\ddot{\omega}$ and $\dot{\omega}$ cannot be co-directional for any point of the herpolhode, that is, the herpolhode has no point of inflection.]

15. On some properties of Euler's and Prepared Bernoullian Numbers—by Messrs. C. Krishnamachari and M. Bhimasena Rau. [Read by Mr. K. B. Madhava].

[Form a Table thus :—

$1 . n.$	$3n+2.$	$3n^2+2n.$	$15n^2+20n+16.$	$15n^3+20n^2+16n.$
$2(n+1).$	$6n+8.$	$12n^2+18n+16.$		
$3(n+2).$	$12n+20.$			
$4(n+3).$	$15n+40.$			
$5(n+4).$				
.....				
.....				
$r(n+r-1).$				

The second column is obtained from the first by an obvious addition.

$3n + 2 = 1 . n + 2(n + 1)$; $6n + 8 = 3n + 2 + 3(n + 2)$; &c.

The third column is obtained by multiplying the second column by the first. The fourth by addition from the third. The fifth by multiplying the fourth by the first column as before. And so on. Then, it is proved that the quantities at the top in the various odd columns are the numerators of the co-efficients of the various powers of x in the expansion of $\sec^n x$.

Ex. $n = 1$. From a table with $1^3, 2^3, 3^3, \ldots$ in the first column, we get Euler's numbers.

$n = 2$. We get the prepared Bernoullians, viz. $2^{2n}(2^{2n}-1)\,\frac{Bn}{2n}$,

which are the co-efficients in $\sec^2 x$ $(= \frac{d}{dx} \tan x)$.]

At the conclusion of the reading of the papers, Prof. Devi Dayal, President of the Punjab University Mathematical Association, proposed that the best thanks of this Joint Sessions of the Indian Mathematical Society and the Punjab University Mathematical Association are due to His Excellency the Governor, the Vice-Chancellor, the University authorities, and the Hon'ble Members and Ministers for their sympathy, co-operation and financial help. Prof. Devi Dayal put this proposition before the audience in a short felicitous speech[1] and on being seconded by Mr. Balakram, it was carried with acclamation.

It was next proposed by Mr. Balakram and seconded by Prof. P. V. Seshu Aiyar that the best thanks of the Indian Mathematical Society are due to the Punjab University Mathematical Association for the generous manner in which they had taken upon themselves the onerous task of allowing the Society to hold so successfully its Conference at Lahore. Also similar thanks are due to His Excellency the Governor, the Vice-Chancellor, the Members of Council, and the Ministers of the Province for their kind co-operation in this matter; also to Profs. Devi Dayal, Das Gupta, Hemraj, Mehr Chand Suri, and to students and volunteers for their selfless help, without which the Society would not have been able to hold its Conference at Lahore. The proposition was carried unanimously.

Finally, with a hearty vote of thanks to the Chair, proposed by Prof. Devi Dayal and seconded by Prof D. D. Kapadia, the proceedings terminated.

In the afternoon at about 3 P.M., the visitors were taken over to the North-Western Railway's Workshops, nearly the biggest workshops in the whole of India. The inspection of these workshops on the extensive railway grounds was somewhat tiresome, but nevertheless it was most instructive and interesting. After mutual greetings, and hopes of meeting again on similar occasions at future conferences, members bade good-bye to one another.

[1] Prof. Devi Dayal's speech is given on p. 108.

Address of Welcome by the Hon'ble Sirdar Bahadur Sundar Singh Majithia.

"Your Excellency: When in the unavoidable absence of the Hon'ble Khan Bahadur Mian Fazl-i-Hussain, Minister of Elucation, Punjab, I was approached by my friends to take up the happy duties of welcoming Your Excellency and the distinguished gentlemen who come from such long distances to attend to-day's Conference of the Indian Mathematical Society, I was assured that no difficult sum would be set before me to do. I explained to my friends that when I left the Government College in 1893, I did so with a sigh of relief; not that I disliked staying in that institution where I had spent two happy years of my life, but on account of the subject which will be deliberated upon to-day by the present day Indian savants in that science. In the first few months of my College life I had the honour to sit at the feet of the well-known Babu S. B. Mukerji, a mathematician of some fame. I must frankly admit that he was beyond my poor grasp. He was a brilliant mathematician and it was hard for a young student like myself to scale the giddy heights to which he jumped omitting the intermediate steps without which it is hard for a beginner to proceed. Then I was placed in charge of the well-known Professor Mr. G. N. Chatterjee who wished his students to follow him with the speed of a mail train. I am not in any way speaking with disrespect of my revered Professors but of the difficulty that I personally had to contend with. The courses were finished in the first three months and then we were regaled with happy and pleasant stories of his wide travels which kept us enchanted and the pleasant memories of the period I find it hard to forget. Of course there was a revision of the subjects later on. To the above was added my school boys dread of the subject. So I am sure Your Excellency and the other distinguished gentlemen present here to-day will forgive me if, at the outset, I say that I find myself in an anomalous position as regards the subject-matter of to-day's Conference. In my final examination when the time drew near for me to sit for my Intermediate test, I was not at all sanguine of success and as I did not want to extend the period of my Court of Wards' supervision I approached my respected Professor, Mr. Chatterjee, for advice. He always found some of us sitting at our fixed places as permanent members of what he called the bench of honour. I put my difficulties before him and asked him how I could pass the examination. He asked me if I could cram up some portions of Euclid, Trigonometry and Conic sections. I told him I had no taste for Algebra but would do my best in the above subjects to carry out his kind advice. He in his humorous vein advised me to throw my Algebra book in the *chohla* and this I confess I did though not by actually throwing the book in the first place but by laying it aside never to be

opened again. I could do a little of Arithmetic, the additions and subtractions which came in very handy when I was Honorary Secretary of the Khalsa College and also in keeping my private accounts. In my Matriculation, I knew the four books of Euclid by heart. I could repeat them from beginning to end and I had not forgotten these. The remaining two or three books—I don't remember now how many they were—I also committed to memory. I still remember the formula of sin (A + B) = sin A cos B + cos A sin B, in Trigonometry. I confess I only remember by cram. I hope I have remembered and expressed it aright.

If not, then the distinguished audience will excuse me as I can assure them that since closing my book with a sigh of relief in 1893, I have not had an occasion to open it. My inability to grasp the subject lay in my own weakness: as a boy I was not very much attached to my subject and in my school days it was not so lucidly explained as I saw it explained when I was Honorary Secretary of the Khalsa College by Professor Nihal Singh. If it had been taught me in school in that way, my aversion to Algebra might not have been there. And I was not able to climb the lofty heights of Professor Mukerjee, or to keep pace with the mail train speed of Mr. Chatterjee—perhaps owing to my dulness in the subject.

All the same, I admit that the science of Mathematics is one which could not be ignored. Even the world is running smoothly in its mathematical precision in the courses chalked out or ordained for it by the Almighty. But for this, the land of Five Rivers, the home of the Rishis of old, would not produce the abundance of wheat which feeds the world. With the aid of this science, the Punjab rivers have been harnessed and the jungle tracts—once the habitation of the King of Forests—and the barren Bars of Lyallpur and Montgomery, the latter being called the Ganji Bar and where antelopes even died of thirst, are teeming with green fields all around. So I must say that though I am unfortunate in not having benefited myself and taken advantage of the opportunities that were afforded me in my College days in the personalities of the two eminent Professors at whose feet I had the honour to sit, but still I consider the subject one that we could not ignore. Your Excellency, we have many a difficult problem to solve these days and it would be out of place for me to mention those. But if the deliberations of my talented friends to-day solve the questions they have set before themselves correctly, which I have no doubt they will do, they will be adding to the knowledge of the world in a subject which I may safely say is indispensable to mankind. Let me further hope that my friends will apply their store of knowledge to the finding of a solution which will bring them nearer to their Maker.

If one could only have a ray of light from that store of all knowledge, the life in this toiling and moiling world will be a happy and a contented one and realising our relationship of the Fatherhood of God and the Brotherhood of Man, we shall lead a happy and a graceful life in this world.

Let me once more offer Your Excellency and my talented friends gathered together in this Hall from the four corners of my country, my most sincere and warm welcome, in the full hope that they will solve the problem of life successfully. I would now request Your Excellency to be so good as to declare the Conference open."

The Opening Address of His Excellency Sir Edward Douglas Maclagan, K.C.S.I., Governor of the Punjab.

"I need hardly say, gentlemen, that I greet the Members of this Conference with much warmth and I may add that I also do so with much awe. I am aware that the gentlemen who tried to teach mathematics in my youth would be much shocked if they could learn that I had attempted even to approach the threshold of a learned Conference such as this. I do not wish to discourage the examination of candidates for the public service in mathematics, but I am free to confess that I have myself managed to get through a longish service without using any other mathematics than arithmetic. I was indeed once ordered as a Census Officer to use a formula which my friends told me entailed the use of the Differential Calculus. I think it was to ascertain the average distance of one Post Office from another, or some similar useful information, and I am afraid that I was reduced to reporting somewhat evasively that the average distance was not very different to what it had been before. I am proud, however, to say that our services in the Province have from time to time included some admirable mathematicians, more especially among that fine body of Engineers who construct and maintain the celebrated system of Punjab Canals. We have also had mathematicians in the Civil Service, and we are just now arranging to give facilities to one of our best mathematicians in that service to devote some time to the scientific examination of the forecasting of crops.

"I am glad to see here to-day not a few of the best known mathematicians in India, and I feel that a great honour has been conferred on the city of Lahore by their visit. I do not know whether this part of India had any special claims to mathematical distinction before the era of Prof. Chowla and before we started the Honour Schools of Mathematics under a Professor with a European reputation. We have all of us however heard something of the long and honoured career of

Mathematics in India generally. I notice that a recent writer on exact science in India is of opinion that the pure mathematics of the ancient Hindus was on the whole not only in advance of the Greeks but anticipated in some remarkable instances the European discoveries of the 16th and 18th centuries. The origin of the decimal system is attributed to them; and they are said to be the real inventors of Algebra, and I am told that some very early Indian mathematical treatises contain spirited attempts to square the circle. With these historical foundations stretching so far below him, it is no wonder that the modern mathematician in this country has risen so high.

"We have not so long ago lost that brilliant genius, Mr. Ramanujan, but we have I hope others of similar calibre still among us, possibly among us here to-day.

"With the spirit of Brahmagupta and Bhaskara behind him, as well as the teachings of Laplace and Newton, the modern Indian mathematician has a start which he will doubtless worthily maintain, and I hope that the distinguished gentlemen who have met here to-day will by their mutual intercourse intensify the favour and success of studies in India.

"I could wish too that you would leave some of your enthusiasm behind you for our beneft in the Punjab. Speaking as a mere administrator, I am bound to say that our most crying practical need here is something far beneath your notice, namely, an improved knowledge by our peasants of the methods of calculating compound interest, but there are others and larger spheres of practical and intellectual value in which I trust that your presence among us may be found of much real use to the Province.

"I shall not detain you longer, gentlemen, and wish you every success in your Conference."

Report of the Hon. Jt. Secretary, Mr. D .D. Kapadia.

"Your Excellency and Gentlemen—

On behalf of the Committee of the Indian Mathematical Conference, it is my pleasant duty to extend to you all a very hearty and cordial welcome on the occasion of this—the Third Conference of the Indian Mathematical Society—and to lay before you a brief report of the work done by the Society during the last two years. The Society cannot but feel extremely thankful to you, both members and well-wishers, for having taken the trouble to be present at the Conference and contributing your individual quota to make it a success.

We consider ourselves extremely fortunate that on each occasion on which we held a Conference we have been uniformly successful in securing the kind and patronising co-operation of the Governors of the Provinces within whose jurisdiction we had the good fortune to meet. His Excellency Lord Pentland opened our First Conference at Madras in December 1916: His Excellency Sir George Lloyd was to open our Bombay Conference in Bombay, but owing to His Excellency's unavoidable absence, at his own express wish, Sir George Carmichael, the Senior Member of the Bombay Executive Council, performed the opening ceremony in January 1919, and to-day on the occasion of the Third Conference, we have been fortunate in having Your Excellency amongst us to perform the opening ceremony and set the seal of approval of the Government of the Punjab on the humble work which our Society has been trying to do for the last 14 years. Your Excellency's presence amongst us to-day will inevitably act as a strong stimulant to our efforts; and our Society cannot but feel highly thankful to Your Excellency for this kind condescension on your part.

When our Society was founded in 1907, it was but a very remote possibility to hold conferences. With great hesitation and diffidence, the first experiment was tried at Madras. The experiment turned out to be so very satisfactory, that two years later the Second Conference was held at Bombay, and after another two years we meet here to-day at Lahore. We feel very grateful to the Punjab University Mathematical Association for giving us this opportunity to meet here, as the Association have been doing the utmost in their power to make this Conference a success.

In the last report of the Committee before the Bombay Conference, the Committee had expressed themselves in some such words as these, *viz.*, "the dire war has now ended and peace is within sight and the humanity will soon be again taking a breath of relief. People will now have leisure and energy to devote to their domestic problems, and our Committee hoped to achieve some tangible results in the near future." These were our pious wishes. But the Committee regret very much to say that they have not been able to do what they so sanguinely hoped to achieve. The world-wide after-war reactionary movements had their dire effect even on the working of our peaceful Society, and beyond doing the actual routine work which the Society had been doing previously, nothing new of any very striking importance could be done during the last two years. Owing to printing difficulties, our journal (which is as some members are pleased to say, the living monument of the existence of our Society), had to go through several vicissitudes, and had it not been for the indefatigable exertions of our Editors, Messrs. M. T. Naraniengar and P. V. Seshu

Aiyar, it would have remained far more in arrears than what it was at present. With the beginning of this year, we have entered into the 15th year of our existence, and our journal commences its thirteenth volume.

Whsn we met last, the strength of the Society was 197 members—viz., 1 patron, 1 hon. member, 9 life members and 186 ordinary members. To-day our net strength is 215 members, viz., 1 patron, 1 hon. member, 11 life members and 202 ordinary members.

During the interval of two years, we have to record with regret among others, the sad death of H. H. the Maharaja Bhawinghji of Bhavnagar and Mr. S. Ramanujam, F. R. S. His late Highness the Maharaja of Bhavangar was along with His late Highness the Maharaja of Mourbhanj Baripada, one of the foremost life-members of our Society, and in the sad and untimely death of the late Mr. S. Ramanujan, F. R. S., at the early age of 32, India has lost a most talented Mathematician, whose place is very difficult to be filled up. Shortly after the late Mr. Ramanujan's return from England in April 1919, our Society elected him as an Hon. Member, but to the bitter disappointment and regret of the whole of the scientific world, he breathed his last in April 1920, and to-day our Society is going to discuss a proposal to commemorate his memory. But whatever the outcome of that proposal is going to be, the name of Ramanujan shall ever be cherished fondly by India and his loss shall ever be deeply mourned.

In the previous report, the Committee have already touched upon the important essentials wanting both in connection with our Journal and our Library and hence it is not thought advisable to speak at length on those subjects. The Society is well cognisant of its deficiencies in these directions. To-day immediately after the formal Sessions of our Conference are over, there will be a discussion at our general meeting of the various suggestions put forward by our President, Mr. Balak Ram, for the general improvement of the get-up of our journal and the subject-matter for our journal. It is suggested that our journal should be published in two parts—one Elementary, for the ordinary students of Mathematics, and the other Advanced for research scholars. When our Society was started in 1907, there was no other society of a similar type in existence in India. Following in its wake, the Calcutta Mathematical Society was started, and recently the Benares Mathematical Society was started at Benares. Also the Punjab University Mathematical Association, as also a similar Association at Madras are now in existence. The number of Indians with European qualifications in Mathematics, as well as the number of students going to Cambridge and Europe and America was then limited. But now the number is steadily increasing; and suffice it to say

that talented Indians are now taking a more vivid and keener interest in mathematics than was possible in those days and hence the original lines on which the Society has been doing its work may require some modifications and we are going to discuss to-day the ways and means by which the objects for which the Society has been originally started can be achieved better in the present times.

Our Library has also its similar deficiencies and the number of books in our Library is only 500, and the number of volumes of Mathematical Periodicals is only 375. The Library receives nearly 30 mathematical magazines from Europe and America mostly in single copies, and at present there are nearly on an average 22 demands for each magazine. If then each member passes on the magazine after keeping it in his possession only for 10 days, the last member would receive it after 210 days? But human nature is weak and prone to err and it is found on inquiries that hardly 10 per cent. of the magazines issued to the members during the last two years have yet come back to the Library. There are some very interesting statistics and the report of the working of the Library submitted by our Librarian, will also be placed before our meeting for information. Suffice it to say that our Library is suffering both for want of funds and for want of active co-operation on the part of our members. Unless we canvass sufficient funds, we shall not be able to buy back numbers of at least some of the important mathematical journals and also a good number of volumes on Advanced Mathematics facilitating mathematical research. Also to ensure quick circulation of current magazines we badly want some important journals in duplicate or more copies. In connection with this, our Committee feel great pleasure in mentioning that our present President, Mr. Balakram, has lately given us a handsome donation of Rs. 1,000 to have back numbers of some of the important journals and the best thanks of the Committee are due to him for the same. At the same time the Committee most earnestly hope that under the present regime, the Indian Ministers in charge of the Departments of Education in the various provinces will not fail to extend their help in the form of library grants to our Society and enable us to fulfil the objects which we have so fondly cherished up to now. To achieve these objects the Committee expect the very active and enthusiastic co-operation of all, both the old and the young members of the Society, more especially the latter on whom the future regeneration of India so intimately depends. The Committee, therefore, take this opportunity to make this strong appeal not only to the members of the Indian Mathematical Society but to all those patriotic Indians interested in the science of mathematics—one of the most ancient and the most useful of the sciences—to contribute their mite towards

the fulfilment of the objects for which the Society was primarily founded.

In conclusion, the Committee once again take this opportunity to record their high sense of appreciation for the trouble taken by Your Excellency to preside on this occasion, and to partake in the deliberations of this Conference in the midst of a multitude of pressing engagements. The Committee also take this opportunity to offer their best thanks to all those who have been ungrudgingly contributing their mite for the welfare of the Society, during these 14 years of its existence. I is also a very gratifying feature to note that some of our members have taken the trouble to undergo the fatigue of a tiresome journey of some 2900 miles to attend this Conference. The Committee take this opportunity to thank cordially these and all others who have worked so selflessly for the successful holding of this Conference at Lahore."

The Presidential Address by Mr. Balak Ram, M.A., I.C.S.

Mr. Balak Ram, M. A., I. C. S., President of the Indian Mathematical Society, delivered the following Presidential Address :

Gentlemen,—Our Society includes among its members a large number of men who were eager students of mathematics in their early days, but whose other occupations leave them hardly any leisure for keeping up their studies. They, however, retain their interest in the subject because of the pleasure they derived and still derive from it, and they are willing to do what little they can to help the Society, feeling honoured when the professional mathematicians among you appreciate their efforts, as for instance, by electing one of them as their President. But, however honoured I personally feel, I confess that I would have felt greater satisfaction if we had been able to carry out our original intention to elect as our President for the current term, Pofessor K. J. Sanjana, who has been one of the most energetic of our members since the Society was started fourteen years ago. We deplore his inability to accept office and trust he will soon be restored to health.

Another stalwart whom we miss to-day is Prof. A. C. L. Wilkinson, my immediate predecessor in office. He presided at the First and Second Conferences of the Society and he has given us freely of his experience and knowledge, first as a member of the Committee and later as President for four years from December 1916.

During the last year, we have lost the active co-operation, due to departure from the country, of Sir John Heaton. Those of us who attended the Conference in Bombay in January 1919, will agree with me that the success of that meeting was largely due to his efforts as the Chairman of the Reception Committee. Sir John Heaton was a member of the Indian Civil Service for nearly 37 years and rose to be a High Court Judge, which post he held for nearly 12 years before retirement in April 1920. At one time he was Vice-Chancellor of the Bombay University on which body he did useful work. His keen analytical mind, his deep and sympathetic knowledge of human nature, his complete freedom from communal and individual prejudice made him a great judge, and his capacity for loyal friendship and wise guidance won for him the affection and respect of all who knew him. He came to India at an early age, as was customary in the eighties, and never had an opportunity of going through a full course of mathematics at a University; but mathematics always fascinated him and he joined our Society on his own initiative as soon as he heard of it. We could always count upon him in time of need. We wish him peace and happiness in his retirement.

Another gentleman whom I mention by name is the Hon'ble Mr. R. P. Paranjpye, our first Honorary Member and our Librarian, who has been prevented from attending the Conference by pressure of his new duties. We congratulate him on being in a position where his powerful personality and abilities will have fuller scope than hitherto. His general career is well-known to all of you, but most people are not aware of one of his activities in pushing forward a scheme which I believe is destined to have an important effect on the future of this country. I refer to his advocacy of the method of Proportionate Representation to provide for communal and minority representation on public bodies. In India, the problem is acute; but the compromise solutions accepted so far are, I am afraid, neither acute nor right, but quite obtuse. It is interesting to note that Mr. Paranjpye's first observations on the subject were published as a short note in our Journal, under the heading "An Electoral Anomaly". After some years of study, he came to the conclusion that the system prevalent in several places in Europe, America and Australia, of Proportionate Representation with a single Transferable Vote gave as satisfactory a solution as was likely to be obtained as long as man was what he is. The working of the scheme is simple as far as the voter is concerned, but rather complicated when the votes are to be counted. A few weeks ago the scheme was adopted by the Council of State for election to their Committee and I have no doubt that one day it will be adopted for all purposes by all elective bodies. But when the scheme was first brought up before

one of the provincial legislatures three years ago, it was rejected with scorn, one indignant opponent calling it "Mr. Paranjpye's Mathematics". So I take it that the final acceptance of the scheme may be legitimately put on the credit side of mathematical ideas as supplied to administrative problems.

Probably some of the audience will be surprised to hear mathematics and administration mentioned together. It is true that the higher development of mathematics does not solve current difficulties of administration; but the same remark applies to ancient history, archæology, and most other branches of knowledge. What is useful here is the particular outlook which the study of science (especially mathematics) tends to produce with peculiar intensity,—a clear enunciation of the fundamental principles, a reduction of the number of those principles to a minimum, exact thinking, and a quantitative appreciation of the degree of exactitude attained by the accepted tentative solutions. These qualities make for success in all spheres; it is, therefore, not surprising that men with sound mathematical training have left their mark on non-mathematical subjects, such as philosophy and law. The contributions of mathematicians to the study of sociology, economics, and biological variations and heredity have been more direct, because these subjects deal with material in the mass and the science of statistics, (which is only another name for the mathematical theory of probability), deals with material in the mass. We all know the scoffer's jeer that statistics will prove anything. The fact is: *in the hands of an ignorant or an unscrupulous man* statistics *will* prove anything; just as in the hands of an ignorant or an unscrupulous man a surgeon's knife will kill anybody. Nor is there more truth in the celebrated classifications: "white lie, lie, statistics." In the hands of an expert, statistics lead to the discovery of unexpectedly accurate laws. An example probably known to you is the working formula obtained by Mr. Jacob, I.C.S., of the Punjab, for predicting the yield of crops. The most striking instance of the power of science in disentangling truth is furnished by Karl Pearson's analysis of mortality figures. He first determined the number of deaths in old age, in middle age, in youth and in childhood, and found them to follow the known laws of distribution of chance events. Eliminating these numbers, he was finally left with the curve of deaths in infancy. The curve was mathematically incomplete as if the initial part of it had been torn off. When completed algebraically, it was found to end at minus 9 months. The inference is that human life commences normally 9 months before civic birth,—a conclusion obvious enough when stated; but the point is that it was arrived at by practically sitting at the cemetery gate, and doing nothing but noting the ages of the dead people. Those who are interested in this investigation will find

full details of the method and the results in Karl Pearson's lecture on "Chances of Death" in his book of that name. The generally accepted opinion is that the only reliable method of finding out laws governing social phenomena is the collection and analysis of statistics. I venture to commend this view to the able administrators who are present here to-day.

Since we met last, we have lost our most prominent member and one of the greatest Indians of modern times. I refer to Srinivasa Ramanujan, whose death at the early age of 32 is mourned by the whole mathematical world. Details of his uneventful life and expert appreciation of his work have been published in scientific and other journals, and are probably known to you. To-day I need only express the unanimous opinion of the Society that we all loved Ramanujan and that India owes a deep debt of gratitude to Professors Hardy and Neville for generously and ably befriending him at a critical time of his career. The Society will, I understand, be asked at the business meeting to-day to consider the question of perpetuating his memory in some form.

Mention of Ramanujan's name suggests the question of the organization and endowment of scientific research in India. Now that the official educational policy is in the hands of Indians and there is an increasing number of eager and capable men anxious to make educational experiments outside the official bounds, it will be our own fault if we do not make our educational system as efficient as that of other countries. Progress has been made in technical education, and of late there has been talk of research institutions, but the research that is attended to is such as is of immediate practical value, which means research which pays dividends. Very few people will deny the claims of technical education; we all want physical comfort; we much prefer travelling in a luxuriously fitted motor car on a smooth dustless road to travelling in what the municipality humorously calls a third-class hackney carriage on a *kutcha* path; we also want the best beloved to wear pretty silks; and if the car is not comfortable enough or the silk not pretty enough, we are willing to set up institutions for improving mechanical engineering, sericulture and the dye industry. This as it should be in the beginning; but it is to be remembered that technical education provides the pupil with a direct means of livelihood. If the trained student becomes a simple manufacturer, he makes goods of the old pattern and sells them; if he, like Edison, is more intellectually inclined, he becomes an inventor and sells his inventions; in both cases, his food problem is solved, and with luck he may even become passing rich on £40,000 a year, if not a millionaire. What I wish to emphasize is that the worldly rewards manufacturers and inventors receive are a sufficient incentive to the pupils of technical institutions and

that after the initial stage is passed, the persons directly benefited by that education may well be required to pay for their training. The very different results following the pursuit of pure learning are vividly described in Browning's poem "The Grammarian's Funeral." There is no money to be earned, and the audience may be fit, but it is certainly "few," and not infrequently the work of a whole lifetime fails to score positive results. The desire for knowledge is, however, one of the primary instincts of the mind; therefore, we find scholars in all settled communities, men like Archimedes and Newton, the stories of whose absorption in their studies have been a tonic to generations of humbler students. Again, every settled community is sub-consciously or consciously convinced that attempts at progress are apt to be wasteful unless guided by thinkers and teachers who have leisure to think and teach clearly; therefore, we find these communities giving special protection and advice to learned men and thus providing them with leisure and the psychological incentives which replace the missing pecuniary reward. I speak of settled communities; but the most telling instance of this view-point is furnished by the rude barbarian Timur, about whom it is related that, when camping, the learned men (without whom he never travelled) were placed in positions of maximum security, while even the women and children had to be content with a less safe place. I do not know if Timur had a coherent theory behind his actions; if he had, he never explained it; he was not of that sort; but the modern man whose attitude towards learning is the opposite of Timur's and is symbolically represented by his thinking that Reading, Writing, and Arithmetic can be described by their initial letters as Three R's, makes no secret of *his* theory. "Research," he says, "is waste of time. Here I am, making piles of money out of sugar or diamonds or musk or whatever it is. What use is it to me, to learn about the stars or electrons or the differential calculus? Look at Professor K. Loy. He is bald and short-sighted and absent-minded, and his socks never match his ties. This comes of algebra and euclid and the fourth dimension." After this he talks patronizingly of schools and says that it is enough if a person is practical and has horse sense; and he further wants to know what those dashed dots are for.

Fortunately, power is passing out of the hands of this type of thinker and it is no longer necessary to explain that scientific research has a practical bearing. Difficulty, however, is experienced when mathematical research comes to be discussed, but there is sufficient evidence to show that though most results in pure mathematics are as a rule not applicable to any purpose of practical value, the work of a pure mathematician has frequently come to the help of physicists and others at unexpected moments. Students of probability know that the science of statistics to

which I have already referred, grew out of an academic study by pure mathematicians of games of chance. The laying of the Atlantic cable, it is said, would have been abandoned as an unprofitable task, if Lord Kelvin had not been able to solve a particular differential equation. Heavyside's skill in solving another differential equation enabled him to surmount a serious difficulty in long distance telephony. The latest instance of an apparently aimless research proving handy for physicists is furnished by Einstein's reconstructing the universe by utilizing Riemann's discoveries in non-euclidean geometry. I believe it is now generally admitted that during the late war pure mathematicians were able to render invaluable help in the development of aircraft, gunnery, and other matters. Instances might be multiplied; as a consolidated argument in favour of the proposition that research pays directly or indirectly, I refer to the example of the hard-headed practical American nation endowing professorships for theoretical research in such unpractical subjects as the Theory of Numbers.

This is our appeal to the moneyed man for endowment of mathematical research. To the mathematical student the study of the subject itself transcends in interest all practical applications. He studies the subject for the sake of the artistic pleasure he obtains in following its rigorous logic and in contemplating the extreme universality of the formulæ. Experience of life or nature gives us the data which the mathematician, instead of writing in English or Hindi, translates into his universally accepted symbols; but as soon as he has finished the work of writing down the symbols, he ignores for the moment their original significance and concerns himself with logical manipulation only. When he gets his final results, he translates them back into the language of the original data, leaving the physicist or the statistician or the engineer to verify the correctness of the data by further experiment. The mathematician is not concerned with the collection or the correctness of data. This aspect of the subject had led our most famous mathematical philosopher, Bertrand Russel, to describe mathematics as a science in which we neither know what we are talking about nor care whether what we say is true or not. The charm of mathematics lies in the fact that the description is accurate.

Whatever the point of view be, I earnestly plead for endowment of research in mathematics and in other sciences. I do not imagine that we shall discover a Ramanujan every decade; but I am sure there are enough people in the country who are willing to follow a career of quiet study and do respectable, if not highly distinguished, work, provided their minds are free from anxiety about the daily bread. I have only to mention the names of Bose, Roy; and (coming to younger men)

of Ganesh Prasad, V. Ramaswami Iyer, C. V. Raman, C. V. H. Rao, Birbal Sahni and Shivram Kashyap, to prove my assertion that there is indeed very promising human material in the country for work in all branches of science. Full opportunity alone is lacking.

I may take the liberty of giving a few suggestions of mine on the subject, not as an elaborated scheme, but as a basis of discussion. In the first place, it has to be noted that the research man cannot work to order any more than a Poet Laureate, even though fortified by an annual cask of wine. You must give the research man as much time as he wants and takes, and also must avoid as far as possible setting him a definite task. He has been well described as nature's loafer, he will neither be hurried nor commandeered, except in times of severe stress, as for instance during the Siege of Syracuse or the late war; and even then you do not get the best out of him unless you leave him alone. It is exceedingly improbable that Newton would have discovered the law of gravitation if he had been placed on special duty, with the usual deputation allowance of 2 %, to study the solar system within six months. Another point to be noted is that the research-man is human and has human needs and that you interfere with his efficiency if you do not provide him with a little comfort and free him from anxiety about the future of himself and his family. In addition to conducting his own research, he may be researchably required to discover his successors. For this purpose, he must be in constant touch with a number of selected students. The scheme that is suggested by these principles is that there should be research scholarships for the beginner and lectureships on a small pay for the student who has proved his worth; the lectureships should not belong to a graded service where the question of seniority or extension of service will cause heart-burnings and depression of mind; and their number should be large and variable, to provide for the appointment of young men without making the older men retire too early. To minimise distractions, there should be a fairly generous provision for individual family pensions and as an additional concession the children of research lecturers of a certain standing may be declared to be entitled to free education up to the highest standard for a certain specified number of years. The duties of the lecturer should be mainly research and control of research by the students, but he should be required to do routine teaching for two or three hours a week, as this would enable him to pick out likely men at an early stage. To allow the students themselves to focus their attention on research, the rules for the M. A. and higher examinations should be made more elastic. Finally, good libraries should be provided with complete files of all the leading scientific journals, and occasional visits by the lecturer to Europe and America should be encouraged.

The above is meant to be a rough sketch only; but it seems to me that anything worked on narrower lines will not give satisfactory results. And even then, it will not give sixteen annas worth for every rupee we spend, if the foundations of education are not properly laid in the school and the intermediate classes. I invite your attention to this urgent problem, and hope you with your experience of school teaching will devote your energies to solving it. The object to be attained is that the logical principles of the subject should be clearly taught to suitably selected boys and habits of clear and concise reasoning inculcated in all boys. Unless we do this, the educational value of mathematics in building up character will be lost.

Considering the wide interest taken by scientists and by laymen in EINSTEIN'S Theory of Relativity, it would not be out of place in a popular address to give a brief exposition of these speculations. The broad results of the theory are not essentially hard to understand; the difficulty, which is insuperable in many cases, consists in removing the old ideas firmly embedded in our minds in our younger days. It is not easy for any but the trained mathematician to realize that a euclidean straight line has no existence outside our conceptual world, and that there is no logical inconsistency in defining straight lines in such a manner that two straight lines can meet in two or three or a hundred distinct points. The euclidean straight line is simpler to argue about; but neither of them is really more straight than the other. Our children, trained differently will see this easily, as they will see that two and two are not always four, and will laugh at us as we laugh at our ancestors who thought the earth was flat. We, handicapped by our old notions, have to do the best we can by digging up the foundations of our mathematical beliefs and putting in as much of the new conceptions as we can. I therefore begin at the beginning.

Our knowledge of the world outside us is built out of our detached sense impressions, and the need for co-ordinating the knowledge establishing a chain of cause and effect arises because of the necessity of self-preservation; for instance, the organism that fails to establish the connection between the sensation of hunger and the effects of eating will die soon. The number of sense-impressions is large; our minds are finite; hence it soon becomes necessary to invent some shorthand formula for economising thought, each formula expressing some relation between two events (*e.g.*, the relation of before-and-after, here-and-there) in terms of an imagined entity (*e.g.*, time and space). When such entities become very well known, the human mind tends to endow them with objective reality. Philosophers may fight about the objective reality of time and space, but

when it comes to describing events or natural phenomena, we quietly ignore these controversies and argue as if time and space had an existence independent of the observer or the group of observers. Hence it came about that geometry was considered to embody absolute truth and many a weary soul found consolation in the thought that even on the farthest stars the three angles of a triangle are equal to two right angles. We have broken away from this faith. We know that geometry *as such*, being only pure mathematics, has no concern with absolute truths, and that the practical correctness of the conclusions is to be verified by the physicist who supplied the geometrician with the data (or postulates), such as the postulate that two intersecting straight lines cannot be parallel to the same line. If further experience of the universe gives a different set of postulates, the geometry that deals with them is different, and the conclusions are also different. It is the physicist who has to decide what sort of geometry best describes the universe. Let us consider the case of a race of beings living on this earth, with senses sharp enough to *see* (with the eye) the variations in the curvature of the surface in a small area but not possessing a worked out geometry. They will draw the shortest distances between two points and call them straight lines, and they will find that no two shortest distances if continued, remain a constant distance apart. They will also find things which fit on the surface of the earth in a smooth place will not fit in a rugged place. Their conception of space will be that it is not flat; and their geometry, when developed from these and other postulates will be a complicated non-euclidean geometry. They may, by a process of abstraction, be able to imagine a flat euclidean plane; but their space of experience (their space of physics) will be a non-euclidean plane.

This example will show from analogy that there is no reason why our space of physics should be euclidean. The difficulties created by electrodynamical phenomena led Einstein to enunciate a theory which was subsequently replaced by a more general theory in which the universe (consisting of space and time) is taken to be non-euclidean. This conception enabled him to dispense with the force of gravitation and to describe the movements of matter as always taking place along lines which were essentially straight; only the type of line which was straight in one place was not necessarily straight at another, as "matter" was nothing but a variation in space curvature altering the type of straightness in its neighbourhood. As a very crude and incomplete analogue in two dimensions of the theory, we may take a wash-hand basin with a sink to let out water. If we think of nothing in the universe except the wash-hand basin and the movement of drops of water in it, the sink will be like a centre of attraction and a drop of water placed on the inner surface of the basin will run to the sink along a shortest line, *i.e.*, along a straight line. But the drop of water will pass through the sink; and here the analogy fails as a particle of matter cannot pass through another.

The universe as imagined in Einstein's theory has four dimensions; and this is the second stumbling block in the path of the layman. The space of our everyday experience has three dimensions: length, breadth, and thickness, and we cannot frame a mental picture of a fourth dimension. To understand the theory, let us again go back to our sense-impressions which form the elements of our knowledge. We notice that every object (real or imaginary) has an existence in space and time, that is to say, it has extension (length, breadth and thickness or height) and duration, in all *four* variables. The mathematician expresses it by saying that every object, when its whole history is taken into account, has *four* dimensions; consequently the statement that the universe is four dimensional is only a new way of expressing a fact known even to cats and dogs and most of the higher animals. But—and here we come to the greatest discovery that the human race has ever made—the time dimension is essentially of the same sort as the three space dimensions, and the four are interchangeable, one man's time being compounded of another man's space and time. The three dimensional analogy is that the separate measures of the three space dimensions of a body change without the body changing in shape or volume. A brick placed on one face, and measuring one inch in the vertical direction, two inches in the northern direction, three inches in the eastern direction, will, if placed on a different face, and measuring 2, 3, 1 inches respectively in the three directions; and supported to stand on a corner the corresponding measures can be varied within wide limits. The length, the breadth and the thickness of the brick have no separate existence; and what exists is a totality of the three. How the totality is broken up by the senses depends upon the circumstances of the individual. Going a step higher, space and time also have no separate existence; what exists is a totality which the mind is not able to grasp as a whole, and consequently the sense breaks it up into four convenient parts, different for different observers. This is not a metaphysical speculation; this is a theory forced upon us by unexpected phenomena displayed by rapidly moving bodies, the details of which are too technical to be described here.

Two objections at once strike us: How can time and space be similar if we do not perceive them in the same way? And how is it that which in space we can move up and down and sideways, in time we can only move forward, and we do so move whether we want to or not? To see that the objections are not valid, we revert to our initial sense-impressions and note that the three dimensions of space are *not* perceived by the same senses. Two of the dimensions become known to us through the eye, when our sight is obstructed; the third dimensions—thickness or distance,—has to be discovered with the help of sight, touch, and muscular

effort and binocular vision. The fact is that the three space dimensions are *not* interchangeable for *all* purposes, and a mile up-hill is *not* the exact equivalent of a mile on the level. The difference is due to the structure of the human body, and it is only when by a mental effort we eliminate part of the personality of the observer that we can say that space is homogeneous. The perceptions of time and of space come to us through different sets of sense-impressions; but the discussion above shows that *that* does not necessarily indicate an essential difference, As regards the monotony and the apparent inevitability of movement in time, the explanation may be the same—our physical and mental circumstances.

We are now in a position to give some account of the picture of the universe that can be built out of Einstein's equations; this picture is not Einstein's, who steadily refuses to go beyond his equations and symbols; but the human mind is not satisfied with merely x and y and insists on knowing whether the symbols have an interpretation in the material universe. The physicist gives the interpretation, but he knows that it is only plausible being based on necessarily incomplete data, that the picture of the universe that emerges is only an approximation to reality,—if there be a reality,—and that the picture may have to be discarded when further data accumulate. Nevertheless, he provides the picture as a help to (his own and other people's) clear thinking; and the picture he gets out of Einstein is something like this. "There is a world-stuff spread in four dimensions, with a three dimensional continuum (our universe) as its section or boundary, much as the earth is stuff spread in three dimensions with a two dimensional surface as its boundary. The continuum, like the surface of the earth, is uneven, the unevenness of the former being of course in four dimensions and not in three like those of the earth. This unevenness we call matter. Our momentary view of the universe changes as if each point of it moved in the straightest line, and we call the change the passage of time". Why the universe seems to change and why the movement in the straightest line, we do, now know.

The theory leads to the result that space, time and matter are essentially one, but it leaves electricity out as a separate entity. Weyl has framed a new theory, based on a geometry more general than that of Riemann, and has included electricity among the property of space, or rather of the world-stuff. But we have not reached the goal in our search for a scientific monism. The entities we call life and mind still stand out of the mechanical scheme. The experimental method is, however, being applied rapidly to biology and psychology, and the next two hundred years might see the end of the isolation.

Prof. Devi Dyal's concluding Speech.

On behalf of the Punjab University Mathematical Association, it is my pleasant duty to offer you our heartiest thanks for coming to attend the Conference. I know that some of you have taken the trouble to travel over 2000 miles to contribute to the success of the gathering. It has indeed been a great pleasure and privilege to meet so many distinguished delegates from all parts of my country. It was only towards the end of February that the Indian Mathematical Society accepted our invitation and decided on holding its third Conference at Lahore. As soon as we learnt of it, we approached the Vice-Chancellor, the Chief Justice and both the Ministers; their sympathy, co-operation and financial help were generously given. In fact, our popular Vice-Chancellor, without whose active support, I fear, it would have been almost impossible to hold the Conference, moved the Syndicate to sanction a handsome donation of Rs. 250 to meet a part of the expenses. You have to-day befittingly recorded a resolution expressing your grateful thanks to him and to the Syndicate of the University.

We are equally grateful to H.E. the Governor who was pleased to accept our invitation to open the Conference. It is needless to say that the Conference felt itself highly honoured by His Excellency's presence at the opening session and flattered by his kindly welcome. Our best thanks are also due to the Hon'ble Khan Bahadur Mian Fazal Hussein, who kindly agreed to be the Chairman of the Reception Committee and to the Hon'ble Sardar Bahadur Sardar Sunder Singh Majithia who, in the unavoidable absence of the Educational Minister, performed his duties with cheerfulness, and invited the members to a splendid tea-party at his bungalow.

Gentlemen, it is thus evident that the success of the Conference has been due to the willing and cheerful co-operation of all. If we have failed to make satisfactory arrangements for your comforts, I request you to forgive us; for, this could not be due to lack of thought. I once more thank you for kindly visiting our province and infusing in us a spirit of mathematical study and research. You have shown us how wide is the field for work. Let me assure you that your visit will long live in our memory and our Association will never forget the honour done to it by your presence.

An Extension of Feuerbach's Theorem

By Mr. M. Bhimasena Rao,
Research Scholar, Central College, Bangalore.

Introductory.

IN the post-script to the paper on "An Extension of Feuerbach's Theorem" read by me at the last Conference of the Indian Mathematical Society and published in the Journal of the above Society for December 1919, I stated as below :—

The following property suggested by Feuerbach's Theorem may be found interesting. My proof is not complete :

> "Of the system of pedal-contact circles, each circle touches four circles which latter is also touched by three more circles of the system."

The proof of the above property is presented in this paper. It is based on the following theorems :—

(i) Of the system of pedal circles of pairs of isogonal conjugates of lines passing through any fixed point, the envelope of the radical axes of any circle and the remaining circles of the system is a conic.

(ii) Isogonal conjugate points on any four lines through the fixed point referred to in (i), which taken two by two belongs to three definite systems of involution, form a pair of Steinerian quadruples.*

The paper is divided into three sections. The first gives the properties of pedal-contact circles which are required for the purpose of this paper. The second section deals with the conic referred to in (i) above, and contains the proof of the main theorem. In the last section, a method of associating each tetrad of circles mentioned in the main theorem with a definite point is given and the relation between the two tetrads of tangent circles is established.

SECTION I.

The Pedal-Contact Circle.

§ 1. The pedal-contact circle is defined to be the pedal circle of a point P whose pedal triangle is in perspective with the triangle of reference ABC.[1] The join of P and its isogonal conjugate P′ passes through

* This result is stated on page 225, Vol. XI, Journal of the Indian Mathematical Society.

[1] Cf. Author's paper, Journal of the Indian Mathematical Society *loc. cit.*

the orthocentre of the antimedial triangle of ABC.[2] The locus of P is the self-isogonal cubic which passes through A, B, C, the circum-centre, the orthocentre, the in-and the ex-centres of ABC and cuts the sides of the triangle ABC again in the meets of AP, BP, CP, with BC, CA, AB, and the circum-circle of ABC in the points diametrically opposite to A, B, C. The circum-centre of ABC is also the centre of the cubic and its asymptotes are the perpendicular bisectors of the sides of ABC.[3] Its trilinear equation is

$$(\cos A - \cos B \cos C)\,\alpha(\beta^2 - \gamma^2) + (\cos B - \cos C \cos A)\,\beta(\gamma^2 - \alpha^2) + (\cos C - \cos A \cos B)\,\gamma(\alpha^2 - \beta^2) = 0.$$

§ 2. Let DEF be the pedal triangle of P and D′E′F′, that of P′, the isogonal conjugate of P. Let EF, and E′F′ intersect in X and EF′ and E′F in X′. If Y, Y′ and Z, Z′ be similar points obtained from the remaining sides of DEF and D′E′F′, the following points are collinear :—

(1) X′, Y′, Z′. (2) X′, Y, Z. (3) Y′, Z, X. (4) Z′, X, Y.

The lines of collinearity are evidently the pascal lines of the hexagons DE′FD′EF′, etc. This holds good for any conic cutting the sides of A, B, C in D, D′; E, E′; F, F′. The lines AX, BY, CZ are concurrent at a point π which is the pole of X′Y′Z′ with respect to the conic. Further if AD, BE, CF meet in a point Q, AD′ BE′ CF′ will meet in another point Q′; the pascal line X′Y′Z′ passes through Q and Q′ and the pascal lines X′YZ, XY′Z and XYZ′ respectively pass through A, B, C. The triangle XYZ will now be self-conjugate with respect to the conic.

The above results for conics are well-known. In applying them to the case of the pedal circle, we note that for any pedal circle the pascal line X′Y′Z′ passes through the isogonal conjugate points P and P′ and hence is a diameter of the pedal circle. The lines AX, BY, CZ are parallel and perpendicular to the line PP′X′Y′Z′,[4] the point π being now the point at infinity in direction at right angles to this line.

§ 3. It is due to the concurrency of AD, BE, CF, that the triangle XYZ becomes self-conjugate with respect to the circle. Hence we have :

(1) In the case of any circle DD′EE′FF′, the triangle XYZ is in perspective with ABC.

[2] Cf. Author's paper on Double Lines, page 125, Vol. XII, Journal of the Indian Mathematical Society.

[3] The geometrical treatment of this cubic forms the subject of a paper under preparation.

[4] For an elementary proof, cf. author's note on 'The Pedal Circle,' page 143 Vol. IV, Journal of the Indian Mathematical Society.

(2) If the circle is a pedal circle, the centre of perspective is at infinity.

(3) If the circle is a contact circle, the triangle is self-conjugate with respect to the circle.

The last of the results stated above is important. From this it follows that

(4) DX, EY, FZ, concur at a point W on the circle, and D'X, E'Y, F'Z concur at a point W' also on the circle.

(5) The line WW' is the triangular polar of π with respect to ABC.

The triangle XYZ is the common diagonal triangle of the quadrilaterals DEFW and D'E'F'W'. If DW and DW' cut YZ in R and R', the ranges (DX, RW) and (D'X, R'W') are harmonic. Therefore DD', *i.e.*, BC, RR', *i.e.*, YZ and WW' are concurrent. Similarly WW' passes through the meets of CA, AB, with ZX, XY respectively, etc.

(6) XYZ is the diagonal triangle of the quadrilateral formed by the sides of ABC and the line WW'.

§ 4. The following properties are special to the pedal-contact circle :—

(7) The points X, Y, Z, lie on the sides of the medial triangle of ABC.[5]

(8) The conic circumscribing the triangle ABC, and passing through Q and its isotomic conjugate is a rectangular hyperbola.[6]

(9) The points W,W' referred to in (4) are the centres of the rectangular hyperbolas ABCQ and ABCQ' and therefore lie on the nine-points circle of ABC.[7]

The main result of this section which is required for the proof of the theorem of this paper may be stated as below :—

The triangle XYZ is self-conjugate with respect to the pedal circle of P and also self-conjugate with respect to the in-conics of the quadrilateral formed by the sides of ABC and WW' (which is the radical axis of the nine-point circle and the pedal circle).

SECTION II.

The Proof of the Theorem.

§ 5. Let us consider the pedal circles of points on the self-isogonal cubic,

$$p\alpha(\beta^2-\gamma^2)+q\beta(\gamma^2-\alpha^2)+r\gamma(\alpha^2-\beta^2)=0. \quad \ldots \quad (1)$$

[5], [6], [7].—These results are taken from the paper referred to in foot-note ([2]).

The radical axis of any fixed circles of the system and any of the remaining circles touches a conic. In order to establish this, it is sufficient to prove that through any arbitrary point, two and not more than two of these radical axes pass. Given any point T, the power of T with respect to S is known, and the locus of a point whose pedal circle has the same power with respect to T is another self-isogonal cubic of the form

$$l\alpha\,(\beta^3 + \gamma^3) + m\beta(\gamma^3 + \alpha^3) + n\gamma(\alpha^3 + \beta^3) + 2s\,\alpha\beta\gamma = 0 \ldots \quad (2)$$

The cubics (1) and (2) intersect in 9 points of which three are the vertices of the triangle of reference ABC. The remaining six arrange themselves into three pairs of isogonal conjugate points of which one pair, namely, the points whose pedal circle is the fixed circle S, is known. Hence there are two pairs of isogonal conjugate points on (1) whose pedal circles with respect to T has the given power; that is to say, only two radical axes pass through T. The cubic (1) is the locus of a point which lies in a line with its isogonal conjugate and a given point (p, q, r). Hence we have the following results :—

(10) Of the system of pedal circles of pairs of isogonal conjugate points whose join passes through a fixed point, the envelope of the radical axes of any given circle and the remaining circles of the system is a conic [8].

This conic may be briefly denoted as the envelope-conic associated with the given pedal circle.

(11) A given pedal circle of the system touches *four* pedal circles of the same system.

This is evident from the above result. The tangents at the points of contact are the common tangents of the given pedal circle and the envelope-conic referred to in (10).

§ 6. Let J be the fixed point in (1) and through it draw a fixed line L and a variable line M. Since the radical axis of the pedal circles, say (L) and (M) of isogonal conjugate points on L and M touches a conic, there is a one-to-one correspondence between the line M and the corresponding radical axis. Hence we see that

(12) Lines through J correspond anharmonically with the radical axis of (L) and (M) and with their points of contact on the envelope-conic.

[8] The envelope of the radical axis of the pedal circle of the fixed point J and the remaining circles of the system is the orthopolar ellipse of J. I have got a more direct proof of the result (10) by the anharmonic properties of conics. This incidentally determines the position of the centre of the envelope-conic.

In the general case when J is an arbitrary point, a pedal circle (L) is touched by four pedal circles (M). Each of these latter is touched by 4 pedal circles of which the circle (L) is one. This result is not of much interest. Suppose that when J is a particular point we are able to prove that the isogonal conjugate points on the four lines M—say M, M_1, M_2, M_3 whose pedal circles touch any circle (L) of the system, form a pair of Steinerian quadruples. In this case, the lines L, L_1, L_2, L_3 corresponding to the circles (L), $(L)_1$, $(L)_2$, $(L)_3$, which touch one of the pedal circles (M) are such that the isogonal conjugate points on them form also a pair of Steinerian quadruples. When one point of a Steinerian quardruple is given, the remaining three are uniquely determined, the circles (L), $(L)_1$, $(L)_2$, $(L)_3$, are repeated for each of the 4 circles (M), $(M)_1$, $(M)_2$, $(M)_3$. Hence each of the four circles (L) will touch each of the four circles (M).

In order to establish the theorem, it is sufficient to prove that :—

> When J is the orthocentre of the anti-medial triangle of ABC, the lines M, M_1, M_2, M_3, corresponding to the line L are such that the isogonal conjugate points on them form a pair of Steinerian quadruples.

§ 7. The condition that the isogonal conjugate points on four lines M, M_1, M_2, M_3, passing through J form a pair of Steinerian quadruples is satisfied when the rectangular hyperbola through J intersects on these lines in points forming a qudrangle having ABC as the diagonal triangle, *i.e.*, when the following systems are in involution :—

$(MM_1,$	$M_2M_3,$	JB, JC)	 (A)
$(MM_2,$	$M_3M_1,$	JC, JA)	 (B)
$(MM_3,$	$M_1M_2,$	JA, JB)	 (C)

Lines drawn from J and through the points on the cubic forming a Steinerian quadruple taken two by two, belong to the above three systems, which may be denoted as the (A), (B), (C) involutions.

The (A) system is determined as follows :—

> Take any point U on the cubic. Let AU produced meet the cubic again in U_1. Then JU and JU_1 are conjugate rays of the (A) system.

Similarly for the (B) and (C) systems.

§ 8. When J is the orthocentre of the anti-medial triangle of ABC, the cubic passes through H, the orthocentre of ABC, and AH, BH, CH are parallel to the asymptotes of the cubic. If $J_a J_b J_c$ denote the perpendiculars on the sides of ABC and J_n denote the Euler line JONH (O circumcentre, N-nine-point centre of ABC), the rays (J_nJ_a), (J_nJ_b), (J_nJ_c), are conjugate rays of the (A), (B), (C) systems respectively, as also the rays (J_bJ_c), (J_cJ_a), (J_aJ_b). Hence the conditions under consideration may be written as :—

$(MM_1,\ M_2M_3,\ J_nJ_a,\ J_bJ_c,)$ (A)
$(MM_2,\ M_3M_1,\ J_nJ_b,\ J_cJ_a)$ (B)
$(MM_3,\ M_1M_2,\ J_nJ_c,\ J_aJ_b,)$ (C)

The whole question has been reduced to one of showing that the M-lines taken two and two are conjugate rays of the pencils in involution determined by the four lines, viz. the perpendiculars from J on the sides of ABC and the Euler line OH of ABC, taken two and two. This is the form best suited for our purpose.

§ 9. The envelope-conic associated with the line L touches the line circles of the pedal-contact system, *viz.*, the side of ABC, and the radical axis WW′ of the pedal circle of isogonal conjugate points on L, which we have denoted by (L), and the nine-point circle. Let α, α_1, α_2, α_3 be the points of contact of the envelope-conic with WW′ and the sides of ABC. By the result stated at the end of the first section, we see that the triangle XYZ is self conjugate with respect to the envelope-conic. Therefore XYZ is the diagonal triangle of the quadrangle $\alpha\alpha_1\alpha_2\alpha_3$. Let the points of contact of the (L) circle with the M circles be β, β_1, β_2, β_3. The common tangents at these points touch the envelope conic. If γ, γ_1, γ_2, γ_3 are the points of contact on the envelope-conic, we see by a known theorem in conic sections, that XYZ is the diagonal triangle of each of the quadrangles $\beta\beta_1\beta_2\beta_3$ and $\gamma\gamma_1\gamma_2\gamma_3$, since it is self-conjugate with respect to the circle (L) and the envelope-conic. Thus we see that the five quadrangles DEFW, D′E′F′W′, $\alpha\alpha_1\alpha_2\alpha_3$, $\beta\beta_1\beta_2\beta_3$, $\gamma\gamma_1\gamma_2\gamma_3$, have a common diagonal triangle XYZ.

Consider the α- and γ-points on the envelope-conic. It is seen that the lines

$\alpha\alpha_1,\ \alpha_2\alpha_3,\ \gamma\gamma_1,\ \gamma_2\gamma_3,$ intersect in X, ... (1)
$\alpha\alpha_2,\ \alpha_3\alpha_1,\ \gamma\gamma_2,\ \gamma_3\gamma_1,$ intersect in Y, ... (2)
$\alpha\alpha_3,\ \alpha_1\alpha_2,\ \gamma\gamma_3,\ \gamma_1\gamma_2,$ intersect in Z. ... (3)

Therefore the points in (1), (2), (3) form three involutions. According to the correspondence stated in § 6 of this section, the α-points correspond anharmonically with WW′ and the sides of ABC and these latter with the lines J_n, J_a, J_b, J_c. The γ-points correspond with the M-lines. It is evident that the pencils

$(MM_1,\ M_2M_3,\ J_nJ_a,\ J_bJ_c)$
$(MM_2,\ M_3M_1,\ J_nJ_b,\ J_cJ_a)$
$(MM_3,\ M_1M_2,\ J_nJ_c,\ J_aJ_b)$

form three involutions. Hence the condition obtained in § 8 of this Section is satisfied and consequently the theorem is established.

SECTION III.

§ 10. In the preceding section, we have shown that a given pedal-contact circle (L) is touched by four similar circles (M), which latter are

uched by three more circles associated with the given circle. We shall w examine the relation between the (L) and (M) circles more closely.

The pedal-contact circles may be arranged into two sets, each set ntaining groups of four pedal circles of points forming a Steinerian uadruple, a group of the first set corresponding with that group of the cond whose members touch the circles of the former group. Any circle a group of either set determines not only the other members belonging the group, but also the circles of the corresponding group of the other t. Hence corresponding groups may be taken to belong to either set, at is to say, the correspondence is mutual. If we can establish a (1, 1) rrespondence between each group and points on a line, the corresponding oints will form an involution. This may be done as follows :—

Let J be the point (l, m, n), where $l = \cos A - \cos B \cos C$, $= \cos B - \cos C \cos A$, $n = \cos C - \cos A \cos B$. The locus of sogonal conjugate points on lines through J lie on the cubic,

$$l\alpha(\beta^2 - \gamma^2) + m\beta(\gamma^2 - \alpha^2) + n\gamma(\alpha^2 - \beta^2) = 0.$$

If P be any point on this cubic, its associated points of the Steinerian uadruple are given by the following construction :—

Let JP meet in R, the first polar of J with respect to the cubic, *viz.*, he rectangular hyperbola.

$$l^2(\beta^2 - \gamma^2) + m^2(\gamma^2 - \alpha^2) + n^2(\alpha^2 - \beta^2) = 0 \quad \ldots \quad (1)$$

nd AQ, BQ, CQ intersect the rectangular hyperbola in Q_1, Q_2, Q_3. If P′ is the isogonal conjugate of P, then AP′, BP′, CP′ intersect respectively Q_1, JQ_2, JQ_3, in the required points, say P_1, P_2, P_3, lying on the cubic. t will be seen therefore that the pedal circles of P, P_1, P_2, P_3 are associated espectively with the lines JQ, JQ_1, JQ_2, JQ_3. The rectangular hyperola (1) is self-conjugate with respect to the triangle ABC. The co-rdinates of the Q-points may be taken to be $(x, \pm y, \pm z)$, satisfying the equation $\Sigma l^2(y^2 - z^2) = 0$.

The four Q points may evidently be associated with a single point R $(px \quad qy^2, rz^2)$, where p, q, r are arbitrary constants.

The locus of R is the line

$$l\left(\frac{\beta}{q} - \frac{\gamma}{r}\right) + m^2\left(\frac{\gamma}{r} - \frac{\alpha}{p}\right) + n^2\left(\frac{\alpha}{p} - \frac{\beta}{q}\right) = 0;$$

i.e.

$$\frac{\alpha}{p}(m^2 - n^2) + \frac{\beta}{q}(n^2 - l^2) + \frac{\gamma}{r}(l^2 - m^2) = 0. \quad \ldots \quad (2)$$

The nine-point-circle and the sides of the triangle form a group for which the Q-points are $-\left(\frac{1}{a}, \pm\frac{1}{b}, \pm\frac{1}{c}\right)$, and R is $\left(\frac{p}{a^2}, \frac{q}{b^2}, \frac{r}{c^2}\right)$. The in- and the ex-circles form the corresponding group for which the

Q-points are $(1, \pm 1, \pm 1)$ and the R′ is (p, q, r). R′ obviously lies on (2) and R also lies on (2) by virtue of the relation

$$\frac{l}{\cos A - \cos B \cos C} = \frac{m}{\cos B - \cos C \cos A} = \frac{n}{\cos C - \cos A \cos B}.$$

To determine the involution to which R, R′ belong, it is necessary to find two more such points. For the group to which the pedal circle of J belongs R is (pl^2, qm^2, rn^2), which evidently lies on (2). To find the corresponding group, bisect internally and externally the angles BJC, CJA, AJB. If the bisectors meet the sides BC, CA, AB in X, X′; Y, Y′; Z, Z′ the pedal circles of isogonal conjugate points on the perpendiculars from J to X′Y′Z′, X′YZ, Y′ZX, Z′XY, from the corresponding group [9]

The co-ordinates of R′ corresponding to this group are found to be

$$\frac{p(2\cos^2 B + 2\cos^2 C - \sin^2 A)}{\sin^2 A}, \quad \&c., \quad \&c.$$

The involution is completely determined by the conjugate points

$$(p, q, r); \left(\frac{p}{\sin^2 A}, \frac{q}{\sin^2 B}, \frac{r}{\sin^2 C}\right);$$

and $$(pl^2, qm^2, rn^2); \left\{\frac{p(2\cos^2 B + 2\cos^2 C - \sin^2 A)}{\sin^2 A}, \ldots\ldots\right\}.$$

Since these points are collinear, we have

$$\frac{p + \frac{kp}{\sin^2 A}}{pl^2} = \frac{q + \frac{kq}{\sin^2 B}}{qm^2} = \frac{r + \frac{kr}{\sin^2 C}}{rn^2}, \quad \ldots \quad (1)$$

and $$\left(p + \frac{k'p}{\sin^2 A}\right) = \frac{p}{\sin^2 A}(2\cos^2 B + 2\cos^2 C - \sin^2 A)$$

$$= \left(q + \frac{k'q}{\sin^2 B}\right) = \frac{q}{\sin^2 B}(2\cos^2 C + 2\cos^2 A - \sin^2 B)$$

$$= \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

From (1) and (2), we find that

$$k = -\frac{\sin^2 A \sin^2 B \sin^2 C}{2\cos A \cos B \cos C},$$

and $$k' = -4\cos A \cos B \cos C;$$

ann hence the constant of involution is given by

$$kk' = 8\sin^2 A \sin^2 B \sin^2 C.$$

The foci of the involution are real. Corresponding to eac! focus, there are therefore four lines through J such that the four pedal circles of isogonal conjugate points on the lines touch one another.

[9] *Vide*: Author's note on page 105, Vol. IV, Journal of the Ind. Math. Soc.

SOLUTIONS.

Question 927.

(M. K. KEWALRAMANI)—A B C D E is a polygon whose sides taken in order are a, b, c, d, e.....If the angles which the sides b, c, d, e...make with the positive direction of the side a, be respectively called ab, ac, ad, ae.....prove that

$$a^n = \sum \frac{n!}{r!\,s!\,t!\,\ldots}\, b^r\, c^s\, d^t \left(\cos \left\{ r.\,ab + s.\,ac + t.\,ad + \ldots \right\} \right)$$

where n is a positive integer and r, s, t are also positive integers including zero but such that $r + s + t + \ldots = n$ and Σ denoting summation of like expressions when r, s, t ... are taken in their entirety subject to the above restriction; of course the right-hand side a not occurring.

Solution by N. Sankara Aiyar.

It can easily be seen that

a = real part of $\Sigma\, b\,(\cos ab + i \sin ab)$ the summation extending to the $(n - 1)$ sides but a. Hence a^n = real part of $\{ \Sigma b\,(\cos ab + i \sin ab) \}^n$. The typical term in the expansion of this expression is given by the multinomial theorem in the form

$\frac{n!}{r!\,s!\,t!\,\ldots}\, b^r\, c^s\, d^t \ldots (\cos ab + i \sin ab)^r\, (\cos ac + i \sin ac)^s\, (\cos ad + i \sin ad)^t \ldots$ where $r + s + t + \ldots = n$.

i.e., $\frac{n!}{r!\,s!\,t!\,\ldots}\, b^r\, c^s\, d^t \ldots (\cos r\,.\,ab + i \sin r\,.\,ab) \times (\cos s\,ac + i \sin s\,.\,ac)\, (\cos t\,.\,ad + i \sin t\,.\,ad) \times \ldots$

i.e., $\frac{n!}{r!\,s!\,t!\,\ldots}\, b^r\, c^s\, d^t \ldots [\cos \{ r\,.\,ab + s\,.\,ac + t\,.\,ad + \ldots \} + i \sin \{ r\,.\,ab + s\,.\,ac + t\,.\,ad + \ldots \}]$.

By summing up the real parts of all the terms we get the required result.

Question 929.

(S. MALHARI RAO):—Express 2595600 as the product of two factors such that the sum of the factors of either factor may be equal to the other factor.

Solution by N. Sankara Aiyar.

The sum of the factors of any number of the form $\alpha^p\, \beta^q\, \gamma^r \ldots$ =

$$\frac{(\alpha^{p+1} - 1)\,(\beta^{q+1} - 1) \ldots}{(\alpha - 1)\,(\beta - 1) \ldots}$$

If AB = 2595600 = $2^4 \times 5^2 \times 3^2 \times 7 \times 103$ and 103 is a factor of A, 104 should be a factor of B. This is clearly impossible. If however the question is:—'the sum of the factors of either factor may be the same,' we get a solution. 3^2 should be a factor of B. So also should 7 be. Similarly 2^4 may be a factor of A or B and then 5^2 will be a factor of B or A. The two solutions are thus $5^2 \times 103$, $2^4 \times 7 \times 3^2$ and $2^4 \times 103$, $5^2 \times 7 \times 3^2$.

Question 1121.

(N. Durai Rajan):—Constant lengths a are measured along the tangents to a curve. If the locus of the extremities be a circle of radius a, show that the curve is the inverse of an involute of a circle of radius $2a$, the radius of inversion being equal to $2a$.

Solution by Martyn M. Thomas and others.

Let a length PQ ($= a$) be measured along the tangent at P to the curve required.

Then, if P be (x, y), Q will be $(x+a\cos\psi, y+a\sin\psi)$, where ψ is the angle made by the tangent with the x-axis.

Since Q lies on a circle of radius a,

$$(x+a\cos\psi)^2 + (y=a\sin\psi)^2 = a^2.$$

$$\therefore \quad x^2+y^2+2a(x\cos\psi + y\sin\psi) = 0$$

$$\therefore \quad x^2 + y^2 + 2a\left(x\frac{dx}{ds} + y\frac{dy}{ds}\right) = 0.$$

$$\therefore \quad -\frac{2s}{a} = \frac{2(xdx + ydy)}{x^2 + y^2} = d[\log(x^2+y^2)] = d[2\log r].$$

$$\therefore \quad -\frac{s}{a} = 2\log r + \text{const.}$$

Differentiating with respect to θ,

$$-\frac{ds}{d\theta} = 2a \cdot \frac{1}{r}\frac{dr}{d\theta}.$$

Squaring $r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{4a^2}{r^2}\left(\frac{dr}{d\theta}\right)^2$.

$$\therefore \quad \frac{dr}{d\theta} = \frac{r^2}{\sqrt{4a^2 - r^2}},$$

which is the polar differential eqn. of the reqd. curve.

Now, the (p, r) equation of an involute of a circle of radius $2a$ is

$$r^2 = p^2 + 4a^2. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Since for inverse curves, $rr' = k^2$ and $\frac{p}{r} = \frac{p'}{r'}$, change

$$r \text{ to } \frac{k^2}{r} \text{ and } p \text{ to } \frac{k^2 p}{r^2}$$

in eqn. (1).

Thus the inverse of the involute of the circle of radius $2a$ is

$$\frac{k^4}{r^2} = \frac{k^4 p^2}{r^4} + 4a^2.$$

Putting $k = 2a$, $r^2 = p^2 + \frac{r^4}{4a^2}$.

$$\therefore \quad \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \equiv \frac{1}{p^2} \equiv \frac{4a^2}{r^2(4a^2 - r^2)}.$$

$$\therefore \quad \left(\frac{dr}{d\theta}\right)^2 = \frac{r^4}{4a^2 - r^2}.$$

Hence $$\frac{dr}{d\theta} = \frac{r^2}{\sqrt{4a^2 - r^2}},$$

which is the same as the polar differential equation found above.

Question 1124.

(G. S. M.) :—Prove that

$$1 + (r, 1)\frac{n}{m} + (r, 2)\frac{n(n-1)}{m(m-1)} + \ldots \text{ to } (n+1) \text{ terms}$$
$$= \frac{(m+1)(m+2)\ldots(m+r)}{(m-n+1)(m-n+2)\ldots(m-n+r)}$$

where $(r, 1)$, $(r, 2)$... are the figurate numbers of the rth order.

Solution by K. J. Sanjana, V. M. Gaitonde and others.

The given left side may be written more generally thus :—

$$1 + {}^nC_1\frac{r}{m} + {}^nC_2\frac{r(r+1)}{m(m-1)} + \ldots,$$

where n is a positive integer, but r, m are any numbers.

We have by Van der Monde's theorem,

$(x + y)_n = x_n + {}^nC_1 x_{n-1} y_1 + {}^nC_2 x_{n-2} y_2 + \ldots$ to $(n + 1)$ terms.

Hence $$\frac{(x+y)_n}{x_n} = 1 + {}^nC_1\frac{y}{x-n+1} + {}^nC_2\frac{y(y-1)}{(x-n+1)(x-n+2)} + \ldots,$$

where x and y are any numbers. Put $x = -m + n - 1$, $y = -r$, and we get

$$\frac{(-m+n-1-r)_n}{(-m+n-1)_n} = 1 + {}^nC_1\frac{-r}{-m} + {}^nC_2\frac{-r(-r-1)}{-m(-m+1)} + \ldots$$
$$= 1 + r\,.\,\frac{n}{m} + \frac{r(r+1)}{1\,.\,2}\,\frac{n(n-1)}{m(m-1)} + \ldots \text{ to } (n+1) \text{ terms.}$$

Also $$\frac{(-m+n-1-r)_n}{(-m+n-1)_n} = \frac{(m+r)(m+r-1)\ldots(m+r-n+1)}{m(m-1)\ldots(m-n+1)}$$

where there are n factors both in the numerator and the denominator.

When r is a positive integer, $\frac{r}{1}$, $\frac{r(r+1)}{1\,.\,2}$, ... are the second, third, ... figurate numbers of the rth order; and we can cancel factors above and below from $m \equiv (m + r - r)$ to $(m - n + r + 1)$, and thus obtain the given result.

QUESTIONS FOR SOLUTION.

1173. (S KRISHNASWAMI AIYANGAR):—If $w_1 + w_2 + w_3 = 0$, show with the usual notation in elliptic functions $s_1c_1d_1 + s_2c_2d_2 + s_3c_3d_3$
$= -(s_1s_2s_3)^{-3}\,[k^4 - (s_1s_2s_3)^2\,(1 - c_1c_2c_3)\,k^2 - 2\,(s_1s_2s_3)]^2$.

1174. (S. KRISHNASWAMI AIYANGAR):—If $w_1 + w_2 + w_3 = K$, show that
$$s_1c_1d_1 + s_2c_2d_2 + s_3c_3d_3$$
$$= \left(\frac{2 - k^2}{k'}\right) c_1c_2c_3 + 3k^2\, s_1s_2s_3\, c_1c_2c_3 - k^4.$$

1175. (M. K. KEWALRAMANI):—Shew that the area of any part of the curve $y(x + m)(x - m) = n$, depends on the rectification of a certain parabola, and find out the general form of curves whose areas will be constant multiples of the corresponding parabolic length, the same parabola being employed in each case. Construct the above curve geometrically by means of the parabola.

1176. (A. NARASINGA RAO):—In Basset's "*Cubic and Quartic Curves*"* it is stated that the orthoptic locus of an Epi-cycloid is an Epi-trochoid. Show that Basset's Epi-trochoid is but part of the complete locus, which consists of $\frac{m+2}{2}$ such epi-trochoids, or $\frac{m+1}{2}$ trochoids and a circle repeated $\frac{m+1}{2}$ times, according as m (the ratio of the fixed to the rolling circle) is even or odd.

Prove also that the orthoptic locus splits up into algebraically distinct curves, whenever a rational curve of class λ has $\tan\frac{\psi}{\lambda}$ as a possible parameter, and that this happens only when all the tgts. from the circular points to the curve, coalesce.

1177. (A. NARASINGA RAO):—If the tangents at P and P^1 of a plane curve meet at right angles at O, show that the curvature of the orthoptic locus at O is $\frac{2c^2 - Pr^1 - P^1r}{c^3}$ where P and P^1 are the radii of curvature at P and P^1, $PO = r$, $P^1O = r^1$ and $PP^1 = c$.

1178. (A. NARASINGA RAO):—Trace the curve whose equation in intrinsic co-ordinates is

(i) $S = e^{a\psi} \sin b\psi$.

(ii) $S = e^{a_1\psi} \sin b_1\psi + e^{a_2\psi} \sin b_2\psi$.

* Pages 213 and 214.

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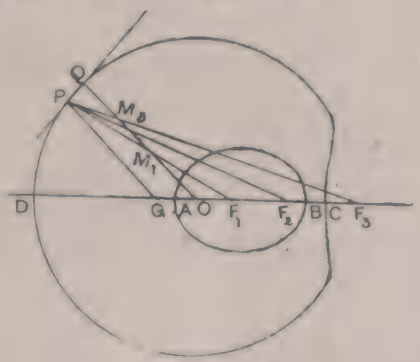
ON THE CARTESIAN OVAL (*Second Paper.*)

By V. Ramaswami Aiyar, M.A.

Trigonometric Analysis.

1. Let O be the triple focus and F_1, F_2, F_3 the single foci of a cartesian oval (Γ), in regard to which we shall generally follow the notation in the previous paper (*J. I. M. S.*, Vol. XI, pp. 123-144.)

Let P be a point on the oval of parameter λ and let θ_1, θ_2, θ_3, denote the angles that F_1P, F_2P, F_3P, make with the axis pointed leftwards, that is, with F_1O, F_2O, F_3O. The trigonometric analysis of these angles leads to a number of interesting results.



2. The cosines of the angles θ_1, θ_2, θ_3 can be found from the triangles OF_1P, OF_2P, OF_3P, whose sides are known. For OF_1, OF_2, OF_3 are a, b, c, which we may speak of as the *focal lengths* of the oval. The side $OP = \sqrt{bc + ca + ab + 2\lambda}$ (Section III, *First Paper*). The lengths F_1P, F_2P, F_3P, are ρ_1, ρ_2, ρ_3, when P is on the outer oval, where $\rho_1 = \sqrt{bc} + \lambda/\sqrt{l}$, &c. If P be on the inner oval, the

lengths are $\rho_1, -\rho_2, -\rho_3$ (Section II, *First Paper*.) On account of this difference, our formulæ will come to differ for the two ovals. But we can avoid this by a simple convention, that is, we shall regard $\surd a$ (the square root of the smallest focal length) to be intrinsically negative for the inner oval. To put it more fully, while $\surd a$ $\surd b$, $\surd c$, are all taken positive for the outer oval, we shall regard $\surd a$ as negative, while $\surd b$ and $\surd c$ are positive, for the inner oval. Further, we shall understand $\surd ab$ as short for $\surd a \surd b$, and $\surd abc$ as short for $\surd a \surd b \surd c$, and so on. On this understanding it is clear that the lengths F_1P, F_2P, F_3P, taken positively, are ρ_1, ρ_2, ρ_3, for both the ovals ; and we obtain

$$\left.\begin{aligned} \cos\theta_1 &= \left(\frac{\lambda^2}{abc} + a - b - c\right) \div 2\rho_1 \\ \cos\theta_2 &= \left(\frac{\lambda^2}{abc} + b - c - a\right) \div 2\rho_2 \\ \cos\theta_3 &= \left(\frac{\lambda^2}{abc} + c - a - b\right) \div 2\rho_3. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1)$$

3. From these equations we find

$$\left.\begin{aligned} \sin\tfrac{1}{2}\theta_1 &= \tfrac{1}{2}\sqrt{\frac{V_1V_4}{\rho_1}}\,;\ \cos\tfrac{1}{2}\theta_1 = \tfrac{1}{2}\sqrt{\frac{V_2V_3}{\rho_1}}\,; \\ \sin\tfrac{1}{2}\theta_2 &= \tfrac{1}{2}\sqrt{\frac{V_2V_4}{\rho_2}}\,;\ \cos\tfrac{1}{2}\theta_2 = \tfrac{1}{2}\sqrt{\frac{V_3V_1}{\rho_2}}\,; \\ \sin\tfrac{1}{2}\theta_3 &= \tfrac{1}{2}\sqrt{\frac{V_3V_4}{\rho_3}}\,;\ \cos\tfrac{1}{2}\theta_3 = \tfrac{1}{2}\sqrt{\frac{V_1V_2}{\rho_3}}, \end{aligned}\right\} \quad \ldots \quad (2)$$

where V_1, V_2, V_3, V_4 denote four new linear functions of λ as follows :—

$$\left.\begin{aligned} V_1 &= \frac{\lambda}{\surd abc} + \surd b + \surd c - \surd a \\ V_2 &= \frac{\lambda}{\surd abc} + \surd c + \surd a - \surd b \\ V_3 &= \frac{\lambda}{\surd abc} + \surd a + \surd b - \surd c \\ V_4 &= \surd a + \surd b + \surd c - \frac{\lambda}{\surd abc}. \end{aligned}\right\} \quad \ldots \quad (3)$$

Hence we get

$$\tan\tfrac{1}{2}\theta_1 = \sqrt{\frac{V_1V_4}{V_2V_3}},\quad \tan\tfrac{1}{2}\theta_2 = \sqrt{\frac{V_2V_4}{V_3V_1}},\quad \tan\tfrac{1}{2}\theta_3 = \sqrt{\frac{V_3V_4}{V_1V_2}} \quad \ldots \quad (4)$$

We also get

$$\left.\begin{aligned} 4\rho_1 &= V_1 V_4 + V_2 V_3 \\ 4\rho_2 &= V_2 V_4 + V_3 V_1 \\ 4\rho_3 &= V_3 V_4 + V_1 V_2. \end{aligned}\right\} \quad \dots \quad (5)$$

Let Y denote the perpendicular from P on the axis.

Then $Y = \rho_1 \sin\theta_1 = 2\rho_1 \sin\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta_1,$

which gives $Y = \tfrac{1}{2}\sqrt{V_1 V_2 V_3 V_4} \quad \dots \quad \dots \quad \dots \quad (6)$

4. It is desirable to note at this stage that the V's (like the ρ's) are all positive, for points on the outer, as well as, the inner oval. For, from equations (2), since the ρ's are all positive, it is seen at once that the V's are all of one sign. And this sign can be seen to be positive by examining that of V_4, say, for each of the ovals.

5. From formulæ (2), we next find that

$$\left.\begin{aligned} \sin\tfrac{1}{2}(\theta_2 - \theta_3) &= \tfrac{1}{2}\sqrt{\frac{V_1 V_4}{\rho_2 \rho_3}}\,(\sqrt{c} - \sqrt{b}) \ ; \\ \cos\tfrac{1}{2}(\theta_2 - \theta_3) &= \tfrac{1}{2}\sqrt{\frac{V_2 V_3}{\rho_2 \rho_3}}\,(\sqrt{c} + \sqrt{b}) \ ; \\ \text{hence} \quad \tan\tfrac{1}{2}(\theta_2 - \theta_3) &= \sqrt{\frac{V_1 V_4}{V_2 V_3}} \cdot \frac{\sqrt{c} - \sqrt{b}}{\sqrt{c} + \sqrt{b}} \ ; \end{aligned}\right\} \quad \dots \quad (7)$$

with similar expressions for the sines, cosines and tangents of $\tfrac{1}{2}(\theta_1 - \theta_3)$ and $\tfrac{1}{2}(\theta_1 - \theta_2)$.

Hence we get

$$\left.\begin{aligned} \frac{\tan\frac{1}{2}(\theta_2 - \theta_3)}{\tan\frac{1}{2}\theta_1} &= \frac{\sqrt{c} - \sqrt{b}}{\sqrt{c} + \sqrt{b}} \\ \frac{\tan\frac{1}{2}(\theta_1 - \theta_3)}{\tan\frac{1}{2}\theta_2} &= \frac{\sqrt{c} - \sqrt{a}}{\sqrt{c} + \sqrt{a}} \\ \frac{\tan\frac{1}{2}(\theta_1 - \theta_2)}{\tan\frac{1}{2}\theta_3} &= \frac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}} \end{aligned}\right\} \quad \dots \quad (8)$$

Thus the expressions on the left continue constant as P moves on the oval.

6. We next proceed to prove the important relations

$$\frac{Y}{\sqrt{\rho_1 \rho_2 \rho_3}} = \frac{\sin\frac{1}{2}(\theta_1 + \theta_2 + \theta_3)}{\frac{\lambda}{\sqrt{abc}}} = \frac{\sin\frac{1}{2}(\theta_2 + \theta_3 - \theta_1)}{\sqrt{a}}$$

$$= \frac{\sin\frac{1}{2}(\theta_3 + \theta_1 - \theta_2)}{\sqrt{b}} = \frac{\sin\frac{1}{2}(\theta_1 + \theta_2 - \theta_3)}{\sqrt{c}} \quad \dots \quad (9)$$

Let $A = \sin \frac{1}{2}\theta_1 \cdot \cos \frac{1}{2}\theta_2 \cdot \cos \frac{1}{2}\theta_3$

$B = \cos \frac{1}{2}\theta_1 \cdot \sin \frac{1}{2}\theta_2 \cdot \cos \frac{1}{2}\theta_3$

$C = \cos \frac{1}{2}\theta_1 \cdot \cos \frac{1}{2}\theta_2 \cdot \sin \frac{1}{2}\theta_3$

$D = \sin \frac{1}{2}\theta_1 \cdot \sin \frac{1}{2}\theta_2 \cdot \sin \frac{1}{2}\theta_3.$

Then we have

$$\sin \tfrac{1}{2}(\theta_1 + \theta_2 + \theta_3) = A + B + C - D,$$
$$\sin \tfrac{1}{2}(\theta_2 + \theta_3 - \theta_1) = -A + B + C + D,$$
$$\sin \tfrac{1}{2}(\theta_3 + \theta_1 - \theta_2) = A - B + C + D,$$
$$\sin \tfrac{1}{2}(\theta_1 + \theta_2 - \theta_3) = A + B - C + D,$$

Calculating their values by (2), we get

$$A, B, C, D = \tfrac{1}{8}\sqrt{\frac{V_1 V_2 V_3 V_4}{\rho_1 \rho_2 \rho_3}}\ (V_1, V_2, V_3, V_4), \text{ respectively.}$$

Therefore we have

$$\sin \tfrac{1}{2}(\theta_1 + \theta_2 + \theta_3) = \tfrac{1}{8}\sqrt{\frac{V_1 V_2 V_3 V_4}{\rho_1 \rho_2 \rho_3}}\ (V_1 + V_2 + V_3 - V_4)$$

$$\sin \tfrac{1}{2}(\theta_2 + \theta_3 - \theta_1) = \tfrac{1}{8}\sqrt{\frac{V_1 V_2 V_3 V_4}{\rho_1 \rho_2 \rho_3}}\ (V_2 + V_3 + V_4 - V_1)$$

&c.

But from equations (3), we get

$$\frac{\lambda}{\sqrt{abc}} = \tfrac{1}{4}(V_1 + V_2 + V_3 - V_4,)$$
$$\sqrt{a} = \tfrac{1}{4}(V_2 + V_3 + V_4 - V_1,) \text{ &c.}$$

Also $$Y = \tfrac{1}{2}\sqrt{V_1 V_2 V_3 V_4}.$$

Hence we get

$$\sin \tfrac{1}{2}(\theta_1 + \theta_2 + \theta_3) = \frac{Y}{\sqrt{\rho_1 \rho_2 \rho_3}} \cdot \frac{\lambda}{\sqrt{abc}}$$

$$\sin \tfrac{1}{2}(\theta_2 + \theta_3 - \theta_1) = \frac{Y}{\sqrt{\rho_1 \rho_2 \rho_3}} \cdot \sqrt{a}, \text{ &c.}$$

which proves the relations stated.

7. The first application we shall make of these equations will be as follows. From O let us draw a line making with the axis pointed leftwards an angle equal to $\frac{1}{2}(\theta_1 + \theta_2 + \theta_3)$ and cutting F_1P, F_2P, F_3P at M_1, M_2, M_3. The line will evidently cut F_1P, F_2P, F_3P at

the angles $\frac{1}{2}(\theta_2 + \theta_3 - \theta_1)$, $\frac{1}{2}(\theta_3 + \theta_1 - \theta_2)$ $\frac{1}{2}(\theta_1 + \theta_2 - \theta_3)$. Now let us see into what parts the radius vector F_1P, for example, is cut at M_1.

From the triangle OF_1M_1 we have

$$\frac{M_1F_1}{F_1O} = \frac{\sin \frac{1}{2}(\theta_1 + \theta_2 + \theta_3)}{\sin \frac{1}{2}(\theta_2 + \theta_3 - \theta_1)}$$

$$= \frac{\frac{\lambda}{\sqrt{abc}}}{\sqrt{a}} = \frac{\frac{\lambda}{\sqrt{bc}}}{a}.$$

But $F_1O = a$. Hence $M_1F_1 = \lambda/\sqrt{bc}$; and thence we get

$$P_1M_1 = \sqrt{bc}.$$

The first property of the triple focal perpendicular on the tangent enables us at once to identify the line. It is *itself* the triple focal perpendicular on the tangent at P.

We have thus proved the *fourth* property of the triple focal perpendicular on the tangent, namely, it makes with the axis pointed leftwards an angle equal to one-half of the sum of the angles that the focal vectors F_1P, F_2P, F_3P make with the axis pointed leftwards.

We further see that it cuts F_1P, F_2P, F_3P at the angles $\frac{1}{2}(\theta_2 + \theta_3 - \theta_1)$, $\frac{1}{2}(\theta_3 + \theta_1 - \theta_2)$, $\frac{1}{2}(\theta_1 + \theta_2 - \theta_3)$.

8. Draw the normal at P cutting the axis in G. We see that the normal GP makes with the axis pointed leftwards an angle equal to $\frac{1}{2}(\theta_1 + \theta_2 + \theta_3)$.

Hence it follows that confocal cartesians intersect orthogonally.

Further we see that if a point P describe a cartesian oval the angular velocity of the normal at P is one-half of the sum of the angular velocities of the three focal vectors of P.*

9. Let Q be the foot of the perpendicular from O on the tangent at P; and let $PQ = t$. From the triangles PM_1Q, PM_2Q, PM_3Q, we get

$$\frac{t}{\sqrt{abc}} = \frac{\sin \frac{1}{2}(\theta_2 + \theta_3 - \theta_1)}{\sqrt{a}} = \&c. \ldots \qquad \ldots (10)$$

That is, we have the term $\frac{t}{\sqrt{abc}}$ added to the list of equal quantities in (9).

* This property holds for bi-circular quartics, as it holds for the bifocal conics and the parabola, with reference to the number of focal vectors possessed in each case, and leads at once to formulæ for the radius of curvature.

In particular, we have

$$\frac{t}{\sqrt{abc}} = \frac{Y}{\sqrt{\rho_1\rho_2\rho_3}} \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

We shall presently meet with many more terms equal to these in value and, for convenience sake, we shall use the letter M to denote any one of them. Using the value of Y, we have

$$M = \tfrac{1}{2}\sqrt{\frac{V_1V_2V_3V_4}{\rho_1\rho_2\rho_3}}. \quad \ldots \quad \ldots \quad (11\text{-}a)$$

For the truth of the results (10) and (11) for the inner oval, we should regard t as intrinsically negative, like $\sqrt{a}$, for that oval.

10. Let us consider our results in connection with the confocal cartesian oval (Γ') passing through P. Its triple focus O′ will be a point in OF_3 produced. Let us denote its focal lengths F_1O', F_2O', F_3O' by a', b', c'. The smallest of them is c'. Let the parameter of P in regard to Γ' be λ'; and let the distance OO′ between the triple foci be denoted by δ, so that we have $a + a' = b + b' = c + c' = \delta$. The θ's of the point P in regard to Γ' are the supplements of the θ's we are dealing with. Hence, noting that Y, ρ_1, ρ_2, ρ_3 mean the same geometrical quantities in connection with P for each of the confocals, we get from (9)

$$M = \frac{Y}{\sqrt{\rho_1\rho_3\rho_2}} = \frac{-\cos\frac{1}{2}(\theta_1 + \theta_2 + \theta_3)}{\frac{\lambda'}{\sqrt{a'b'c'}}} = \frac{\cos\frac{1}{2}(\theta_2 + \theta_3 - \theta_1)}{\sqrt{a'}}$$

$$= \frac{\cos\frac{1}{2}(\theta_3 + \theta_1 - \theta_2)}{\sqrt{b'}} = \frac{\cos\frac{1}{2}(\theta_1 + \theta_2 - \theta_3)}{\sqrt{c'}}, \quad \ldots \quad (12)$$

which adds four more to our list of equivalent terms.

11. Collating now the results

$$M = \frac{\sin\frac{1}{2}(\theta_2 + \theta_3 - \theta_1)}{\sqrt{a}} = \frac{\cos\frac{1}{2}(\theta_2 + \theta_3 - \theta_1)}{\sqrt{a'}}$$

we get, by ratios, $M = \dfrac{1}{\sqrt{a} + \sqrt{a'}}$; that is, $M = \dfrac{1}{\sqrt{\delta}}$ $\quad \ldots \quad (13)$

which makes an important addition to the set.

Taking this result with $M = Y/\sqrt{\rho_1\rho_2\rho_3}$ we see that the distance between the triple foci of the two confocals is given by

$$\delta = \frac{\rho_1\rho_2\rho_3}{Y^2} \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

12. Taking (13) with $M = \frac{t}{\sqrt{abc}}$, we get

$$t = \sqrt{\frac{abc}{\delta}} \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

Observing that t is equal to the perpendicular from O on the normal at P, as (15) holds for all the eight points of intersection of the two confocals and the right-hand side is the same for all such points, it follows that t is the same for all such points and we have the theorem that the normals to the given cartesian Γ at its eight points of intersection with any confocal Γ' all touch a circle X whose centre is the triple focus O.

The radius of this circle is t which, from the value above obtained is

$$\sqrt{\frac{OF_1 . OF_2 . OF_3}{OO'}}.$$

Hence [from (43), *First Paper*], we infer that the circle X has double contact with the confocal Γ'.

Make now Γ and Γ' change roles. We infer that the tangents to Γ at its eight points of intersection with any confocal cartesian Γ' all touch a circle X' whose centre is the triple focus of Γ' and which has double contact with the given cartesian Γ *

13. We next proceed to a noteworthy geometrical theorem in regard to the centre of curvature of the oval at P.

Let PG be the normal and S the centre of curvature at P. Then [by (58), *First Paper*] we have

$$G P : S P = (\lambda + t)^2 : \lambda$$

$$\therefore \quad G S : S P = t^2 : \lambda.$$

Now $G O = abc/\lambda$ [by (28), *First Paper*]

and $OO' = abc/t^2$, by (15).

$$\therefore \quad G O : OO' = t^2 : \lambda.$$

Hence we get $G O : O O' = G S : S P,$

whence O S is paralled to P O'. That is, we have the theorem :

In any cartesian oval the line joining the centre of curvature at any point P to the triple focus is parallel to the line joining P to the triple focus of the confocal passing through P.

13·1. In close connection with this we have another theorem, as follows.

* See Prof. Wilkinson's Paper, *J. I. M. S.* Vol. XI, p. 172.

If Q, Q_1, are the projections of O, O' on the tangent at P, then $PQ.QQ_1 = \lambda$, the parameter of P.

Proof. Since GP, OQ and $O'Q_1$ are parallel, we have

$$PQ : QQ_1 = GO : OO'.$$

Hence $$PQ : QQ_1 = t^2 : \lambda.$$

But $$PQ = t. \qquad \therefore\ QQ_1 = \lambda/t.$$

$$\therefore \qquad PQ.QQ_1 = \lambda.$$

Remembering that $\lambda = PM_1.\ M_1F_1$, &c, this fixes Q_1. Thence we are enabled to fix O' the triple focus of the confocal through P ; and thence S the centre of curvature at P, by the foregoing theorem.

14. We next proceed to prove the following relations from which we can derive a good account of Genochhi's Theorem. The relations are

$$\left.\begin{aligned}
\sqrt{b}.\cos\theta_2 + \sqrt{c}.\cos\theta_3 + \sqrt{a}\cos(\theta_2 - \theta_3) &= \frac{\lambda}{\sqrt{abc}} \\
\sqrt{c}.\cos\theta_3 + \sqrt{a}.\cos\theta_1 + \sqrt{b}\cos(\theta_1 - \theta_3) &= \frac{\lambda}{\sqrt{abc}} \\
\sqrt{a}.\cos\theta_1 + \sqrt{b}.\cos\theta_2 + \sqrt{c}.\cos(\theta_1 - \theta_2) &= \frac{\lambda}{\sqrt{abc}}
\end{aligned}\right\} \qquad (16)$$

Proof. These will be true provided they are true when for $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ and $\frac{\lambda}{\sqrt{abc}}$ we substitute $\sin\frac{1}{2}(\theta_2 + \theta_3 - \theta_1)$, $\sin\frac{1}{2}(\theta_3 + \theta_1 - \theta_2)$ $\sin\frac{1}{2}(\theta_1 + \theta_2 - \theta_3)$ and $\sin\frac{1}{2}(\theta_1 + \theta_2 + \theta_3)$ which are proportional to them. On substitution, the results are seen to become pure trigonometric identities.

15. To deduce Genochhi's Theorem, we require some development of the system of quantities equivalent to M.

We saw that

$$M = \frac{t}{\sqrt{abc}} = \frac{\sin\frac{1}{2}(\theta_2 + \theta_3 - \theta_1)}{\sqrt{a}}$$

$$= \frac{\sin\frac{1}{2}(\theta_3 + \theta_1 - \theta_2)}{\sqrt{b}} = \frac{\sin\frac{1}{2}(\theta_1 + \theta_2 - \theta_3)}{\sqrt{c}},$$

Hence, by ratios, and use of the identical relations

$$\sin^2(x + y) = \sin^2 x + \sin^2 y + 2\sin x \sin y \cos(x + y)$$

$$\sin^2(x - y) = \sin^2 x + \sin^2 y - 2\sin x \sin y \cos(x - y)$$

we get

$$\left.\begin{aligned} M = \frac{t}{\sqrt{abc}} &= \frac{\sin\theta_1}{(b + c + 2\sqrt{bc}\cos\theta_1)^{\frac{1}{2}}} \\ &= \frac{\sin\theta_2}{(c + a + 2\sqrt{ca}\cos\theta_2)^{\frac{1}{2}}} \\ &= \frac{\sin\theta_3}{(a + b + 2\sqrt{ab}\cos\theta_3)^{\frac{1}{2}}} \\ &= \frac{\sin(\theta_2 - \theta_3)}{(b + c - 2\sqrt{bc}\cos\theta_2 - \theta_3)^{\frac{1}{2}}} \\ &= \frac{\sin(\theta_1 - \theta_3)}{(c + a - 2\sqrt{ca}\cos\theta_1 - \theta_3)^{\frac{1}{2}}} \\ &= \frac{\sin(\theta_1 - \theta_2)}{(a + b - 2\sqrt{ab}\cos\theta_1 - \theta_2)^{\frac{1}{2}}} \end{aligned}\right\} \quad \ldots (17)$$

which may be considered to give so many expressions for t in terms of the θ's severally, and of their differences severally.

16. Now let P be a point of the outer oval and let the arc DP be equal to s. Then $\theta_1, \theta_2, \theta_3$ are the angles subtended by this arc at the foci.

When s varies, we have

$$\frac{d\lambda}{ds} = -t.$$

This is the little equation we got in (25), *First Paper*, the negative sign coming in here as the arc is measured from D, and not C.

From the relations (16), taking differentials, and noting $d\lambda = \frac{d\lambda}{ds} \cdot ds$, we get

$$\left.\begin{aligned} &\sqrt{b} \cdot \sin\theta_2 \cdot d\theta_2 + \sqrt{c} \cdot \sin\theta_3\, d\theta_3 + \\ &\quad \sqrt{a}\sin(\theta_2 - \theta_3) \cdot d(\theta_2 - \theta_3) = \frac{t}{\sqrt{abc}} \cdot ds. \\ &\sqrt{c} \cdot \sin\theta_3\, d\theta_3 + \sqrt{a}\sin\theta_1 \cdot d\theta_1 + \\ &\quad \sqrt{b}\sin(\theta_1 - \theta_3) \cdot d(\theta_1 - \theta_3) = \frac{t}{\sqrt{abc}} \cdot ds. \\ &\sqrt{a} \cdot \sin\theta_1\, d\theta_1 + \sqrt{b}\sin\theta_2\, d\theta_2 + \\ &\quad \sqrt{c}\sin(\theta_1 - \theta_2) \cdot d(\theta_1 - \theta_2) = \frac{t}{\sqrt{abc}} \cdot ds. \end{aligned}\right\} \quad \ldots (18)$$

Substitute now for $\sin\theta_1$, $\sin\theta_2$, $\sin\theta_3$, $\sin(\theta_2 - \theta_3)$, $\sin(\theta_1 - \theta_3)$, $\sin(\theta_1 - \theta_2)$ and t, in the above, the quantities proportional to them

forming the denominators in the relations (17). We get

$$\left.\begin{aligned} ds = {} & \sqrt{b}\,(c + a + 2\sqrt{ca}\cos\theta_2)^{\frac{1}{2}}\,d\theta_2 \\ & + \sqrt{c}\,(a + b + 2\sqrt{ab}\cos\theta_3)^{\frac{1}{2}}\,d\theta_3 \\ & + \sqrt{a}\,(b + c - 2\sqrt{bc}\cos\theta_2 - \theta_3)^{\frac{1}{2}}\,d(\theta_2 - \theta_3). \\ ds = {} & \sqrt{c}\,(a + b + 2\sqrt{ab}\cos\theta_3)^{\frac{1}{2}}\,d\theta_3 \\ & + \sqrt{a}\,(b + c + 2\sqrt{bc}\cos\theta_1)^{\frac{1}{2}}\,d\theta_1 \\ & + \sqrt{b}\,(c + a - 2\sqrt{ca}\cos\theta_1 - \theta_3)^{\frac{1}{2}}\,d(\theta_1 - \theta_3). \\ ds = {} & \sqrt{a}\,(b + c + 2\sqrt{bc}\cos\theta_1)^{\frac{1}{2}}\,d\theta_1 \\ & + \sqrt{b}\,(c + a + 2\sqrt{ca}\cos\theta_2)^{\frac{1}{2}}\,d\theta_2 \\ & + \sqrt{c}\,(a + b - 2\sqrt{ab}\cos\theta_1 - \theta_2)^{\frac{1}{2}}\,d(\theta_1 - \theta_2) \end{aligned}\right\} \quad \ldots (19)$$

Hence, by integration, we have

$$\begin{aligned} s = {} & \int_0^{\theta_2} \sqrt{b}\,(c + a + 2\sqrt{ca}\cos\theta)^{\frac{1}{2}}\,d\theta \\ + {} & \int_0^{\theta_3} \sqrt{c}\,(a + b + 2\sqrt{ab}\cos\theta)^{\frac{1}{2}}\,d\theta \\ + {} & \int_0^{\theta_2 - \theta_3} \sqrt{a}\,(b + c - 2\sqrt{bc}\cos\theta)^{\frac{1}{2}}\,d\theta \end{aligned} \quad \ldots (20)$$

and two similar results, proving Genochhi's theorem that the arc of the Cartesian can be expressed in terms of three elliptic arcs; and we see that there are three ways of doing so.

17. By a slightly different treatment we can obtain an expression for s in which the integrals involved are symmetric in θ_1, θ_2, θ_3, but which involves an extra term. The first relation in (16) is

$$\sqrt{b}\cos\theta_2 + \sqrt{c}\cos\theta_3 + \sqrt{a}\cos(\theta_2 - \theta_3) = \frac{\lambda}{\sqrt{abc}}.$$

This may be written

$$\sqrt{a}\cos\theta_1 + \sqrt{b}\cos\theta_2 + \sqrt{c}\cos\theta_3 + \sqrt{a}\,\{\cos(\theta_2 - \theta_3) - \cos\theta_1\} = \frac{\lambda}{\sqrt{abc}}$$

Now

$$\begin{aligned} & \sqrt{a}\,\{\cos(\theta_2 - \theta_3) - \cos\theta_1\} \\ & \quad = 2\sqrt{a}\,\sin\tfrac{1}{2}(\theta_2 + \theta_3 - \theta_3)\sin\tfrac{1}{2}(\theta_1 + \theta_3 - \theta_2) \\ & \quad = 2\sqrt{a}\,.\,\frac{t}{\sqrt{ab}}\,.\,\frac{t}{\sqrt{ac}} = \frac{2t^2}{\sqrt{abc}}. \end{aligned}$$

Hence we have

$$\sqrt{a}\cos\theta_1 + \sqrt{b}\cos\theta_2 + \sqrt{c}\cos\theta_3 + 2.\frac{t^2}{\sqrt{abc}} = \frac{\lambda}{\sqrt{abc}}$$

Taking differentials, and remembering $\frac{d\lambda}{ds} = -t$, we get

$$\sqrt{a}\,.\sin\theta_1\,.\,d\theta_1 + \sqrt{b}\,.\sin\theta_2\,.\,d\theta_2 + \sqrt{c}\,.\sin\theta_3\,.\,d\theta_3 - 4\frac{t}{\sqrt{abc}}\,.\,dt = \frac{t}{\sqrt{abc}}\,.ds\,.$$

Substituting for $\sin\theta_1$, $\sin\theta_2$, $\sin\theta_3$ and t the quantities proportional to them from (17) as before, we get

$$\begin{aligned} ds = {} & \sqrt{a}\,(b + c + 2\sqrt{bc}\cos\theta_1)^{\frac{1}{2}}\,d\theta_1 \\ & + \sqrt{b}\,(c + a + 2\sqrt{ca}\cos\theta_2)^{\frac{1}{2}}\,d\theta_2 \\ & + \sqrt{c}\,(a + b + 2\sqrt{ab}\cos\theta_3)^{\frac{1}{2}}\,d\theta_3 - 4\,dt \quad \ldots (21) \end{aligned}$$

which gives

$$\begin{aligned} s = {} & \int_0^{\theta_1} \sqrt{a}\,(b + c + 2\sqrt{b}\cos\theta)^{\frac{1}{2}}\,d\theta \\ & + \int_0^{\theta_2} \sqrt{b}\,(c + a + 2\sqrt{ca}\cos\theta)^{\frac{1}{2}}\,d\theta \\ & + \int_0^{\theta_3} \sqrt{c}\,(a + b + 2\sqrt{ab}\cos\theta)^{\frac{1}{2}}\,d\theta - 4t \quad \ldots (22) \end{aligned}$$

where $t = \mathrm{P\,Q}$ and we know its value, as already observed, in terms of θ_1, θ_2, or θ_3.

18. P being on the outer oval still, let the arc C P $= s'''$ and let us seek to express it in terms of the angles ϕ_1, ϕ_2, ϕ_3 which it subtends at the foci. Since $(s + s''')$ is constant, we have

$$ds = -ds_3''' \quad \ldots \quad \ldots \quad \ldots (i)$$

Also we have

$$\theta_1 = \pi - \phi_1,\ \theta_2 = \pi - \phi_2,\ \theta_3 = \phi_3\ ; \quad \ldots (ii)$$

$$d\theta_1 = -d\phi_1,\ d\theta_2 = -d\phi_2,\ d\theta_3 = d\phi_3. \quad \ldots (iii)$$

By means of these relations we can transform the four expressions for ds contained in (19) and (21) and obtain the corresponding expressions in for ds''' terms of $d\phi_1$, $d\phi_2$, $d\phi^3$, or of these and dt.

19. Let us next consider P to be on the inner oval. Let the arc A P $= s'$. Then $\theta_1, \theta_2, \theta_3$ will be the angles subtended by this arc at the foci. It can be seen that the investigation for s' is exactly the same as for s. All the steps in the latter as well as the final results hold good ; only, we should consider $\sqrt{a}$ and t to be intrinsically negative for the inner oval. Consequently explicit expressions for ds' are obtained from those for ds in (19) by changing $\sqrt{a}$ into $-\sqrt{a}$; or from that in (21) by changing both $\sqrt{a}$ and t into their negatives.

Finally, we observe that if the arc B P in this case be s'', and it subtend angles ϕ_1, ϕ_2, ϕ_3 at the foci, then expressions for ds'' could be derived from those for ds' in the manner of § 18.

ON THE EXPANSION OF CERTAIN FUNCTIONS*

(WITH PROPERTIES OF ASSOCIATED CO-EFFICIENTS.)

By C. Krishnamachari.

Introduction.—Various series involving inverse powers of natural numbers have been summed up by mathematicians. We have the well-known summations in terms of Bernoulli's and Euler's numbers. Mr. S. R. Ranganathan in his paper, "Bernoulli's Polynomial and Fourier Series," (*vide* J. I. M. S., Vol. XI, No. 2), sums a number of interesting series. *The object of this paper is to express the sums of various series* (including those summed by Mr. Ranganathan) *in terms of coefficients of various trigonometric expansions*, which possess interesting properties similar to Bernoulli's and Euler's numbers. Since writing this paper, I find that Dr. Glaisher has anticipated me in the 'Quarterly Journal of Mathematics,' Vol. XXIX, a volume which I have not been able to procure even now from any Indian Library. On page 187 of the Quarterly Journal, Vol. XLV, Dr. Glaisher makes a reference to an article in Vol. XXIX entitled "On the Bernoullian Function." From the title I conclude that Dr. Glaisher's method of proof is different from mine, and that the A_{2n} in § 1 of my paper is Dr. Glaisher's P_n; and my A_{2n-1} is his Q_n. Dr. Glaisher further points out that

$$\frac{2e^x}{e^{2x}+1} = 1 - A_1 x - A_2 \cdot \frac{x^2}{2!} + A_3 \cdot \frac{x^3}{3!} + A_4 \cdot \frac{x^4}{4!} - \ldots\ldots$$

§ 1. Let

$$\frac{1}{\cos x - \sin x} = 1 + A_1 x + A_2 \frac{x^2}{2!} + \ldots\ldots + A_n \frac{x^n}{n!} + \ldots\ldots,$$

so that, cross-multiplying and equating to zero the coefficients of the various powers of x, we obtain,

$$\left.\begin{aligned} A_{2n} &= \binom{2n}{1} A_{2n-1} + \binom{2n}{2} A_{2n-2} - \binom{2n}{3} A_{2n-3} \\ &\quad - \ldots\ldots + (-1)^{n-1} \binom{2n}{2n-1} A_1 + (-1)^{n-1} \\ A_{2n+1} &= \binom{2n+1}{1} A_{2n} + \binom{2n+1}{2} A_{2n-1} - \binom{2n+1}{3} A_{2n-2} \\ &\quad - \ldots\ldots + (-1)^{n-1} \binom{2n+1}{2n} A_1 + (-1)^n, \end{aligned}\right\} \quad \ldots (1)$$

* Presented to the Third Conference of the Indian Mathematical Society, March, 1921.

where $\binom{n}{r}$ denotes the number of combinations of n things r at a time.

In particular

$$\left.\begin{aligned} &A_1 = 1, \quad A_2 = 3, \quad A_3 = 11, \quad A_4 = 57, \\ &A_5 = 361, \quad A_6 = 2763, \quad A_7 = 24611, \quad A_8 = 250737. \end{aligned}\right\}^{*} \ldots (2)$$

Now $$\frac{1}{\cos x - \sin x} = \frac{1}{\sqrt{2}} \cdot \sec\left(\frac{\pi}{4} + x\right)$$

$$= \frac{1}{\sqrt{2}}\left\{\frac{2}{\pi - 2\left(\frac{\pi}{4} + x\right)} + 2\sum_{1}^{\infty}(-1)^{n-1}\left[\frac{1}{2n\pi - \pi + 2\left(\frac{\pi}{4} + x\right)}\right.\right.$$

$$\left.\left. - \frac{1}{2n\pi + \pi - 2\left(\frac{\pi}{4} + x\right)}\right]\right\}$$

$$= \frac{1}{\sqrt{2}}\left\{\frac{4}{\pi} \cdot \frac{1}{1 - \frac{4x}{\pi}} + \sum_{1}^{\infty}(-1)^{n-1}\left[\frac{4}{(4n-1)\pi} \cdot \frac{1}{1 + \frac{4x}{(4n-1)\pi}}\right.\right.$$

$$\left.\left. - \frac{4}{(4n+1)\pi} \cdot \frac{1}{1 - \frac{4x}{(4n+1)\pi}}\right]\right\}$$

$$= \frac{1}{\sqrt{2}}\left\{\begin{aligned} &\frac{4}{\pi}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \ldots\ldots\right) \\ &+ \frac{4^2}{\pi^2} \cdot x\left(1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \ldots\ldots\right) \\ &+ \frac{4^3}{\pi^3} \cdot x^2\left(1 + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} - \ldots\ldots\right) + \\ &+ \frac{4^{2n-1}}{\pi^{2n-1}} \cdot x^{2n-2}\left(1 + \frac{1}{3^{2n-1}} - \frac{1}{5^{2n-1}} - \frac{1}{7^{2n-1}} + \ldots\ldots\right) \\ &+ \frac{4^{2n}}{\pi^{2n}} \cdot x^{2n-1}\left(1 - \frac{1}{3^{2n}} - \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{9^{2n}} - \ldots\ldots\right) + \ldots \end{aligned}\right\} \ldots (3)$$

Hence by equating the co-efficients of the coresponding powers of x on both sides, we obtain,

$$\left.\begin{aligned} &\frac{A_{2n-1}}{2n-1!} \cdot \left(\frac{\pi}{4}\right)^{2n} \cdot \sqrt{2} = 1 - \frac{1}{3^{2n}} - \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{9^{2n}} - \ldots \\ &\frac{A_{2n}}{2n!} \cdot \left(\frac{\pi}{4}\right)^{2n+1} \cdot \sqrt{2} = 1 + \frac{1}{3^{2n+1}} - \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \ldots \end{aligned}\right\}^{\dagger} \ldots (4)$$

* Dr. Glaisher has tabulated the values of forty of these numbers. *Quarterly Journal*, Vol. XLV. See Introduction.

† These formulæ are arrived at by Dr. Glaisher presumably by a different method. See Introduction.

§ 2. The following generalisation of this method is fruitful of various results in summation of series :

$$\frac{1}{\cos x - \tan \alpha \,.\, \sin x} = \cos \alpha \,.\, \sec (x + \alpha)$$

$$= \cos \alpha \left\{ \frac{2}{\pi - 2(x+\alpha)} + 2 \sum_{1}^{\infty} (-1)^{n-1} \left[\frac{1}{2n\pi - (\pi - 2x + \alpha)} - \frac{1}{2n\pi + \pi - 2(x + \alpha)} \right] \right\}$$

$$= 2 \cos \alpha \left\{ \frac{1}{\pi - 2\alpha} \cdot \frac{1}{1 - \frac{2x}{\pi - 2\alpha}} + \sum_{1}^{\infty} (-1)^{n-1} \left[\frac{1}{(2n-1)\pi + 2\alpha} \cdot \frac{1}{1 + \frac{2x}{(2n-1)\pi + 2\alpha}} - \frac{1}{(2n+1)\pi - 2\alpha} \cdot \frac{1}{1 - \frac{2x}{(2n+1)\pi - 2\alpha}} \right] \right\}$$

$$= 2 \cos \alpha \left\{ \left(\frac{1}{\pi - 2\alpha} + \frac{1}{\pi + 2\alpha} - \frac{1}{3\pi - 2\alpha} - \frac{1}{3\pi + 2\alpha} + \frac{1}{5\pi - 2\alpha} + \frac{1}{5\pi + 2\alpha} - \cdots \right) \right.$$

$$+ 2x \left(\frac{1}{(\pi - 2\alpha)^2} - \frac{1}{(\pi + 2\alpha)^2} - \frac{1}{(3\pi - 2\alpha)^2} + \frac{1}{(3\pi + 2\alpha)^2} + \frac{1}{(5\pi - 2\alpha)^2} - \cdots \right)$$

$$+ 2^3 x^3 \left[\frac{1}{(\pi - 2\alpha)^3} + \frac{1}{(\pi + 2\alpha)^3} - \frac{1}{(3\pi - 2\alpha)^3} - \frac{1}{(3\pi + 2\alpha)^3} + \cdots \right]$$

$$+ \cdots\cdots\cdots\cdots\cdots$$

$$+ 2^{2n-1} x^{2n-1} \left[\frac{1}{(\pi - 2\alpha)^{2n}} - \frac{1}{(\pi + 2\alpha)^{2n}} - \frac{1}{(3\pi - 2\alpha)^{2n}} + \frac{1}{(3\pi + 2\alpha)^{2n}} + \frac{1}{(5\pi - 2\alpha)^{2n}} - \cdots \right]$$

$$+ 2^{2n} x^{2n} \left[\frac{1}{(\pi - 2\alpha)^{2n+1}} + \frac{1}{(\pi + 2\alpha)^{2n+1}} - \frac{1}{(3\pi - 2\alpha)^{2n+1}} - \frac{1}{(3\pi + 2\alpha)^{2n+1}} + \cdots \right]$$

$$\left. + \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \right\} \quad (5)$$

after expanding the partial fractions and re-arranging in powers of x, We have taken the absolutely convergent series for sec θ.

Hence by writing the expansion in the form

$$1 + \Sigma A_n (\tan \alpha) \frac{x_n}{n!},$$

and equating the co-efficients of the powers of x, we have ... (6)

$$\frac{A_{2n-1}(\tan \alpha)}{(2n-1)!} = 2^2 \cos \alpha \left[\frac{1}{(\pi - 2\alpha)^{2n}} - \frac{1}{(\pi+2\alpha)^{2n}} - \frac{1}{(3\pi-2\alpha)^{2n}} + \frac{1}{(3\pi+2\alpha)^{2n}} + \ldots \right] \quad \ldots (7)$$

$$\frac{A_{2n}(\tan \alpha)}{2n!} = 2^{2n+1} \cos \alpha \left[\frac{1}{(\pi-2\alpha)^{2n+1}} + \frac{1}{(\pi+2\alpha)^{2n+1}} - \frac{1}{(3\pi-2\alpha)^{2n+1}} - \frac{1}{(3\pi+2\alpha)^{2n+1}} + \ldots \right] \quad \ldots (8)$$

In a paper on "A Table of Values of thirty Eulerian Numbers based on a New Method" by Mr. M. Bheemasena Rao and myself, it is proved that, if we write, $\tan \alpha = a$, then we have

$$\left.\begin{aligned} A_n(\tan \alpha) &= F(a) \\ &= n!\, a^n + (n-2)! \Sigma(n-1)^2 a^{n-2} \\ &+ (n-4)! \Sigma(n-3)^2 \Sigma(n-2)^2 a^{n-4} + \ldots \\ &+ (n-2r)! \Sigma(n-2r+1)^2 \Sigma(n-2r+2)^2 \ldots \\ &\qquad \Sigma(n-r)^2 a^{n+2r} + \ldots \end{aligned}\right\} \quad (9)$$

where the 'Σ' notation is used in a particular sense.

The above equations are fundamental, and a number of interesting summations can be obtained by giving particular values to α. We may note the following :—

(*a*) Put $\alpha = \frac{\pi}{4}$, then $\tan \alpha = 1$. We obtain the series of section 1. The properties of these co-efficients will be discussed in another paper. We observe that

$$A_n = n! + (n-2)! \Sigma(n-1)^2 + (n-4)! \Sigma(n-3)^2 \Sigma(n-2)^2 + \ldots + (n-2r)! \Sigma(n-2r+1)^2 \Sigma(n-2r+2)^2 \ldots \Sigma(n-r)^2 + \ldots$$

(*b*) Put $\alpha = \frac{\pi}{3}$, $\tan \alpha = \sqrt{3}$. We easily find that, if the expansion be written in the form,

$$1 + b_1 x + b_2 \frac{x^2}{2!} + \ldots + b_n \frac{x^n}{n!} + \ldots,$$

then $$1 - \frac{1}{5^{2n}} - \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \frac{1}{13^{2n}} - \ldots = \frac{\pi^{2n}}{6^{2n}} \, 2. \, \frac{b_{2n-1}}{(2n-1)!} \qquad (10)$$

$$1 + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} - \frac{1}{11^{2n+1}} + \ldots = \frac{\pi^{2n+1}}{6^{2n+1}} \, 2. \, \frac{b_{2n}}{2n!}.$$

We find also that b_n is to be calculated from the equations,

$$\left.\begin{aligned} b_{2n} &= \binom{2n}{1} b_{2n-1}\sqrt{3} + \binom{2n}{2} b_{2n-2} - \ldots \\ &\quad + (-1)^{n-1}\binom{2n}{2n-1} b_1\sqrt{3} + (-1)^{n-1}, \\ b_{2n+1} &= \binom{2n+1}{1} b_{2n}\sqrt{3} + \binom{2n+1}{2} b_{2n-1} - \ldots \\ &\quad + (-1)^{n-1}\binom{2n+1}{2n-1} b_2\sqrt{3} + (-1)^{n-1}\binom{2n+1}{2n} b + (-1)^n. \end{aligned}\right\} \qquad (11)$$

The values of the first few co-efficients are,

$$b_1 = \sqrt{3},\ b_2 = 7,\ b_3 = 23\sqrt{3},\ b_4 = 305,\ b_5 = 1685\sqrt{3},\ \ldots$$

(*c*) Put $\alpha = \frac{\pi}{6}$, then $\tan\alpha = \frac{1}{\sqrt{3}}$. If we write the expansion as

$$1 + a_1 x + a_2 \frac{x^2}{2!} + \ldots + a_n \frac{x^n}{n!} + \ldots,$$

$$\left.\begin{aligned} \text{then } 1 - \frac{1}{2^{2n}} - \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} - \ldots &= \frac{\pi^{2n}}{3^{2n}} \cdot \frac{2}{\sqrt{3}} \cdot \frac{a_{2n-1}}{(2n-1)!} \\ 1 + \frac{1}{2^{2n+1}} - \frac{1}{4^{2n+1}} - \frac{1}{5^{2n+1}} + \frac{1}{7^{2n+1}} + \frac{1}{8^{2n-1}} - \ldots &= \\ \frac{\pi^{2n+1}}{3^{2n+1}} \cdot \frac{2}{\sqrt{3}} \cdot \frac{a_{2n}}{2n!}. & \end{aligned}\right\} \qquad (12)$$

Also $$a_1 = \frac{1}{\sqrt{3}},\ a_2 = \frac{5}{3},\ a_3 = \frac{7}{\sqrt{3}}, \ldots$$

By comparing with the formulæ of Mr. Ranganathan (*loc. cit.*), we may write

$$\left.\begin{aligned} a_{2n} &= (-1)^{n+1} \cdot 3^{2n+1} \cdot \frac{(2^{2n+1}+1)}{2n+1} \cdot \phi_{2n+1}\left(\tfrac{1}{3}\right), \\ a_{2n+1} &= \frac{\sqrt{3}}{4(n+1)} \cdot (3^{2n+2}-1)(2^{2n+1}-1)\, B_n. \end{aligned}\right\} \qquad (13)$$

(*d*) If $\alpha = \frac{\pi}{8}$, we get

$$\left.\begin{aligned} &\frac{1}{3^{2n}} - \frac{1}{5^{2n}} - \frac{1}{11^{2n}} + \frac{1}{13^{2n}} + \frac{1}{19^{2n}} - \dots \dots \\ &\qquad = \frac{\pi^{2n}}{4^{2n}} \cdot \frac{1}{2^{2n-1}} \cdot \frac{K_{2n-1}}{(2n-1)!} \cdot \frac{1}{\sqrt{2+\sqrt{2}}}, \\ &\frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{11^{2n+1}} - \frac{1}{13^{2n+1}} + \dots \\ &\qquad = \frac{\pi^{2n+1}}{4^{2n+1}} \cdot \frac{1}{2^{2n-1}} \cdot \frac{K_{2n}}{2n!} \cdot \frac{1}{\sqrt{2+\sqrt{2}}}, \end{aligned}\right\} \quad (14)$$

where the K's are the corresponding co-efficients.

§ 3. Next consider the expansion of $\frac{\sin x + \tan \alpha \cdot \cos x}{\cos x - \tan \alpha \cdot \sin x}$ as an ascending power series in x. Write the expansion in the form,

$$\Sigma \cdot L_n \cdot \frac{x^n}{n!}, \qquad \dots \qquad \dots \qquad \dots \quad (15)$$

Now we have

$$\frac{\sin x + \tan \alpha \cdot \cos x}{\cos x - \tan \alpha \cdot \sin x} = \tan (x + \alpha)$$

$$= 2 \sum_1^\infty \left\{ \frac{1}{(2n-1)\pi - 2(x+\alpha)} - \frac{1}{(2n-1)\pi + 2(x+\alpha)} \right\}$$

$$= 2 \sum_1^\infty \left\{ \frac{1}{(2n-1)\pi - 2\alpha} \cdot \frac{1}{1 - \frac{2x}{(2n-1)\pi - 2\alpha}} \right.$$

$$\left. - \frac{1}{(2n-1)\pi + 2\alpha} \cdot \frac{1}{1 + \frac{2x}{(2n-1)\pi + 2\alpha}} \right\}$$

$$= 2 \left\{ \left(\frac{1}{\pi - 2\alpha} - \frac{1}{\pi + 2\alpha} + \frac{1}{3\pi - 2\alpha} - \frac{1}{3\pi + 2\alpha} + \dots \right) \right.$$

$$\left. + 2x \left[\frac{1}{(\pi - 2\alpha)^2} + \frac{1}{(\pi + 2\alpha)^2} + \frac{1}{(3\pi - 2\alpha)^2} + \frac{1}{(3\pi + 2\alpha)^2} + \dots \right] \right.$$

$$+ 2^3x^3\left[\frac{1}{(\pi-2\alpha)^3}-\frac{1}{(\pi+2\alpha)^3}+\frac{1}{(3\pi-2\alpha)^3}-\frac{1}{(3\pi+2\alpha)^3}+\ \ldots\right]$$
$$+\ \ldots\ldots$$
$$+ 2^{2n-1}x^{2n-1}\left[\frac{1}{(\pi-2\alpha)^{2n}}+\frac{1}{(\pi+2\alpha)^{2n}}+\frac{1}{(3\pi-2\alpha)^{2n}}\right.$$
$$\left.+\frac{1}{(3\pi+2\alpha)^{2n}}+\ \ldots\right]$$
$$+ 2^{2n}x^{2n}\left[\frac{1}{(\pi-2\alpha)^{2n+1}}-\frac{1}{(\pi-2\alpha)^{2n+1}}+\frac{1}{(3\pi-2\alpha)^{2n+1}}\right.$$
$$\left.-\frac{1}{(3\pi+2\alpha)^{2n+1}}+\ \ldots\ldots\right]+\ \ldots\ldots\Big\}\quad\ldots\quad(16)$$

after re-arranging in powers of x.

Hence, by equating co-efficients, we obtain,

$$\left.\begin{aligned}\frac{L_{2n-1}}{(2n-1)!}&=2\,.\,2^{2n-1}\left[\frac{1}{(\pi-2\alpha)^{2n}}+\frac{1}{(\pi+2\alpha)^{2n}}+\frac{1}{(3\pi-2\alpha)^{2n}}\right.\\&\qquad\left.+\frac{1}{(3\pi+2\alpha)^{2n}}+\ \ldots\ldots\right]\ \ldots\\ \frac{L_{2n}}{2n!}&=2\,.\,2^{2n}\left[\frac{1}{(\pi-2\alpha)^{2n+1}}-\frac{1}{(\pi+2\alpha)^{2n+1}}+\frac{1}{(3\pi-2\alpha)^{2n+1}}\right.\\&\qquad\left.-\frac{1}{(3\pi+2\alpha)^{2n+1}}+\ \ldots\ldots\right].\end{aligned}\right\}(17)$$

These equations are again fundamental, and by giving different values to α, various summations may be obtained. The following are interesting. The general expression for L_n as a function of $\tan\alpha$ has yet to be investigated.

(*a*) Put $\alpha=\frac{\pi}{4}$, then $\tan\alpha=1$. The function to be expanded becomes

$$\frac{\sin x+\cos x}{\cos x-\sin x}=\frac{1+\sin 2x}{\cos 2x}=\sec 2x+\tan 2x.$$

We obtain the well-known summations in terms of Euler's and Bernoulli's numbers.

(*b*) Put $\alpha=\frac{\pi}{3}$, $\tan\alpha=\sqrt{3}$. Hence if we write

$$\frac{\sin x+\sqrt{3}\cos x}{\cos x-\sqrt{3}\sin x}=\sqrt{3}\left\{1+c_1x+\frac{c_2}{2!}.x^2+\ \ldots\ +\frac{c^n}{n!}.x^n+\ \ldots\right\},$$

it follows that,

$$\sqrt{3}\cdot\frac{c_{2n-1}}{(2n-1)!}\cdot\frac{\pi^{2n}}{6^{2n}} = 1+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\frac{1}{11^{2n}}+\frac{1}{13^{2n}}+\ldots\ldots\ldots$$

$$\sqrt{3}\cdot\frac{c_{2n}}{2n!}\cdot\frac{\pi^{2n+1}}{6^{2n+1}} = 1-\frac{1}{5^{2n+1}}+\frac{1}{7^{2n+1}}-\frac{1}{11^{2n+1}}+\ldots\ldots\ldots \quad (18)$$

It will be seen that

$$c_1=\frac{4}{\sqrt{3}},\ c_2=8,\ c_3=\frac{80}{\sqrt{3}},\ c_4=352,\ c_5=\frac{5824}{\sqrt{3}},\ c_6=38528,\ldots\ldots$$

and that

$$1+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{11^6}+\ldots\ldots = \frac{91\,\pi^6}{87480} \quad \ldots \quad \ldots \quad (19)$$

The result given by Mr. Ranganathan in his paper for the sum of this series seems to be incorrect. By reference to equations given in continuation of this paper, we have

$$c_{2n} = (-1)^n\cdot 2\cdot 6^{2n}\cdot\psi_{2n}\left(\tfrac{1}{3}\right).$$

$$c_{2n-1} = \frac{1}{\sqrt{3}}\cdot\left\{(-1)^n\,\psi_{2n-1}\left(\tfrac{1}{3}\right)+\frac{2^{2n}-1}{3^{2n}}\cdot\frac{B_n}{2n}\right\}$$

or

$$\frac{2^{2n-1}}{2n}\cdot(3^{2n}-1)\,(2^{2n}-1)\,\frac{B_n}{\sqrt{3}}$$

from Mr. Ranganathan's formula (*loc. cit.*)

(c) Put $\alpha=\frac{\pi}{6}$, $\tan\alpha=\frac{1}{\sqrt{3}}$. If we write the expansion as

$$\frac{1}{\sqrt{3}}\cdot\left\{1+d_1x+d_2\frac{x^2}{2!}+\ldots d_n\cdot\frac{x^n}{n!}+\ldots\right\},$$

we find $d_1=\frac{4}{\sqrt{3}}$, $d_2=\frac{8}{3}$, $d_3=\frac{16}{\sqrt{3}}$, $d_4=32,\ldots$ $\ldots$ (20)

$$d_{2n}=\binom{2n}{1}d_{2n-1}\cdot\frac{1}{\sqrt{3}}+\binom{2n}{2}d_{2n-2}-\binom{2n}{3}d_{2n-3}\frac{1}{\sqrt{3}}-\ldots$$
$$+(-1)^{n-2}\binom{2n}{2n-2}d_2+(-1)^{n-1}\binom{2n}{2n-1}d_1\cdot\frac{1}{\sqrt{3}}$$

$$d_{2n+1}=\binom{2n+1}{1}d_{2n}\cdot\frac{1}{\sqrt{3}}+\binom{2n+1}{2}d_{2n-1}-\ldots$$
$$+(-1)^{n-1}\binom{2n+1}{2n}d_1+(-1)^n\cdot\frac{1}{\sqrt{3}}+(-1)^n\cdot\sqrt{3}$$

and also,

$$\left.\begin{aligned}\frac{1}{\sqrt{3}}\cdot\frac{d_{2n-1}}{(2n-1)!}\cdot\frac{\pi^{2n}}{3^{2n}} &= 1+\frac{1}{2^{2n}}+\frac{1}{4^{2n}}+\frac{1}{5^{2n}}+\ldots\ldots\\ \frac{1}{\sqrt{3}}\cdot\frac{d_{2n}}{2n!}\cdot\frac{\pi^{2n+1}}{3^{2n+1}} &= 1-\frac{1}{2^{2n+1}}+\frac{1}{4^{2n+1}}-\frac{1}{5^{2n+1}}+\ldots\ldots\end{aligned}\right\} \quad (21)$$

(*d*) If $\alpha = \frac{\pi}{8}$, and l_n denote the corresponding co-efficient, we have

$$\left.\begin{aligned} \frac{l_{2n-1}}{(2n-1)!} \cdot \frac{\pi^{2n}}{8^{2n}} &= \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{11^{2n}} + \frac{1}{13^{2n}} + \ldots\ldots \\ \frac{l_{2n}}{2n!} \cdot \frac{\pi^{2n+1}}{8^{2n+1}} &= \frac{1}{3^{2n+1}} - \frac{1}{5^{2n+1}} + \frac{1}{11^{2n+1}} - \frac{1}{13^{2n+1}} + \ldots\ldots \end{aligned}\right\} \quad (22)$$

§ 4. The above formulæ may also be established by an application of Cauchy's Theorem of residues, as under :

Consider the expansion of the function $f(z) = \frac{1}{\cos z - \sin z}$ in ascending powers of z. Since $| z f(z) | \to 0$ uniformly as $z \to \infty$, it follows by Cauchy's theorem that the sum of the residues of the function at all its poles is zero. Its poles are given by $\tan z = 1$, that is,

$$z = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4} \ldots -\frac{3\pi}{4}, -\frac{7\pi}{4}, -\frac{11\pi}{4} \ldots$$

Let $$\frac{1}{\cos z - \sin z} = 1 + \Sigma A_n \frac{z^n}{n!}$$

Then $\frac{A_{n-1}}{(n-1)!}$ is the residue at the origin of the function

$$\phi(z) = \frac{1}{z^n (\cos z - \sin z)}.$$

Residue of $\phi(z)$ at $\frac{\pi}{4} = \left[\frac{4}{\pi}\right]^n \cdot \frac{1}{-\sqrt{2}}$.

" " $\frac{5\pi}{4} = \left[\frac{4}{5\pi}\right]^n \cdot \frac{1}{\sqrt{2}}$

" " $\frac{9\pi}{4} = \left[\frac{4}{9\pi}\right]^n \cdot \frac{1}{-\sqrt{2}}$.

" " $-\frac{3\pi}{4} = \left[-\frac{4}{3\pi}\right]^n \cdot \frac{1}{\sqrt{2}}$.

" " $-\frac{7\pi}{4} = \left[-\frac{4}{7\pi}\right]^n \cdot \frac{1}{-\sqrt{2}}$, etc.

Hence $$\frac{A_{n-1}}{(n-1)!} + \left[\frac{4}{\pi}\right]^n \cdot \frac{1}{\sqrt{2}} \cdot \left\{ -\frac{1}{1^n} + \frac{1}{5^n} - \frac{1}{9^n} + \ldots + (-1)^n \cdot \left(\frac{1}{3^n} - \frac{1}{7^n} + \frac{1}{11^n} - \ldots\right) \right\} = 0.$$

$$\therefore \quad \frac{1}{1^n} - \frac{1}{5^n} + \frac{1}{9^n} - \ldots + (-1)^{n-1}\left[\frac{1}{3^n} - \frac{1}{7^n} + \frac{1}{11^n} - \ldots\right]$$

$$= \frac{A_{n-1}}{(n-1)!} \cdot \left[\frac{\pi}{4}\right]^n \sqrt{2}$$

Hence we have the formula established in § 1.

§ 5. *Application to Definite Integrals.*

We know that $\int_0^\infty e^{-ax} x^{n-1}\, dx = \frac{\Gamma(n)}{a^n}$.

Hence $\frac{1}{(\pi-2\alpha)^{2n}} - \frac{1}{(\pi+2\alpha)^{2n}} - \frac{1}{(3\pi-2\alpha)^{2n}} + \frac{1}{(3\pi+2\alpha)^{2n}}$

$+ \frac{1}{(5\pi-2\alpha)^{2n}} - \frac{1}{(5\pi+2\alpha)^{2n}} - \frac{1}{(7\pi-2\alpha)^{2n}} + \ldots\ldots\ldots$

$$= \frac{1}{\Gamma(2n)}\int_0^\infty x^{2n-1}\left\{ e^{-x(\pi-2\alpha)} - e^{-x(\pi+2\alpha)} - e^{-x(3\pi-2\alpha)} \right.$$

$$\left. + e^{-x(3\pi+2\alpha)} + e^{-x(5\pi-2\alpha)} - \ldots \right\} dx .$$

$$= \frac{1}{\Gamma(2n)}\int_0^\infty x^{2n-1} \cdot \frac{e^{2\alpha x} - e^{-2\alpha x}}{e^{\pi x} + e^{-\pi x}} \cdot dx.$$

Hence from §2, we obtain,

$$\frac{A_{2n-1}(\tan\alpha)}{(2n-1)!} \cdot \frac{1}{2^{2n}\cos\alpha} = \frac{1}{\Gamma(2n)}\int_0^\infty x^{2n-1} \frac{e^{2\alpha x} - e^{-2\alpha x}}{e^{\pi x} + e^{-\pi x}}\, dx.$$

$$\therefore \quad \frac{A_{2n-1}(\tan\alpha)}{2^{2n}\cos\alpha} = \int_0^\infty x^{2n-1} \frac{e^{2\alpha x} - e^{-2\alpha x}}{e^{\pi x} + e^{-\pi x}}\, dx. \qquad (23)$$

Similarly we have,

$$\frac{A_{2n}(\tan\alpha)}{2^{2n+1}\cos\alpha} = \int_0^\infty x^{2n} \frac{e^{2\alpha x} + e^{-2\alpha x}}{e^{\pi x} + e^{-\pi x}}\, dx. \qquad (24).$$

It is to be remembered that A_n as a function of $\alpha = \tan\alpha$, has been written down already in § 2. The following are examples :—

(a) $\alpha = \frac{\pi}{4}$. We have,

$$\frac{A_{2n-1}}{2^{2n}} \cdot \sqrt{2} = \int_0^\infty x^{2n-1} \cdot \frac{\left(e^{\frac{\pi x}{2}} - e^{-\frac{\pi x}{2}}\right)}{e^{\pi x} + e^{-\pi x}}\, dx,$$

which is transformed by the substitution $\frac{\pi x}{2} = y$ into

$$A_{2n-1}\left(\frac{\pi}{4}\right)^{2n}\cdot\sqrt{2} = \int_0^\infty \frac{\sinh x}{\cosh 2x}\cdot x^{2n-1}dx$$

$$= \int_0^1 \frac{1-x^2}{1+x^4}\cdot\left(\log\frac{1}{x}\right)^{2n-1}dx \qquad [y = e^{-x} \qquad (25)$$

$$= \int^{\pi/2} \frac{\cos x}{1+\cos^2 x}\left(\log\cot\frac{x}{2}\right)^{2n-1}dx$$

[$x = \tan y$ in the last.

A_{2n-1} being the same as in § 1.

Similarly

$$\frac{A_{2n}}{2^{2n+1}}\cdot\sqrt{2} = \int_0^\infty \frac{e^{\frac{\pi x}{2}} + e^{-\frac{\pi x}{2}}}{e^{\pi x} + e^{-\pi x}}\cdot x^{2n}\,dx$$

which may be expressed as

$$A_{2n}\left(\frac{\pi}{4}\right)^{2n+1}\cdot\sqrt{2} = \int_0^\infty \frac{\cosh x}{\cosh 2x}\,x^{2n}\,dx$$

$$= \int_0^1 \frac{1+y^2}{1+y^4}\cdot\left(\log\frac{1}{y}\right)^{2n}dy$$

$$= \int_0^{\pi/2} \frac{\left(\log\cot\frac{x}{2}\right)^{2n}}{1+\cos^2 x}\,dx. \qquad \ldots \quad (26)$$

by the same substitutions as above. Since the values of A_n are known, we can write down particular integrals by giving different values to n.

(*b*) Put $\alpha = \frac{\pi}{3}$. We obtain from § 2 (*b*) that

$$\frac{b_{2n-1}}{(2n-1)!}\cdot\frac{1}{2^{-n-1}} = \frac{1}{\Gamma(2n)}\cdot\int_0^\infty x^{2n-1}\cdot\frac{e^{\frac{2\pi}{3}x} - e^{-\frac{\pi}{3}x}}{e^{\pi x} + e^{-\pi x}}\,dx$$

which may be expressed as

$$b_{2n-1}\cdot 2\cdot\left(\frac{\pi}{6}\right)^{2n} = \int_0^\infty \frac{\sinh 2x}{\cosh 3x}\cdot x^{2n-1}\,dx$$

$$= \int_0^1 \frac{1-x^4}{1+x^6}\cdot\left(\log\frac{1}{x}\right)^{2n-1}dx$$

$$= \int_0^{\pi/2} \frac{2\cos x}{1+3\cos^2 x}\cdot\left(\log\cot\frac{x}{2}\right)^{2n-1}dx. \qquad (27)$$

Similarly,

$$b_{2n} \cdot 2 \cdot \left(\frac{\pi}{6}\right)^{2n+1} = \int_0^\infty \frac{\cosh 2x}{\cosh 3x} \cdot x^{2n}\, dx$$

$$= \int_0^1 \frac{1 + x^4}{1 + x^6} \cdot \left(\log \frac{1}{x}\right)^{2n} dx$$

$$= \int_0^{\pi/2} \frac{1 + \cos^2 x}{1 + 3\cos^2 x} \cdot \left(\log \cot \frac{x}{2}\right)^{2n} dx. \qquad (28)$$

(*c*) Put $\alpha = \frac{\pi}{6}$. We obtain,

$$a_{2n-1} \cdot \left(\frac{\pi}{6}\right)^{2n} \cdot \frac{2}{\sqrt{3}} = \int_0^\infty \frac{\sinh x}{\cosh 3x} \cdot x^{2n-1}\, dx$$

$$= \int_0^1 \frac{1 - y^2}{1 + y^6} \cdot y \left(\log \frac{1}{y}\right)^{2n-1} dy$$

$$= \int_0^{\pi/2} \frac{\sin x \cos x}{1 + 3\cos^2 x} \cdot \left(\log \cot \frac{x}{2}\right)^{2n-1} dx. \qquad (29)$$

Similarly

$$a_{2n} \left(\frac{\pi}{6}\right)^{2n+1} \cdot \frac{2}{\sqrt{3}} = \int_0^\infty x^{2n} \cdot \frac{e^x + e^{-x}}{e^{3x} + e^{-3x}} \cdot dx.$$

$$= \int_0^\infty x^{2n} \cdot \frac{\cosh x}{\cosh 3x} \cdot dx.$$

$$= \int_0^1 \frac{1 + x^2}{1 + x^6} \cdot x \left(\log \frac{1}{x}\right)^{2n} dx. \qquad (30)$$

$$= \int_0^{\pi/2} \frac{\sin x}{1 + 3\cos^2 x} \cdot \left(\log \cot \frac{x}{2}\right)^{2n} dx.$$

(*d*) Put $\alpha = \frac{\pi}{m}$ generally. If X_n denotes the corresponding coefficient, we have

$$X_{2n-1} \left(\frac{\pi}{2m}\right)^{2n} \cdot \frac{1}{\cos \frac{\pi}{m}} = \int_0^\infty x^{2n-1} \frac{e^{2x} - e^{-2x}}{e^{mx} + e^{-mx}} \cdot dx. \qquad (31)$$

$$X_{2n} \cdot \left(\frac{\pi}{2m}\right)^{2n+1} \cdot \frac{1}{\cos \frac{\pi}{m}} = \int_0^\infty x^{2n} \cdot \frac{e^{2x} + e^{-2x}}{e^{mx} + e^{-mx}} \cdot dx. \qquad (32)$$

§ 6. We have again from § 3,

$$\frac{1}{(\pi-2\alpha)^{2n}} + \frac{1}{(\pi+2\alpha)^{2n}} + \frac{1}{(3\pi-2\alpha)^{2n}} + \frac{1}{(3\pi+2\alpha)^{2n}} + \frac{1}{(5\pi-2\alpha)^{2n}} + \frac{1}{(5\pi+2\alpha)^{2n}} + \dots \dots$$

$$= \frac{1}{\Gamma(2n)} \int_0^\infty x^{2n-1} \left\{ e^{-x(\pi-2\alpha)} + e^{-x(\pi+2\alpha)} + e^{-x(3\pi-2\alpha)} + e^{-x(3\pi+2\alpha)} + \dots \right\} dx$$

$$= \frac{1}{\Gamma(2n)} \int_0^\infty x^{2n-1} \cdot \frac{e^{2\alpha x} + e^{-2\alpha x}}{e^{\pi x} - e^{-\pi x}} dx.$$

Hence we obtain,

$$\frac{L_{2n-1}}{2^{2n}} = \int_0^\infty x^{2n-1} \cdot \frac{e^{2\alpha x} + e^{-2\alpha x}}{e^{\pi x} - e^{-\pi x}} dx. \qquad (33)$$

Similarly,

$$\frac{L_{2n}}{2^{2n+1}} = \int_0^\infty x^{2n} \frac{e^{2\alpha x} - e^{-2\alpha x}}{e^{\pi x} - e^{-\pi x}} dx. \qquad (34)$$

These formulae are general, and we can obtain particular cases as follows :

(*a*) Put $\alpha = \frac{\pi}{4}$. We get the familiar integrals for B_n and E_n. We need not dilate on these.

(*b*) Put $\alpha = \frac{\pi}{3}$. We obtain,

$$\sqrt{3}\, \frac{c_{2n-1}}{(2n-1)!} \cdot \frac{1}{2^{2n}} = \frac{1}{\Gamma(2n)} \int_0^\infty x^{2n-1} \frac{e^{\frac{2\pi x}{3}} + e^{-\frac{2\pi x}{3}}}{e^{\pi x} - e^{-\pi x}} dx \qquad (35)$$

or what is the same thing,

$$\sqrt{3} \cdot c_{2n-1} \cdot \left(\frac{\pi}{6}\right)^{2n} = \int_0^\infty x^{2n-1} \cdot \frac{e^{2x} + e^{-2x}}{e^{3x} - e^{-3x}} dx$$

$$= \int_0^1 \frac{1 + x^4}{1 - x^6} \cdot \left(\log \frac{1}{x}\right)^{2n-1} dx$$

$$= \int_0^{\pi/2} \frac{1 + \cos^2 x}{\cos x (3 + \cos^2 x)} \cdot \left(\log \cot \frac{x}{2}\right)^{2n-1} dx. \qquad (36)$$

Similarly,

$$\sqrt{3} \cdot c_{2n} \cdot \left(\frac{\pi}{6}\right)^{2n+1} = \int_0^\infty x^{2n} \cdot \frac{e^{2x} - e^{-2x}}{e^{3x} - e^{-3x}}\, dx. \quad (37)$$

(*c*) Put $\alpha = \frac{\pi}{6}$. We obtain from § 3, (*c*) that

$$\frac{1}{\sqrt{3}} \cdot d_{2n-1} \cdot \left(\frac{\pi}{6}\right)^{2n} = \int_0^\infty x^{2n-1} \cdot \frac{e^x + e^{-x}}{e^{3x} - e^{-3x}}\, dx.$$

$$\frac{1}{\sqrt{3}} \cdot d_{2n} \cdot \left(\frac{\pi}{6}\right)^{2n+1} = \int_0^\infty x^{2n} \cdot \frac{e^x - e^{-x}}{e^{3x} - e^{-3x}}\, dx. \quad (38)$$

(*d*) Put $\alpha = \frac{\pi}{m}$ generally. Then if γ_n denotes the corresponding co-efficient, we have

$$\frac{\gamma_{2n-1}}{2^{2n}} \cdot \left(\frac{\pi}{m}\right)^{2n} = \int_0^\infty x^{2n-1} \cdot \frac{e^{2x} + e^{-2x}}{e^{mx} - e^{-mx}}\, dx.$$

$$\frac{\gamma_{2n}}{2^{2n+1}} \cdot \left(\frac{\pi}{m}\right)^{2n+1} = \int_0^\infty x^{2n} \cdot \frac{e^{2x} - e^{-2x}}{e^{mx} - e^{-mx}}\, dx. \quad \ldots \quad (39)$$

§ 7. The following are a few additional integrals. Substituting for the series in § 2 the definite integrals obtained in the last section, we have

$$\sec(x + \alpha) = 2\left[\Sigma \frac{2^{2n-1} x^{2n-1}}{(2n-1)!} \int_0^\infty z^{2n-1} \frac{e^{2\alpha z} - e^{-2\alpha z}}{e^{\pi z} + e^{-\pi z}}\, dz \right.$$

$$\left. + \Sigma \frac{2^{2n} x^{2n}}{2n!} \int_0^\infty z^{2n} \frac{e^{2\alpha z} + e^{-2\alpha z}}{e^{\pi z} + e^{-\pi z}}\, dz\right]$$

$$= 2\int_0^\infty \frac{\sinh 2xz \cdot \sinh 2\alpha z}{\cosh \pi z}\, dz + 2\int_0^\infty \frac{\cosh 2xz \cdot \cosh 2\alpha z}{\cosh \pi z}\, dz.$$

$$\therefore \quad \sec(x + \alpha) = 2\int_0^\infty \frac{\cosh(x + \alpha) 2z}{\cosh \pi z}\, dz. \quad \ldots \quad (40)$$

This is a well-known integral for sec x. Similarly from § 3, and § 4, we obtain the well-known integral for tan x.

N.B.—[The co-efficients A_n introduced in this Note have interesting properties similar to Euler's and Bernoulli's numbers. These will be discussed in a future paper.]

SHORT NOTES.

Linear Systems of the third Order on a Conic.

[Reference: 'Linear Systems of Points,' *J. I. M. S*, June 1919].

1. A general involution relation on a conic might be visualised as the relation between the extremities of a chord passing through a fixed point in the plane. But pairs of points in involution form a particular case of linear systems and there arises the question of representation on the conic of linear systems of higher order, especially the third. There are two types of systems of the third order—the complete and the incomplete. The former consists of ∞^2 triads (or in the geometrical representation of ∞^2 inscribed triangles) and the latter, of ∞ triads.

2. *Representation of the incomplete system or the pencil of inscribed triangles.*

The incomplete system of the third order (*i.e.*, as we have termed it, the pencil of inscribed triangles) corresponds to a straight line in three dimensions and is thus determined by any two of its members. Now let $A_1B_1C_1$, $A_2B_2C_2$ be two triangles inscribed in the fundamental conic S and let S′ be the unique conic inscribed to each of the triangles. Then it is known that there are an infinity of triangles inscribed in and circumscribed to S, S′ respectively, and if *any one* vertex of any of these triangles is known the triangle itself is determined uniquely; hence the set of these triangles constitutes the pencil determined by $A_1B_1C_1$, $A_2B_2C_2$.

It might be shewn similarly that the set of triangles inscribed in S so as to be self-conjugate to another conic form a pencil.

Null Pencils. If the pencil is a *focal* pencil, S′ must have double contact with S.

Since every group of the third order is self-harmonic, it follows that if *two* triangles of a pencil are harmonic, then *any two* triangles of the pencil are harmonic. Such a pencil will be called a *null pencil.*

Since a pencil corresponds to a straight line in the representative three-space with a fundamental twisted cubic,* null pencils should be represented by lines specially related to the cubic; in fact by the lines of the linear complex determined by the cubic. A null pencil containing a given triad p should be represented by a straight line through the

* The standard representation of linear systems as flat spaces in higher dimensions with a fundamental normal curve is dealt with in the paper mentioned in the title.

corresponding point P lying wholly in the polar plane of P and *therefore* meeting the unique line-in-two-planes contained in that polar plane. Hence the null pencil should be the pencil determined by p and a triad of the form $(\alpha\beta\gamma)$ where $\beta\gamma$ are the indeterminate pair (Hessian points) of p and α may be arbitrary. Hence the null pencil containing a given inscribed triangle of S is the set of triangles inscribed in S and circumscribed to S′ where S′ may be any conic, inscribed in the triangle and *touching the polar line of the triangle* with respect to S′; that is, S′ is a conic such that the invariants θ, θ' of S and S′ both vanish; or, as we might agree to say, S′ is *harmonic* to S. We thus reach the theorem:

The null pencil of inscribed triangles of S is the set of triangles circumscribed to (or self-conjugate to) a harmonic conic of S.

The Harmonic Relation. Two inscribed triangles Δ_1, Δ_2 are harmonic when the pencil which they determine is a null pencil; hence

Two inscribed triangles Δ_1, Δ_2 of S are harmonic when the conic inscribed to Δ_1 and Δ_2 is harmonic to S.

3. *First representation of the Complete System.* The geometrical meaning of the harmonic relation obtained above leads immediately to a representation of a complete system. The complete system whose foci are the points ABC on S is the set of all triangles harmonic to ABC, that is, the set of all triangles inscribed in S and circumscribed to the various conics of the 4-line system determined by the sides of ABC and the polar line of ABC with respect to S.

To represent a complete system with a double focus.

Let the foci be ABB and let the tangents at A, B to S meet in T. Then the complete system consists of triangles inscribed in S and circumscribed to the conics touching AB at A and TB at T.

4. *Second representation of the Complete System.* Let P be a point on S and Q, R points not on S. Then the system of conics passing through PQR intersect S again in triads which evidently belong to a complete system. If QR intersects S in UU′, then it is obvious from elementary geometry that

(1) UU′ is the indeterminate pair of the complete system.

(2) If Q′R′ be two points on QR such that QR, Q′R′, UU′ belong to an involution, then the complete system determined on S in the above manner by PQ′R′ is exactly the same as the system determined by PQR.

Let ABC be the foci of the complete system determined as above by PQR. Then the line QR should be the polar line of ABC with respect

to S. Let PA, PB, PC and the tangents at ABC cut QR in $A_1B_1C_1$, $A_2B_2C_2$, respectively. Then, by elementary geometry and the fact that A, B, C are foci of the system, it follows that the involution determined by QR and UU′ contains the pairs A_1A_2, B_1B_2, C_1C_2.

Hence in representing a *given* complete system with foci ABC in this manner, the point P is entirely arbitrary, the line QR is determined by ABC only, the points QR are unspecified except in that they belong to an involution which always contains UU′ and which is further completely determined when P is chosen.

We can shew that no material alteration in the terms of our representation is necessary when the system has a double focus.

5. *The complete system determined by three inscribed triangles* Δ_1, Δ_2, Δ_3.

Let S_1, S_2, S_3 be the conics inscribed to every two of these triangles and let L_1, L_2, L_3 be the polar lines of these triangles with respect to S. There is a fourth common tangent to S_2 and S_3 besides the sides of Δ_1; call this fourth common tangent L_r. The intersections of L_r with S have one correspondent in the pencil (Δ_1, Δ_2) and another correspondent in the pencil (Δ_1, Δ_3). Hence these intersections have more than one correspondent in the complete system (Δ_1 Δ_2 Δ_3) and therefore must form the *indeterminate pair* of the complete system. We thus have the theorm :—

The fourth common tangent to any two of the conics S_1, S_2, S_3 *is the same line* L.

Further, if Δ be the focal triangle, the pencil (Δ, Δ') is a null pencil, hence the conic inscribed to Δ and Δ_1 must touch L and L_1 (since these are the polar lines of Δ and Δ_1). Hence if P_r be the conic touching L, L_r and the sides of Δ_r, we have the theorem :

The conics P_1 P_2 P_3 *belong to a four-line system. Their common tangents other than* L *form a triangle which is inscribed in* S *and which is, in fact, the focal triangle of the complete system* (Δ_1 Δ_2 Δ_3).

6. *The Parabola.*

The feet of concurrent normals is a complete system of the third order whose foci are A ∞ ∞ (where A is the vertex and ∞, the point at infinity on the parabola). We call this the *co-normal system.* The triangle formed by the feet of concurrent normals will be termed 'a co-normal triangle.' The triads of points the centres of curvature at which are collinear form another complete system with the foci AA ∞. This we call the *co-central system.*

From the first representation of complete systems, we have—

(1) Any co-normal triangle is circumscribed to a parabola with the same vertex A and with its axis perpendicular to the axis of the given parabola.

(2) Any co-central triangle is circumscribed to a rectangular hyperbola the asymptotes of which are the tangent at the vertex and the axis.

The second representation gives

(3) If a conic with definite asymptotic directions circumscribes a co-normal triangle, it cuts the parabola again in a fixed point P, such that the involution determined at ∞ by the asymptotic directions and AP and the tangent at P has the point at ∞ on the axis for a fixed point.

In particular, the circum-circles of co-normal triangles pass through the vertex.

More generally, a conic circumscribed to a co-normal triangle with one of its axes parallel to the axis of the parabola, necessarily passes through the vertex. In particular, the *other* parabola which passes through the vertex and is circumscribed to a co-normal triangle has its axis perpendicular to the axis of the given parabola.

(4) A conic which has an asymptote parallel to the axis and is circumscribed to a co-central triangle intercepts on the tangent at the vertex a length which is bisected at the vertex.

Any conic having the tangent at the vertex for an asymptote and circumscribing a co-central triangle is a rectangular hyperbola.

We may also prove the following :

(5) A co-normal triangle may also be in addition co-central. This happens when the point of concurrence of the normals lies on the perpendicular to the axis at the cusp of the evolute ; further in this case the line of collinearity of the centres of curvature is a *diameter* of the parabola. The axis of the parabola is midway between the point of concurrence and the line of collinearity.

The polar line of a co-normal co-central triangle with respect to the parabola is the axis.

(6) Two co-normal triangles are harmonic when the join of their points of concurrence is a diameter ; two co-central triangles are harmonic when their lines of collinearity meet on the perpendicular to the axis at the cusp of the evolute. It is also easy to find the relation between the point of concurrence and the line of collinearity of two harmonic triangles one of which is co-normal and the other co-central.

R. VAIDYANATHASWAMI.

SOLUTIONS.

Question 754.

(S. RAMANUJAN):—Show that

$$\frac{e^x\,\Gamma(1+x)}{x^x\,\sqrt{\pi}} = \sqrt[6]{8x^3 + 4x^2 + x + \mathrm{E}}$$

where E lies between $\frac{1}{100}$ and $\frac{1}{80}$ for all positive values of x.

Remarks by E. H. Neville and C. Krishnamachary.

Mr. Madhava's method (vol. xii, pp. 101-102) leads much nearer to the solution than he supposes, for in passing from the first formula on p. 102 to the third he has overlooked the alternations of sign on the left-hand side. On making the correction, we have

$$\mathrm{E} = \tfrac{1}{80},\quad \mathrm{F} = -\tfrac{11}{2400}.$$

With these values, Mr. Ramanujam's assertion is seen to be credible, but more powerful means must be used if it is to be proved.

Questions 908 and 1012.

(K. APPUKUTTAN ERADY, M.A.):—Show that $(a_0a_1 \ldots a_{2n})\,(x, y)^{2n}$ is reducible to the form

$$p_1(x + b_1y)^{2n} + p_2(x + b_2y)^{2n} + \ldots\ldots + p_n(x + b_ny)^{2n}$$

if

$$\begin{vmatrix} a_0 & a_1 & \ldots & \ldots & a_n \\ a_1 & a_2 & \ldots & \ldots & a_{n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n & a_{n+1} & \ldots & \ldots & a_{2n} \end{vmatrix} = 0.$$

(N. SANKARA AIYAR):—Solve

(1) $x^6 + 12x^5 + 45x^4 + 100x^3 + 120x^2 + 78x + 21 = 0$;

(2) $2x^4 + 12x^3 + 30x^2 + 32x + 12 = 0$

by expressing the left-hand members in the forms $\mathrm{A}^6 + \mathrm{B}^6$ and $\mathrm{A}^4 + \mathrm{B}^4$ respectively.

Additional solution by N. Sankara Aiyar.

Expanding $\Sigma\, p_r(x + b_ry)^{2n}$ and equating coefficients of x, we get

$$\Sigma\, p_r = a_0,\quad \Sigma\, p_rb_r = a_1,\quad \Sigma\, p_rb_r^k = a_k \ldots\ldots$$

the number of these equations being $(2n + 1)$.

Eliminating $p_1 p_2 \ldots\ldots p_n$ from the first $n + 1$ of them, we get an equation giving $b_1 b_2 \ldots\ldots b_n$, viz,

$$\begin{vmatrix} 1 & 1 & \ldots & \ldots & a_0 \\ b_1 & b_2 & \ldots & \ldots & a_1 \\ b_1^2 & b_2^2 & \ldots & \ldots & a_2 \\ \ldots & & \ldots & \ldots & \ldots \\ \ldots & & \ldots & \ldots & \ldots \\ b_1^n & b_2^n & \ldots & \ldots & a_n \end{vmatrix} = 0.$$

This can be expanded and written in the form

$$\Sigma\, a_r A_r = 0 \text{ where } A_r \text{ is the minor of } a_r.$$

Again from the second $n + 1$ equations we shall get

$$\begin{vmatrix} b_1 & b_2 & \ldots & \ldots & a_1 \\ b_1^2 & b_2^2 & \ldots & \ldots & a_2 \\ \ldots & & \ldots & \ldots & \ldots \\ \ldots & & \ldots & \ldots & \ldots \\ b_1^{n+1} & b_2^{n+1} & \ldots & \ldots & a_{n+1} \end{vmatrix} = 0.$$

which may be written in the form $\Sigma\, a_{r+1} A_r = 0$.

From the $n + 1$ equations of the type $\Sigma\, a_{r+k} A_r = 0$ we get by eliminating the minors the required relation between the a's. If this relation is satisfied the $(n + 1)$ equations are consistent and we get the values of A_r and hence by solving the equations $A_r = k$ we can get the values of the b's and lastly of the p's. As an example let us take the function $x^4 + 8x^3 + 18x^2 + 20x + 8$, where the coefficients are easily seen to satisfy the equation.

Hence the equations for b_1, b_2 are

$$2b_1 b_2 - 3(b_1 + b_2) + 5 = 0$$
$$b_1 b_2 - 2(b_1 + b_2) + 3 = 0.$$

Since $b_1 \neq b_2$ this gives $b_1 = \dfrac{\sqrt{5} + 1}{2}$ and $b_2 = \dfrac{1 - \sqrt{5}}{2}$.

Substituting in the equations for the p's

we get $p_1 = \dfrac{5 - \sqrt{5}}{2}$ and $p_2 = \dfrac{\sqrt{5} - 3}{2}$.

Hence the given expression can be written as

$$\frac{5 - \sqrt{5}}{2}\left(x + \frac{\sqrt{5} + 1}{2}\right)^4 + \frac{\sqrt{5} - 3}{2}\left(x + \frac{1 - \sqrt{5}}{2}\right)^4.$$

Solution to Q. 1012, by K. B. Madhava.

[NOTE.—Applying this method to Q. 1012, it is easily seen that the polynomial expressions there given are equivalent to

$$\frac{5+3\sqrt{5}}{10}\left(x+\frac{\sqrt{5}+1}{2}\right)^6+\frac{5-3\sqrt{5}}{10}\left(x+\frac{1-\sqrt{5}}{2}\right)^6=0,$$

$$\frac{5+2\sqrt{5}}{5}\left(x+\frac{\sqrt{5}+1}{2}\right)^4+\frac{5-2\sqrt{5}}{5}\left(x+\frac{1-\sqrt{5}}{2}\right)^4=0;$$

from which the solutions of these equations are obvious.

In general, if the polynomial is of the form

$$x^{2n}+2nc_1\,.\,A_1\,x^{2n-1}+2nc_2\,.\,A_2\,x^{2n-2}+\ldots\ldots+A_{2n}=0,$$

where $A_{r+1}=A_r+A_{r-1}$ with $A_1=2$ and $A_2=3$;

it can be expressed in the form

$$\frac{5+3\sqrt{5}}{10}\left(x+\frac{\sqrt{5}+1}{2}\right)^{2n}+\frac{5-3\sqrt{5}}{10}\left(x+\frac{1-\sqrt{5}}{2}\right)^{2n}=0,$$

and the solution of the equation in this form is immediate.]

Question 957.

(MARTYN M. THOMAS, M.A.):—From a flexible envelope in the form of a surface of revolution formed by the curve $s=f(y)$ revolving about the axis of x, that part between two meridians the planes of which are inclined to each other at an angle $\frac{2\pi}{m}$ is cut away, and the edges are then sewed together. Prove that the meridian curve of the new envelope will be $s=f\left(\frac{my}{m-1}\right)$.

Hence show that, if a lune of angle $\frac{4\pi}{5}$ be cut off from an oblate spheroid, the minor axis of whose generating ellipse is c, and eccentricity $\frac{3}{5}$, the meridian curve of the new surface of revolution will be the curve of sines $y=\frac{3}{4}c\sin\frac{x}{c}$.

Solution by N. Sundaram Aiyar.

Let y be the radius of the circular section at distance x from the origin, and y' that of the new section at the same distance from the origin. The two perimeters of the section are in the ratio

$$\frac{m}{m-1},\text{ so that }\frac{2\pi y}{2\pi y'}=\frac{m}{m-1}.$$

But the surface of revolution is formed by the curve $s = f(y)$. So, the meridian curve of the new envelope is $s = f\left(\frac{my}{m-1}\right)$.

The equation to the meridian curve of the oblate spheroid is $\frac{x^2}{c^2} + \frac{y^2}{a^2} = 1$ where a is the semi-major axis, so that $a^2 - c^2 = a^2 \cdot \frac{9}{25}$;

or $a = \frac{5c}{4}$; also $m = \frac{5}{2}$.

The equation to the curve is therefore $25x^2 + 16y^2 = 25c^2$.

$$\therefore \quad \frac{dx}{dy} = -\frac{16y}{25x} \text{ or } \left(\frac{ds}{dy}\right)^2 = 1 + \frac{256y^2}{25(25c^2 - 16y^2)}.$$

$$\therefore \quad \frac{ds}{dy} = \phi(y) = \sqrt{\frac{625c^2 - 144y^2}{25(25c^2 - 16y^2)}}.$$

$$\therefore \quad s = \int_0^y \phi(y)\, dy = \lambda(y) \text{ say.}$$

Then the meridian curve of the new surface of revolution is given by

$$s = \lambda\left(\frac{my}{m-1}\right) = \lambda\left(\frac{5y}{3}\right),$$

or
$$\frac{ds}{dy} = \frac{5}{3}\phi\left(\frac{5y}{3}\right) = \frac{5}{3}\sqrt{\frac{625c^2 - 16.25y^2}{25(25c^2 - 16.25y^2/9)}}$$

$$= \sqrt{\frac{25c^2 - 16y^2}{9c^2 - 16y^2}}.$$

$$\therefore \quad \left(\frac{dx}{dy}\right)^2 = \frac{16c^2}{9c^2 - 16y^2}$$

or
$$x = \int_0^y \frac{4c\,dy}{\sqrt{9c^2 - 16y^2}}$$

$$= c \sin^{-1}\frac{4y}{3c}$$

or
$$y = \frac{3c}{4}\sin\frac{x}{c}.$$

Question 996.

Find the conditions that the conics

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and
$$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$$

have one asymptote in common.

Solution by P. R. Venkatakrishna Iyer and others.

Let $y = mx + n$ be the common asymptote. The x-co-ordinate of the points of intersection of $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and $y = mx + n$ are given by

$$ax^2 + 2hx(mx + n) + b(mx + n)^2 + 2gx + 2f(mx + n) + c = 0,$$

i.e., $x^2(a + 2hm + bm^2) + 2x(hn + bmn + g + fm) + bn^2 + 2fn + c = 0.$

If $y = mx + n$ is an asymptote, then

$$a + 2hm + bm^2 = 0, \quad \ldots \quad \ldots \quad \text{(i)}$$

and

$$hn + bmn + g + fm = 0. \quad \ldots \quad \ldots \quad \text{(ii)}$$

Similarly, if $y = mx + n$ is an asymptote of the other conic, then

$$a' + 2h'm + b'm^2 = 0, \quad \ldots \quad \ldots \quad \text{(iii)}$$

$$h'n + b'mn + g' + f'm = 0. \quad \ldots \quad \ldots \quad \text{(iv)}$$

Eliminating m between (i) and (iii), we get

$$H^2 = 4A.B, \quad \ldots \quad \ldots \quad \text{(I)}$$

where $A = bh' - b'h$; $B = ha' - h'a$; $H = ab' - a'b$.

From (ii) and (iv)

$$n = -\frac{g + fm}{h + bm} = -\frac{g' + f'm}{h' + b'm},$$

i.e., $m^2(b'f - bf') + m(b'g - bg' + h'f - hf') + (h'g - hg') = 0.$ (v)

Eliminating m^2 and m between (ii), (iii) and (v), we have

$$\begin{vmatrix} a & 2h & b \\ a' & 2h' & b' \\ (h'g - hg') & (b'g - bg' + h'f - hf') & (h'g - hg') \end{vmatrix} = 0,$$

which on simplification reduces to

$$(g\sqrt{A} + f\sqrt{B})(b'\sqrt{B} + h'\sqrt{A}) = (g'\sqrt{A} + f'\sqrt{B})(b\sqrt{B} + h\sqrt{A}). \quad \text{(II)}$$

The conditions (I) and (II) are the required conditions for a common asymptote.

Question 1010.

(K. J. Sanjana, M.A.) :—Solve the equation $x^3 + y^3 = a^3 + b^3$, where a and b are given positive quantities and x and y are required to be positive.

Examples. $x^3 + y^3 = (\frac{5}{2})^3 + (\frac{3}{2})^3$, $x^3 + y^3 = 91 = 4^3 + 3^3$.

Solution by N. B. Mitra.

A particular solution of the problem may be given by following Vieta's method.

Let $u^3 - v^3 = a^3 + b^3$.

Assume $u = a + z,\ v = kz - b$.

Then $z^2(1 - k^3) + 3z(a + bk^2) + 3(a^2 - b^2k) = 0$

To get a particular solution assume $k = a^2/b^2$.

Then $z = \dfrac{3ab^3}{a^3 - b^3}$. $\therefore\ u = \dfrac{a(a^3 + 2b^3)}{a^3 - b^3},\ v = \dfrac{b(2a^3 - b^3)}{a^3 - b^3}$.

Now we have $x^3 + y^3 = u^3 - v^3$.

If $u^3 > 2v^3$, assume $x = p - v,\ y = u - rp$;
then, $p^2(1 - r^3) + 3p^2(r^2u - v) + 3p(v^2 - u^2r) = 0$.

A particular solution is obtained by putting $r = v^3/u^3$.

We get

$$p = \frac{3vu^3}{u^3 + v^3}$$

and finally

$$x = \frac{v(2u^3 - v^3)}{u^3 + v^3},\ y = \frac{u(u^3 - 2v^3)}{u^3 + v^3}.$$

If, however, $u^3 < 2v^3$, we proceed thus:

Assume $u_1^3 - v_1^3 = u^3 - v^3$.

Let $u_1 = p_1 - v$ and $v_1 = r_1p_1 - u$; then proceeding in the same way, we get

$$u_1 = \frac{v(2u^3 - v^3)}{u^3 + v^3},\ v_1 = \frac{u(2v^3 - u^3)}{u^3 + v^3};$$

and finally $x = \dfrac{v_1(2u_1^3 - v_1^3)}{u_1^3 + v_1^3}$ and $y = \dfrac{u_1(u_1^3 - 2v_1^3)}{u_1^3 + v_1^3}$.

Examples: (1) $a = \frac{5}{2},\ b = \frac{3}{2}$: then

$$u = \frac{895}{196},\ v = \frac{669}{196},$$

and $x = \dfrac{1134416441}{196 \cdot 1016335684},\ y = \dfrac{895 \cdot 118080757}{196 \cdot 1016335684}$.

(2) $a = 4,\ b = 3$: then

$$u = \frac{472}{37},\ v = \frac{303}{37},$$

and $x = \dfrac{303 \cdot 182489969}{132972175 \cdot 37},\ y = \dfrac{472 \cdot 49517794}{132972175 \cdot 37}$.

Question 1030.

(SELECTED) :—If $x^{2n-1} + y^{2n-1} + a_1^{2n-1} + a_2^{2n-1} + \ldots + a_{2r}^{2n-1} = 0$, for all integral values of n from 1 to r inclusive, then

$$(x+a_1)(x+a_2) \ldots\ldots (x + a_{2r}) = (y + a_1)(y + a_2) \ldots\ldots (y + a_{2r}).$$

Solution by F. H. V. Gulasekharam, and N. G. Leather.

Let S_m denote the sum of the m^{th} powers, and p_m, the sum of the m—ary products of the quantities $x, y, a_1, a_2, \ldots\ldots\ldots a_{2r}$. Let P_m denote the sum of the m —ary products of the quantities $a_1, a_2 \ldots\ldots\ldots a_{2r}$.

By the question $S_1 = S_3 = \ldots\ldots\ldots\ldots\ldots = S^{2r-1} = 0.$

Hence $p_1 = p_3 = \ldots\ldots\ldots\ldots\ldots = p_{2r-1} = 0.$

These are equivalent to the following $(r - 1)$ equations :—

$$\begin{aligned}
(x + y) + p_1 &= 0 \\
p_1 \cdot xy + p_2 \cdot (x + y) + p_3 &= 0 \\
p_3 \cdot xy + p_4 \cdot (x + y) + p_5 &= 0 \\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots & \\
p_{2r-3} \cdot xy + p_{2r-2} \cdot (x + y) + p_{2r-1} &= 0.
\end{aligned}$$

Now, denoting $x^m - y^m$ by D_m, multiply the above $(r - 1)$ equations in order by $D_{2r}, D_{2r-2}, \ldots\ldots D_2$; and add the results. Then

$$D_{2r} \cdot (x + y) + p_1 [D_{2r} + xy \cdot D_{2r-2}] + p_2 D_{2r-2}(x + y) + p_3 [D_{2r-2} + xy\, D_{2r-4}] + \ldots\ldots = 0 \ldots \quad (1)$$

Now remembering that $D_{2m} + xy \cdot D_{2m-2} \equiv (x+y) D_{2m-1}, \ldots \quad (2)$

the equation (1) may be written in the form

$$(x+y)[D_{2r} + p_1 D_{2r-1} + p_2 D_{2r-2} + \ldots\ldots + p_{2r-2} D_2 + p_{2r-1} D_1] = 0$$

Hence,

$$\begin{aligned}
x^{2r} + p_1 x^{2r-1} + \ldots\ldots\ldots + p_{2r-1} x \\
= y^{2r} + p_1 y^{2r-1} \ldots + p_{2r-1} y.
\end{aligned}$$

$$\therefore (x + a_1)(x + a_2) \ldots\ldots (x + a_{2r}) = (y + a_1)(y + a_2) \ldots\ldots\ldots (y + a_{2r}).$$

Question 1031.

(C. KRISHNAMACHARI) :— With the usual notation in elliptic functions, show that if $u + v + w = 0$, then

$$p(u)\, p(v)\, p(w) = \tfrac{1}{4} g_3 + \tfrac{1}{4} \left\{ \frac{p'(u)\, p(v) - p'(v)\, p(u)}{p(u) - p(v)} \right\}^2$$

Deduce that

$$p(z)\,[p(z+w_1)+p(z+w_2)+p(z+w_3)]$$
$$=\tfrac{1}{4}\,p'(z)\left\{\frac{1}{e_1-p(z)}+\frac{1}{e_2-p(z)}+\frac{1}{e_3-p(z)}\right\}.$$

Solution by F. H. V. Gulasekharam; and V. Tiruvenkatachari.

Since $u+v+w=0$,

the points $\{p(u), p'(u)\}, \{p(v), p'(v)\}, \{p(w), p'(w)\}$,

of the curve $y^2=4x^3-g_2x-g_3$ are in a straight line.

Let the straight line be $y=Ax+B$.

$$\therefore\quad Ap(u)+B=p'(u);\quad Ap(v)+B=p'(v).$$

$$\therefore\quad B=-\frac{p'(u)\,p(v)-p'(v)\,p(u)}{p(u)-p(v)},$$

also $p(u)$, $p(v)$, $p(w)$, are the roots of the cubic equation

$$4x^3-g_2x-g_3-(Ax+B)^2=0.$$

$$\therefore\quad p(u)\,p(v)\,p(w)=\tfrac{1}{4}\,g_3+\tfrac{1}{4}\,B^2$$
$$=\tfrac{1}{4}\,g_3+\tfrac{1}{4}\left\{\frac{p'(u)\,p(v)-p(u)\,p'(v)}{p(u)-p(v)}\right\}^2.$$

The second part of the question seems to be inaccurate.

For, $$p(z)\,p(z+w_1)\,e_1=\tfrac{1}{4}\,g_3+\tfrac{1}{4}\,e_1{}^2.\frac{p'^2(z)}{(p(z)-e^1)^2}.$$

Hence $p(z)\{p(z+w_1)+p(z+w_2)+p(z+w_3)\}+\tfrac{1}{4}\,g_2$ (after some easy reduction) is equal to

$$\frac{1}{4}\,p'^2(z)\cdot\sum\frac{e_r}{\{p(z)-e_r\}^2}$$

[Note.—That the question is inaccurate can be easily seen by applying Liouville's Theorem; for, a pole, say $z=0$, is of order 2 for the left side, while it is only simple for the right-hand expression: K. B. M.]

QUESTIONS FOR SOLUTION.

1179. (F. H. V. Gulasekharam):—From an external point T (x, y), tangents TP, TQ are drawn to the conic $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, prove that the area of the triangle TPQ is

$$\frac{S^{\frac{3}{2}}\sqrt{-\Delta}}{\Delta - CS};$$

and that the area of the quadrilateral OPTQ, where O is the centre of the conic, is

$$\frac{1}{2}\frac{\sqrt{-\Delta S}}{\Delta - CS}\left[x\frac{\partial S}{\partial x} + y.\frac{\partial S}{\partial y}\right],$$

where $\Delta = \begin{vmatrix} a, & h, & g, \\ h, & b, & f, \\ g, & f, & c, \end{vmatrix}$, and $C = ab - h^2$.

1180. (F. H. V. Gulasekharam):—If g and h are the lengths of the bisectors of the angles B and C respectively of a triangle ABC, prove that

$$4a^2 > g^2 + h^2.$$

1181. (C. N. Sreenivasa Iyengar):—A'B'C' is the triangle formed by joining the midpoints of Δ ABC; A''B''C'' is similarly formed from A'B'C', A'''B'''C''' from A''B''C'' and so on. If S, S', S'', etc., N, N', N'', etc., and O, O', O'', etc., denote the corresponding circum-centres, N.P. centres and ortho-centres respectively, then the three sets of points all tend to the same limit. Find the exact position of this limit in relation to the triangle ABC.

1182. (B. B. Bagi):—The sides taken in order of an n-gon (n being odd) circumscribed to a circle are $a_1, a_2, a_3, \ldots a_n$. Prove that the radius of the inscribed circle is given by

(i) $$\tan^{-1}\frac{x}{s - a_1 - a_3 - a_5 \ldots - a_{n-2}}$$

$$+ \tan^{-1}\frac{x}{s - a_2 - a_4 - a_6 - \ldots - a_{n-1}}$$

$$+ \tan^{-1} \frac{x}{s - a_3 - a_5 - a_7 - \ldots - a_n} + \ldots$$

$$+ \tan^{-1} \frac{x}{s - a_n - a_2 - a_4 - a_{n-3}} = \frac{\pi}{2}(n-2),$$

(ii) and that the area Δ is determined by

$$\tan^{-1} \frac{\Delta}{s(s - a_1 - a_3 - a_5 - \ldots - a_{n-2})}$$

$$+ \tan^{-1} \frac{\Delta}{s(s - a_2 - a_4 - \ldots - a_{n-1})}$$

$$+ \tan^{-1} \frac{\Delta}{s(s - a_3 - a_5 - a_7 - \ldots - a_n)} + \ldots$$

$$+ \tan^{-1} \frac{\Delta}{s(s - a_n - a_2 - a_4 \ldots - a^{s-n})}$$

$$= \frac{\pi}{2}(n-2),$$

where $2s = a_1 + a_2 + a_3 + \ldots + a_n$.

1183. (B B. Bagi):—The circles round AQR, BRP, CPQ where P, Q and R are points in order on the sides BC, CA, AB of a triangle ABC, meet in O. If A′, B′, C′ are the middle points of the arcs QOR, ROP, POQ then show that A′B′C′ is a triangle similar to the triangle of the ex-centres of ABC, and also that A′, B′, C′ and the in-centre of ABC are concyclic.

1184. (G. S. Mahajani):—In any triangle, we know that

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

Similarly, in any polygon of n sides $(a_1\ a_2\ \ldots\ a_n)$

$$a_n^2 = \sum_1^{n-1} a_r^2 - 2 \sum a_r a_s \cos \hat{a_r a_s},$$

r and s being unequal and taking all integral values from 1 to $(n-1)$.

Symmetrically, the sides and angles of a polygon are connected by the following relation :—

$$\sum_1^n a_r^2 - 2 \sum a_r a_s \cos \hat{a_r a} = 0,$$

r s being unequal and taking all integral values from 1 to n.

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EXTENSION OF DR. STEWART'S THEOREMS TO NON-EUCLIDEAN GEOMETRY.

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[Dr. Stewart's "Theorems" have been proved geometrically by ral persons. Mr. Robert Leslie Ellis, Senior Wrangler 1840, gave ytical proofs by means of a formula which is of a general interest. It ie object of this note to use this formula in the case of some closed es and to find out the analogues of the theorems in non-euclidean space.]

1. We shall first state the analytical formula of Mr. Ellis and give roof in the words of the author.

Lemma. If $f(\phi)$ is a rational integral function of $\sin\phi$ and $\cos\phi$, a value may always be assigned to n, such that

$$S \equiv f(\phi) + f\left(\phi + \frac{2\pi}{n}\right) + \ldots\ldots + f\left(\phi + \frac{n-1}{n}2\pi\right)$$

be independent of ϕ.

If $$Df(\phi) = f\left(\phi + \frac{2\pi}{n}\right),$$

preceding expression is equivalent to

$$(1 + D + \ldots D^{n-1})f(\phi),$$

$$= \frac{D^n f(\phi) - f(\phi)}{D - 1}$$

$$= \frac{1}{D-1}\{f(\phi + 2\pi) - f(\phi)\} = \Delta^{-1}(0).$$

Now obviously,

$$\Delta_{-1}(0) = \Sigma a_m \sin mn\phi + \Sigma b_m \cos mn\phi, \ (m \text{ integral}).$$

Hence $$S = \Sigma a_m \sin mn\phi + \Sigma b_m \cos mn\phi.$$

Let the index of the highest power of $\sin\phi$ or $\cos\phi$ in $f(\phi)$ be p then it is easily seen that when $f(\phi)$ is developed as it may always be in a series of sines and cosines of multiples of ϕ, $(p\phi)$ will be the largest angle that can enter into the development; but if n is greater than p, $mn\phi$ will be greater than $p\phi$, except when $m = 0$. Hence the development

$$\Sigma a_m \sin mn\phi + \ldots\ldots$$

cannot coincide with that obtained by summing the separate development of

$$f(\phi),\ f\left(\phi + \frac{2\pi}{n}\right), \text{ etc., unless } a_m = b_m = 0$$

in every case except when $m = 0$. Hence, as $\sin mn\phi = 0$ when $m = 0$, the expression will be reduced to b_0 and we shall have, when $n > p$,

$$f(\phi) + \ldots\ldots f\left(\phi + \frac{n-1}{n}\,2\pi\right) = b_0 = \text{a constant}.$$

The constant b_0 will of course be the sum of the constant parts of the development of $f(\phi)$, $f\left(\phi + \frac{2\pi}{n}\right)$, etc., and as these are n in number, it will be n times the constant part of $f(\phi)$; but this is by Fourier's theorem equal to

$$\frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi.$$

Hence

$$f(\phi) + f\left(\phi + \frac{2\pi}{n}\right) + \ldots f\left(\phi + \frac{n-1}{n}\,2\pi\right) = \frac{n}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi,$$

when $n > p$, which is the analytical formula of Mr. Ellis.

2. We shall now give some of its applications to simple curves before stating Dr. Stewart's theorems and finding their analogues in *Elliptic Space*.

Ex. (1). If any n radii vectores be drawn from the centre of an ellipse at equal angles to one another, the sum of the squares of their reciprocals is equal to n times the square of the reciprocal of that radius vector which is equally inclined to the major and minor axes.

[For, we have

$$\frac{1}{r^2} = \frac{1}{b^2}(1 - e^2\cos^2\theta).$$

$$\therefore \quad \sum\frac{1}{r^2} = \frac{n}{b^2} - \frac{e^2}{b^2}\sum\cos^2\theta\,;$$

and $$\sum\cos^2\theta = \frac{n}{2\pi}\int_0^{2\pi}\cos^2\theta\,d\theta = \frac{n}{2}, \text{ by § 1.}$$

$$\therefore \quad \sum\frac{1}{r^2} = \frac{n}{b^2} - \frac{ne^2}{2b^2} = \frac{n}{b^2}\left(1 - \frac{e^2}{2}\right) = \frac{n}{b^2}\left(1 - e^2\cos^2\frac{\pi}{4}\right),$$

which proves the result.

Ex. (2). In the previous example, consider the fourth powers of the reciprocals of the radii vectores :

$$\frac{1}{r^4} = \frac{1}{b^4}(1 - e^2\cos^2\theta)^2 = \frac{1}{b^4}\{1 - 2e^2\cos^2\theta + e^4\cos^4\theta\}.$$

$$\therefore \quad \sum\frac{1}{r^4} = \frac{n}{b^4} - 2\frac{e^2}{b^4}\sum\cos^2\theta + \frac{e^4}{b^4}\sum\cos^4\theta.$$

Now $$\sum\cos^4\theta = \frac{n}{2\pi}\int_0^{2\pi}\cos^4\theta\,d\theta = \frac{3n}{8}.$$

$$\therefore \quad \sum\frac{1}{r^4} = \frac{n}{b^4} - \frac{2e^2}{b^4}\cdot\frac{n}{2} + \frac{e^4}{b^4}\cdot\frac{3n}{8} = \frac{n}{b^4}\left(1 - e^2 + \frac{3e^4}{8}\right).$$

In general, $$\sum\frac{1}{r^{2m}} = \frac{1}{b^{2m}}(1 - e^2\cos^2\theta)^m$$

$$= \frac{1}{b^{2m}}\Sigma[1 - {}^mC_1\,e^2\cos^2\theta + {}^mC_2\,e^4\cos^4\theta - \ldots\ldots \pm e^{2m}\cos^{2m}\theta]$$

$$= \frac{n}{b^{2m}} - {}^mC_1\frac{e^2}{b^{2m}}\sum\cos^2\theta + \ldots\ldots \pm \frac{e^{2m}}{b^{2m}}\sum\cos^{2m}\theta,$$

where $$\sum\cos^{2m}\theta = \frac{n}{2\pi}\int_0^{2\pi}\cos^{2m}\theta\,d\theta$$

$$= \frac{n}{2\pi}\left[\frac{(2m-1).(2m-3)\ldots\ldots 1}{m.(m-1)\ldots\ldots 1}.2\pi\right].$$

Similar properties exist for the limacon and the lemniscate as may be readily seen.

Ex. (3). If n radii vectores be drawn from the centre of an ellipse equally inclined to one another to meet its inverse with respect to the centre as base, shew that $\Sigma\rho^{\frac{2}{3}}r^3$ is constant, where r is the radius vector

to the inverse curve and ρ the radius of curvature at the point on the ellipse determined by r;

$$\text{For} \quad \rho = \frac{(\text{normal})^3}{l^2}.$$

$$\therefore \quad \rho^{\frac{2}{3}} = \frac{(\text{normal})^2}{l^{\frac{4}{3}}} = (ab)^{\frac{4}{3}} r_1^2 \left(\frac{\sin^2\theta}{b^4} + \frac{\cos^2\theta}{a^4} \right),$$

where r_1 is the corresponding radius vector of the ellipse.

$$\therefore \quad \rho^{\frac{2}{3}} r^2 = (ab)^{\frac{4}{3}} k^4 \left\{ \frac{1}{b^4} - \left(\frac{1}{b^4} - \frac{1}{a^4} \right) \cos^2\theta \right\},$$

k being the radius of inversion.

$$\therefore \quad \sum \rho^{\frac{2}{3}} r^2 = (ab)^{\frac{4}{3}} k^4 \left\{ \frac{n}{b^4} - \frac{n}{2} \left(\frac{1}{b^4} - \frac{1}{a^4} \right) \right\}$$

which proves the result.

As significantly remarked by Mr. Ellis, every curve could have an interpretation for the analytical formula. The interpretation that we have given is applicable only to closed curves.

3. We shall now state Dr. Stewart's theorems and find out corresponding results in Elliptic Space.

The following are the theorems of Dr. Stewart :—

(1) From any point in the circumference of a circle draw perpendiculars p, p_1, ... to the sides of a regular n-sided circumscribed polygon; then if r is the radius,

$$2\Sigma p^3 = 5nr^3. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Dem. Let the assumed point subtend at the centre an angle ϕ from the adjacent point of contact. Then

$$p = r(1 - \cos\phi), \; p_1 = r\left\{ 1 - \cos\left(\phi + \frac{2\pi}{n} \right) \right\}, \text{ etc.}$$

$$\therefore \quad \Sigma(p^3) = r^3 \Sigma(1 - \cos\phi)^3.$$

By the general formula

$$\sum (1 - \cos\phi)^3 = \frac{n}{2\pi} \int_0^{2\pi} (1 - \cos\phi)^3 \, d\phi$$

$$= \frac{n}{2\pi} \cdot 2^4 \int_0^{2\pi} \sin^6\theta \, d\theta \ldots\ldots (\theta = \tfrac{1}{2}\phi)$$

$$= \frac{n}{\pi} \cdot 2^3 \cdot \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \pi = \frac{5}{2^5} \cdot 2\pi.$$

$$\therefore \quad 2\Sigma p^3 = 5nr^3.$$

Cor. If the assumed point be at a distance l from the centre, instead of on the circumference, prove that

$$2\Sigma p^3 = 2nr^3 + 3nl^2r . \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

(2) A regular n-sided polygon is inscribed in the circle and lines c_1, c_2, ... are drawn from its corners to a point assumed on the circumference, then $\Sigma c^4 = 6nr^4$. (3)

Cor. If the assumed point be at a distance l from the centre, shew that $\Sigma c^4 = nr^4 + 4nr^2l^2 + nl^4$. (4)

(3) We return to the circumscribed polygons and determine the sum of the fourth powers of the perpendiculars

$$8\Sigma p^4 = 35nr^4. \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Cor. In the general case when the assumed point is at a distance l from the centre, to shew that

$$8\Sigma p^4 = 8nr^4 + 24nr^2l^2 + 3nl^4. \quad \ldots \quad \ldots \quad (6)$$

(4) To find now the sum of the m^{th} powers of the perpendiculars; that is, to shew that

$$\Sigma p^m = n \cdot \frac{2m-1 \,.\, 2m-3 \,\ldots\ldots\, 1}{m \,.\, m-1 \,\ldots\ldots\, 1} r^m. \quad \ldots \quad (7)$$

Cor. If the assumed point is at a distance l from the centre, to shew that

$$\Sigma p^m = nr^m + n \cdot \frac{m.m-1}{1.2} \cdot \tfrac{1}{2} r^{m-2} l^2 + n. \frac{m.m-1.m-2.m-3}{4!} \cdot \frac{3.1}{4.2} r^{m-4} l^4.$$

$$+ \ldots\ldots \quad (n > m). \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

(5) Lastly, consider the $2m^{\text{th}}$ powers of the chords in the case of the inscribed polygons:

$$c^2 = 2r^2(1 - \cos\phi).$$

$$\therefore \quad \Sigma c^{2m} = 2^m r^m \Sigma(1 - \cos\phi)^m$$

$$= n \cdot \frac{2m-1 \,.\, 2m-3 \,\ldots\ldots\, 1}{m \,.\, m-1 \,\ldots\ldots\, 1} r^{2m} \ldots \quad \ldots \quad (9)$$

Cor. In the general case when the assumed point is at a distance l from the centre, we have

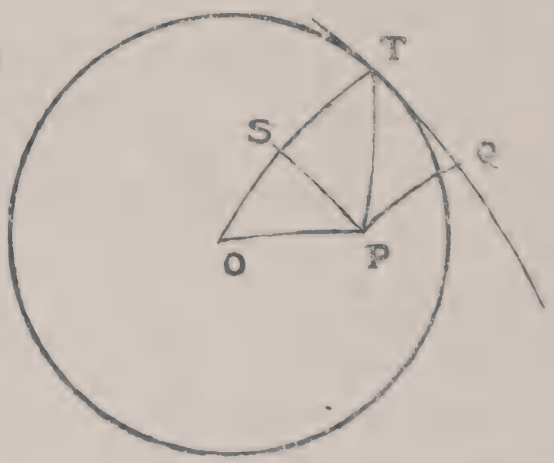
$$c^2 = r^2 + l^2 - 2rl\cos\phi.$$

$$\therefore \quad \Sigma c^{2m} = \Sigma(r^2 + l^2 - 2rl\cos\phi)^m$$
$$= n.(r^2 + l^2)^m + {}^mC_2.(r^2 + l^2)^{m-2}(2rl)^2\Sigma\cos^2\phi$$
$$+ {}^mC_4(r^2 + l^2)^{m-4}(2rl)^4\Sigma\cos^4\phi + \ldots\ldots$$
(odd powers vanish in the summation)
$$= n(r^2 + l^2)^m + {}^mC_2(r^2 + l^2)^{m-2}(2rl)\frac{n}{2}$$
$$+ {}^mC_4(r^2 + l^2)^{m-4}(2rl)^4.\frac{3n}{8} + \ldots\ldots \quad \ldots (10)$$

[Proofs of the above depend upon the following :—

$$\Sigma\cos^m\phi = \frac{n}{2\pi}\int_0^{2\pi}\cos^m\phi\,d\phi$$
$$= \frac{n}{2\pi}.4.\int_0^{\pi/2}\cos^m\phi\,d\phi \text{ (when } m \text{ is even integer)}$$
$$= 2\frac{n}{\pi}.\frac{m-1.m-3\ldots\ldots1}{2.4.6\ldots\ldots m}.\frac{\pi}{2}\Big].$$

4. To extend these propositions to non-euclidean space :



In elliptic* space, let a circle have a radius r. P, any point at a distance l from O and TQ the tangent at T and PQ $\perp$ from P to the tangent and equal to p say. Draw a $\perp$ PS on OT. Then the quadrilateral S'PQT is tri-rectangular ; if we write the parts

$$\left[-\frac{\pi}{2} + \angle SPQ,\ \frac{SP}{k},\ \frac{\pi}{2} - \frac{ST}{k},\ \frac{\pi}{2} - \frac{TQ}{k},\ \frac{PQ}{k}\right]$$

* Results in hyperbolic geometry will be obtained by changing k into ik in the results of elliptic geometry and consequently need not be proved separately.

in cyclic order, then we have the rule :

sine of the middle part = product of the cosines of the opposite parts.

Hence we get

$$\sin \frac{PQ}{k} = \cos \frac{SP}{k} \sin \frac{ST}{k}$$

i.e. $$\sin \frac{p}{k} = \cos \frac{SP}{k} \sin \frac{(r - OS)}{k}$$

$$= \cos \frac{SP}{k} \left\{ \sin \frac{r}{k} \cos \frac{OS}{k} - \cos \frac{r}{k} \sin \frac{OS}{k} \right\}$$

$$= \sin \frac{r}{k} \cos \frac{SP}{k} \cos \frac{OS}{k} - \cos \frac{r}{k} \cos \frac{SP}{k} \sin \frac{OS}{k}$$

$$= \sin \frac{r}{k} \cos \frac{l}{k} - \cos \frac{r}{k} \cos \frac{l}{k} \tan \frac{OS}{k}$$

(by solving the right-angled Δ OSP)

$$= \sin \frac{r}{k} \cos \frac{l}{k} - \cos \frac{r}{k} \cos \frac{l}{k} \tan \frac{l}{k} \cos \phi$$

$$= \sin \frac{r}{k} \cos \frac{l}{k} - \cos \frac{r}{k} \sin \frac{l}{k} \cos \phi. \quad \ldots \quad \ldots \quad (1)$$

In particular, if the point be on the circumference, we get

$$2 \sin \frac{p}{k} = \sin \frac{2r}{k} (1 - \cos \phi). \quad \ldots \quad \ldots \quad (2)$$

If the chord PT be called c, we get by solving the ΔOPT,

$$\cos \frac{c}{k} = \cos \frac{r}{k} \cos \frac{l}{k} + \sin \frac{r}{k} \sin \frac{l}{k} \cos \phi ; \quad \ldots \quad (3)$$

and when P is on the circumference,

$$\cos \frac{c}{k} = \cos^2 \frac{r}{k} + \sin^2 \frac{r}{k} \cos \phi. \quad \ldots \quad \ldots \quad (4)$$

5. The following are the analogues of Dr. Stewart's theorems :

(1) $$\sin \frac{p}{k} = \frac{1}{2} \sin \frac{2r}{k} (1 - \cos \phi),$$

$$\sin \frac{p_1}{k} = \frac{1}{2} \sin \frac{2r}{k} \left\{ 1 - \cos \left(\phi + \frac{2\pi}{r} \right) \right\}, \text{ etc.}$$

$$\therefore \sum \sin^3 \frac{p}{k} = \frac{1}{8} \sin^3 \frac{2r}{k} \sum (1 - \cos \phi)^3 = \frac{1}{8} \sin^3 \frac{2r}{k} \cdot \frac{5n}{2}$$

$$= \frac{5n}{16} \sin^3 \frac{2r}{k}.$$

Cor. $\sin \frac{p}{k} = \sin \frac{r}{k} \cos \frac{l}{k} - \cos \frac{r}{k} \sin \frac{l}{k} \cos \phi$, &c.

$$\therefore \sum \sin^3 \frac{p}{k} = n \sin^3 \frac{r}{k} \cos^3 \frac{l}{k} - 3 \sin^2 \frac{r}{k} \cos^2 \frac{l}{k} \cos \frac{r}{k} \sin \frac{l}{k} \sum \cos \phi$$

$$+ 3 \sin \frac{r}{k} \cos \frac{l}{k} \cos^2 \frac{r}{k} \sin^2 \frac{l}{k} \sum \cos^2 \phi - \cos^3 \frac{r}{k} \sin^3 \frac{l}{k} \sum \cos^3 \phi$$

$$= \frac{1}{2} n \sin \frac{r}{k} \cos \frac{l}{k} \left(2 \sin^2 \frac{r}{k} \cos^2 \frac{l}{k} + 3 \cos^2 \frac{r}{k} \sin^2 \frac{l}{k} \right).$$

$$\therefore 2 \sum \sin^3 \frac{p}{k} = n \sin \frac{r}{k} \cos \frac{l}{k} \left(2 \sin^2 \frac{r}{k} \cos^2 \frac{l}{k} + 3 \cos^2 \frac{r}{k} \sin^2 \frac{l}{k} \right).$$

(2) $\cos \frac{c}{k} = \cos^2 \frac{r}{k} + \sin^2 \frac{r}{k} \cos \phi$, &c.

$$\therefore \sum \cos^2 \frac{c}{k} = n \cos^4 \frac{r}{k} + 2 \cos^2 \frac{r}{k} \sin^2 \frac{r}{k} \sum \cos \phi + \sin^4 \frac{r}{k} \sum \cos^2 \phi$$

$$= n \cos^4 \frac{r}{k} + \frac{n}{2} \sin^4 \frac{r}{k}.$$

Cor. $\cos \frac{c}{k} = \cos \frac{r}{k} \cos \frac{l}{k} + \sin \frac{r}{k} \sin \frac{l}{k} \cos \phi$, &c.

$$\therefore \sum \cos^2 \frac{c}{k} = n \cos^2 \frac{r}{k} \cos^2 \frac{l}{k} + 2 \sin \frac{r}{k} \sin \frac{l}{k} \cos \frac{r}{k} \cos \frac{l}{k} \sum \cos \phi$$

$$+ \sin^2 \frac{r}{k} \sin^2 \frac{l}{k} \sum \cos^2 \phi$$

$$= n \cos^2 \frac{r}{k} \cos^2 \frac{l}{k} + \frac{n}{2} \sin^2 \frac{r}{k} \sin^2 \frac{l}{k}.$$

(3) $$\sum \sin^4 \frac{p}{k} = \frac{1}{16} \sin^4 \frac{2r}{k} \sum (1 - \cos \phi)^4$$

$$= \frac{35n}{128} \sin^4 \frac{2r}{k}.$$

Cor. $\sin \frac{p}{k} = \sin \frac{r}{k} \cos \frac{l}{k} - \cos \frac{r}{k} \sin \frac{l}{k} \cos \phi$, &c.

$$\therefore \sum \sin^4 \frac{p}{k} = n \sin^4 \frac{r}{k} \cos^4 \frac{l}{k} - 4 \sin^3 \frac{r}{k} \cos^3 \frac{l}{k} \cos \frac{r}{k} \sin \frac{l}{k} \sum \cos \phi$$

$$+ 6 \sin^2 \frac{r}{k} \cos^2 \frac{l}{k} \cos^2 \frac{r}{k} \sin^2 \frac{l}{k} \sum \cos^2 \phi$$

$$- 4 \sin \frac{r}{k} \cos \frac{l}{k} \sin^3 \frac{l}{k} \cos^3 \frac{r}{k} \sum \cos^3 \phi + \cos^4 \frac{r}{k} \sin^4 \frac{l}{k} \sum \cos^4 \phi$$

$$= n \sin^4 \frac{r}{k} \cos^4 \frac{l}{k} + 3n \sin^2 \frac{r}{k} \cos^2 \frac{l}{k} \cos^2 \frac{r}{k} \sin^2 \frac{l}{k} + \frac{3n}{8} \cos^4 \frac{r}{k} \sin^4 \frac{l}{k}.$$

(4) $\sin \frac{p}{k} = \frac{1}{2} \sin \frac{2r}{k} (1 - \cos \phi)$, &c.

$$\therefore \sum \sin^m \frac{p}{k} = \frac{1}{2^m} \sin^m \frac{2r}{k} \sum (1 - \cos \phi)^m$$

$$= \frac{1}{2^m} \sin^m \frac{2r}{k} \cdot n \cdot \frac{2m - 1 \cdot 2m - 3 \ldots\ldots 1}{m \cdot m - 1 \ldots\ldots\ldots 1}.$$

Cor. $\sin \frac{p}{k} = \sin \frac{r}{k} \cos \frac{l}{k} - \cos \frac{r}{k} \sin \frac{l}{k} \cos \phi$, &c.

$$\therefore \sum \sin^m \frac{p}{k} = n \sin^m \frac{r}{k} \cos^m \frac{l}{k}$$

$$+ n \cdot {}^mC_2 \cdot \frac{1}{2} \sin^{m-2} \frac{r}{k} \cos^{m-2} \frac{l}{k} \cos^2 \frac{r}{k} \sin^2 \frac{l}{k}$$

$+ \ldots\ldots$ where $n > m$.

(5) $\cos \frac{c}{k} = \cos^2 \frac{r}{k} + \sin^2 \frac{r}{k} \cos \phi$, &c.

$$\therefore \sum \cos^m \frac{c}{k} = n \cos^{2m} \frac{r}{k} + \frac{n}{2} {}^mC_2 \cdot \cos^{2m-4} \frac{r}{k} \sin^4 \frac{r}{k}$$

$$+ n \cdot \frac{3 \cdot 1}{4 \cdot 2} {}^mC_4 \cos^{2m-8} \frac{r}{k} \sin^8 \frac{r}{k} + \ldots$$

Cor. When the assumed point is at a distance l from the centre, we can likewise calculate $\sum \cos^m \frac{c}{k}$.

ON A CERTAIN THEOREM IN GEOMETRY.

By F. H. V. Gulasekharam, M.A.

[1. A point P in the plane of the triangle ABC is defined by the angles PAB, PBC, PCA called angular co-ordinates and denoted by α, β, γ respectively.

2. The normal co-ordinates of P and its isogonal conjugate P′ are deduced in terms of α, β, γ and α', β', γ', where $\alpha' = A - \alpha$, &c.

3. The equation to the *inconic* of ABC having P and P′ for foci is deduced in the form $\Sigma\,[(\cot\alpha + \cot\alpha')\,x]^{\frac{1}{2}} = 0$. (areal)

(*a*) The co-ordinates of the centre of this inconic (*referred to the medial* triangle) are determined in a simple form.

(*b*) The semi-axes of the inconic are found to be

$$\rho = \frac{R \sin A \sin B \sin C}{(\Sigma \cot\alpha + \Sigma \cot\alpha')\sin\alpha \sin\beta \sin\gamma}$$

$$= \frac{R \sin A \sin B \sin C}{\cos\alpha \cos\beta \cos\gamma + \cos\alpha' \cos\beta' \cos\gamma'}$$

$$\rho'^2 = \frac{4R_2 \sin A \sin B \sin C}{\Sigma \cot\alpha + \Sigma \cot\alpha'}.$$

(*c*) The semi latus rectum $l = 4R \sin\alpha \sin\beta \sin\gamma$.

4. Two of Mr. V. Ramaswami Iyer's theorems and two of Mr. M. Bhimasena Rao's theorems are proved from the above results.

5. *Finally, the following new theorem is proved* :—

"*In any triangle, the radical axis of the Director circle of an inconic and the N. P. circle cuts the latter at an angle equal to the angle TOT′, where OT, OT′ are the tangents from the circum-centre O, to the inconic.*"]

1. Any point P in the plane of a triangle ABC may be defined by the angles PAB, PBC, PCA (denoted by α, β, γ respectively) which may be called the angular co-ordinates of P.

The identical relation satisfied by (α, β, γ). Let x, y, z be the *absolute* normal co-ordinates of P with respect to ABC.

Then $\frac{y}{z} = \frac{\sin(A-\alpha)}{\sin\alpha}$; $\frac{z}{x} = \frac{\sin(B-\beta)}{\sin\beta}$; $\frac{x}{y} = \frac{\sin(C-\gamma)}{\sin\gamma}$.

$\therefore$ $\sin\alpha \sin\beta \sin\gamma = \sin(A-\alpha)\sin(B-\beta)\sin(C-\gamma)$... (1.1)

Writing α', β', γ' for A—α, B—β, C—γ respectively, the relation (1.1) becomes

$$\sin\alpha \sin\beta \sin\gamma = \sin\alpha' \sin\beta' \sin\gamma'. \quad \ldots \quad \ldots \quad (1{\cdot}2)$$

NOTE. α', β' γ' are the angular co-ordinates of P', the isogonal conjugate of P with respect to ABC.

2. If (α, β, γ) $(\alpha', \beta' \gamma')$ be the angular co-ordinates of an isogonal pair of points, the following identities are interesting and useful in what follows :—

If $S = \Sigma(\alpha)$, $S' = \Sigma'(\alpha')$, so that $S + S' = \pi$;

$\Sigma_s = \Sigma \sin 2\alpha$; $\Sigma_{s'} = \Sigma \sin 2\alpha'$;

$\Sigma_c = \Sigma \cos 2\alpha$; $\Sigma_{c'} = \Sigma \cos 2\alpha'$;

$\sigma = \Sigma \cot\alpha + \Sigma \cot\alpha'$;

then (*a*) $$\frac{\Sigma_s + \Sigma_s'}{\sin S} = \frac{\Sigma_c - \Sigma_c'}{\cos S} = 4\sigma \sin\alpha \sin\beta \sin\gamma$$
$$= 4\cos\alpha\cos\beta\cos\gamma + 4\cos\alpha'\cos\beta'\cos\gamma'.$$

(*b*) $$4\Sigma(\sin\alpha \sin\alpha' \sin A) + 4 \sin A \sin B \sin C = \Sigma_s + \Sigma_s'.$$

To prove (*a*) : From (1.2),

$$4 \sin\alpha \sin\beta \sin\gamma = 4 \sin\alpha' \sin\beta' \sin\gamma' ;$$

and $$\sin S = \sin S' ;$$

$$\therefore \Sigma \sin(\beta + \gamma - \alpha) = \Sigma' \sin(\beta' + \gamma' - \alpha').$$

$$\therefore \Sigma \sin(S - 2\alpha) = \Sigma' \sin(S' - 2\alpha').$$

Hence remembering that $\sin S = \sin S'$; and $\cos S = -\cos S'$, we have

$$\frac{\Sigma_s + \Sigma_s'}{\sin S} = \frac{\Sigma - \Sigma_c'}{\cos S} = k \text{ (say)}. \quad \ldots \quad \ldots \quad (2.1)$$

Again

$$4\sigma \sin\alpha \sin\beta \sin\gamma = 4 \sin\alpha \sin\beta \sin\gamma \Sigma \cot\alpha + 4 \sin\alpha' \sin\beta' \sin\gamma' \Sigma \cot\alpha'$$
$$= 4\Sigma \cos\alpha \sin\beta \sin\gamma + 4\Sigma' \cos\alpha' \sin\beta' \sin\gamma'.$$

Hence remembering that $\cos(\alpha + \beta + \gamma) = -\cos(\alpha' + \beta' + \gamma')$,

$$4\sigma \sin\alpha \sin\beta \sin\gamma = 4\cos\alpha\cos\beta\cos\gamma + 4\cos\alpha'\cos\beta'\cos\gamma' \ldots (2.2),$$
$$= \Sigma \cos(\beta + \gamma - \alpha) + \Sigma' \cos(\beta' + \gamma' - \alpha')$$
$$= \Sigma \cos(S - 2\alpha) + \Sigma' \cos(S - 2\alpha')$$
$$= \cos S [\Sigma_c - \Sigma_c'] + \sin S [\Sigma_s + \Sigma_s'].$$
$$= k \text{ (from 2.1)} \quad \ldots \quad \ldots \quad \ldots \quad (2.3)$$

Hence the results (*a*).

To prove (*b*).

$$4 \sin \alpha \sin \alpha' \sin A = \sin 2\alpha + \sin 2\alpha' - \sin 2(\alpha + \alpha')$$

$$\therefore \quad 4 \Sigma (\sin \alpha \sin \alpha' \sin A) = \Sigma_s + \Sigma_s' - \Sigma \sin 2A.$$

$$\therefore \quad 4 \Sigma (\sin \alpha . \sin \alpha' . \sin A) + 4 \sin A \sin B \sin C = \Sigma_s + \Sigma_s'. \quad \ldots \quad \ldots \quad (2.4).$$

3. *To find the absolute normal co-ordinates of P and P' (isogonal pairs) in terms of their angular co-ordinates.*

As before let (α, β, γ), $(\alpha', \beta', \gamma')$ be the angular co-ordinates; and (x_1, y_1, z_1), $(x_2, y_2, z_2,)$ the *absolute* normal co-ordinates of P and P' respectively. So that

$$x_1 x_2 = y_1 y_2 = z_1 z_2 = \rho'^2, \text{ say.}$$

Let a, b, c be the sides of the triangle of reference.

$$\left.\begin{aligned} &\text{Now} \quad a x_1 = \rho'^2 . \frac{a}{x_2} = \rho'^2 (\cot \beta' + \cot \gamma) \\ &\text{Similarly} \quad b y_1 = \rho'^2 . \frac{b}{y_2} = \rho'^2 (\cot \gamma' + \cot \alpha), \\ &\text{and} \quad c z_1 = \rho'^2 . \frac{c}{z_2} = \rho'^2 (\cot \alpha' + \cot \beta). \end{aligned}\right\} \quad \ldots \quad (3.1).$$

Hence from (3 . 1), if M denotes the area of the triangle ABC,

$$2M = a x_1 + b y_1 + c z_1 = \rho'^2 . \sigma.$$

$$\therefore \quad \rho'^2 = \frac{2M}{\sigma}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.2).$$

Hence from (3 . 1) and (3 . 2), the absolute normal co-ordinates x_1, y_1, z_1 are determined in terms of α, β, γ.

Similarly

$$\left.\begin{aligned} a x_2 &= \rho'^2 (\cot \beta + \cot \gamma) \\ b y_2 &= \rho'^2 (\cot \gamma + \cot \alpha') \\ c z_2 &= \rho'^2 (\cot \alpha + \cot \beta'). \end{aligned}\right\} \quad \ldots \quad \ldots \quad (3.3).$$

NOTE. — *The quantity ρ' of this section expresses the length of the semi-minor axis of the inconic of ABC having P and P' for its foci.*

4. *To determine the normal co-ordinates of S, the middle point of P P'.*

From (3 . 1) and (3 . 3), the absolute normal co-ordinates of S are immediately obtained.

Referred to the medial triangle of ABC, the relative normal co-ordinates of S are

$$(\operatorname{cosec}\alpha \operatorname{cosec}\alpha',\ \operatorname{cosec}\beta \operatorname{cosec}\beta',\ \operatorname{cosec}\gamma \operatorname{cosec}\gamma'),$$

or

$$\left(\frac{\sin\beta\sin\gamma}{\sin\alpha'},\ \frac{\sin\gamma\sin\alpha}{\sin\beta'},\ \frac{\sin\alpha\sin\beta}{\sin\gamma'}\right). \quad \ldots \quad (4)$$

5. *The inconic of ABC having P and P' for its foci.*

(*a*) Its *centre* is the point S of § 4.

(*b*) From the co-ordinates of the centre, the *areal equation to the conic referred to ABC is*

$$\Sigma\,[\cot\alpha + \cot\alpha')\,x\,]^{\frac{1}{2}} = 0. \quad \ldots \quad \ldots (5.1)$$

(*c*) *The lengths of the axes of the inconic:*

From (5 . 1), the tangential areal equation of the inconic is

$$(\cot\alpha + \cot\alpha')\,mn + (\cot\beta + \cot\beta')\,nl + (\cot\gamma + \cot\gamma')\,lm = 0.$$

Hence, if ρ^2, ρ'^2 be the squares of the semi-axes of the conic, Prof. Wilkinson's equation (*vide* Q. 1038. *J. I. M. S.*) immediately gives

$$\rho^2\rho'^2 = \frac{4\,R^4\sin^3 A\sin^3 B\sin^3 C}{\sigma^3\sin^2\alpha\sin^2\beta\sin^2\gamma}.$$

But from § 3. Note, $\rho'^2 = \dfrac{4\,R^2\sin A\sin B\sin C}{\sigma}$. $\quad \ldots \quad (5.2).$

$$\therefore \quad \rho = \frac{R\sin A\sin B\sin C}{\sigma\sin\alpha\sin\beta\sin\gamma}. \quad \ldots \quad \ldots (5.3)$$

(*d*) *The length l of the semi-latus-rectum:*

$$l = \rho'^2/\rho = 4R\sin\alpha\sin\beta\sin\gamma. \quad \ldots (5.4)$$

6. *The equation to the line PP'.*

We have $2\Delta\,APP' = AP.\,AP'\sin(\alpha - \alpha')$

$$= \rho'^2\cdot\frac{\sin(\alpha-\alpha')}{\sin\alpha\sin\alpha'} \quad \ldots \quad \ldots (6.1)$$

Hence, if p, q, r be the lengths of the perpendiculars from A, B, C respectively on PP',

$$p : q : r = \frac{\sin(\alpha-\alpha')}{\sin\alpha\sin\alpha'} : \frac{\sin(\beta-\beta')}{\sin\beta\sin\beta} : \frac{\sin(\gamma-\gamma')}{\sin\gamma\sin\gamma'}.$$

Hence the *trilinear* equation to PP′, is

$$\sum \sin A. \frac{\sin(\alpha - \alpha')}{\sin\alpha \sin\alpha'} x = 0. \qquad \ldots (6.2)$$

7. *The pedal circle of the point P. w. r. t. the triangle ABC.*

Since the pedal circle is the auxiliary circle of the conic of § 5, its centre S and radius ρ are known from § 5.

The *absolute normal co-ordinates of S referred to the medial triangle are easily proved to be*

$$\frac{\rho \sin\beta \sin\gamma}{\sin\alpha'}, \quad \frac{\rho \sin\gamma \sin\alpha}{\sin\beta'}, \quad \frac{\rho \sin\alpha \sin\beta}{\sin\gamma'}. \qquad \ldots (7.1)$$

Hence the *pedal circle cuts the sides of the medial triangle of ABC at angles whose cosines are respectively*

$$\frac{\sin\beta \sin\gamma}{\sin\alpha'}, \quad \frac{\sin\gamma \sin\alpha}{\sin\beta'}, \quad \frac{\sin\alpha \sin\beta}{\sin\gamma'}. \qquad \ldots (7.2)$$

8. *To prove Mr. V. Ramaswami Iyer's Theorem* : "*The pedal circle of P cuts the N. P.* circle *at an angle equal to* $\frac{\pi}{2} - (\alpha + \beta + \gamma)$."

J. I. M. S., Q. 863.

By a well-known result, if d be the distance of a point (whose *absolute* normal co-ordinates are x, y, z) from the circum-centre of the triangle of reference, then

$$(R^2 - d^2) \sin A \sin B \sin C = \Sigma yz \sin A.$$

Hence from (7 . 1), if D be the distance of P from the N. P. centre,

$$(R\,2/4 - D^2) \sin A \sin B \sin C = \rho^2 \Sigma \sin\alpha \sin\alpha' \sin A.$$

Again if θ be the angle of intersection required,

$$R\rho \cos\theta = \rho^2 + R^2/4 - D^2.$$

$$\therefore 4R\rho \cos\theta. \sin A \sin B \sin C = 4\rho^2 \sin A \sin B \sin C + 4\rho^2 \Sigma \sin\alpha \sin\alpha' \sin A.$$

$$= \rho^2. [\Sigma_s + \Sigma_s'] \text{ from } (2.4).$$

$$\therefore \cos\theta = \frac{\Sigma_s + \Sigma_s'}{4\sigma \sin\alpha \sin\beta \sin\gamma}$$

$$= \sin(\alpha + \beta + \gamma) \text{ from § 2. (a)}$$

$$\therefore \theta = \tfrac{1}{2}\pi - (\alpha + \beta + \gamma).$$

9. *To prove Mr. M. Bhimasena Rao's result*: "*The area of the triangle OPP' is* $R\rho \cos(\alpha + \beta + \gamma)$, *where O is the circum-centre of ABC*" (*J. I. M. S.*, Vol. IX., p. 168).

From (6 . 2), $\dfrac{\Delta\, OPP'}{\Delta\, APP'} = \dfrac{R \sum \cos A \sin A . (\cot \alpha' - \cot \alpha)}{\dfrac{2M}{a} . \sin A (\cot \alpha' - \cot \alpha)}$

$$\therefore \Delta\, OPP' = \frac{R^2}{\sigma} \sum \frac{\sin A \cos A . \sin(\alpha - \alpha')}{\sin \alpha . \sin \alpha'}$$

from (6 . 1)

$$= \frac{R^2}{4\sigma \sin\alpha \sin\beta \sin\gamma} . V,$$

where $V = \sum [\cos A \sin A (4 \cos \alpha' \sin \beta' \sin \gamma' - 4 \cos \alpha \sin \beta \sin \gamma)]$

$$= \sum [\cos A \sin A (\cos \overline{S - 2\alpha} - \cos \overline{S' - 2\alpha'} - \cos \overline{S - 2\beta} + \cos \overline{S' - 2\beta'} - \cos \overline{S - 2\gamma} + \cos \overline{S' - 2\gamma'} + \cos S')];$$

whereas in § 2, $S = \alpha + \beta + \gamma$; $S' = \alpha' + \beta' + \gamma'$; $S + S' = \pi$.

Hence V can be thrown into the form

$$V = 4 \sin A \sin B \sin C . \cos S + U \cos S + W \sin S,$$

where $U = \sum [(\sin A \cos A - \sin B \cos B - \sin C \cos C)(\cos 2\alpha + \cos 2\alpha')]$

$$= -2 \cos A \cos B \cos C . \sum 2 \sin A \cos(\alpha - \alpha')$$
$$= -2 \cos A \cos B \cos C (\Sigma_s + \Sigma'_s)$$
$$= -2k \cos A \cos B \cos C . \sin S, \text{ from (2 .1)}$$

and $W = \sum [(\sin A \cos A - \sin B \cos B - \sin C \cos C)(\sin 2\alpha - \sin 2\alpha')]$

$$= 2 \cos A \cos B \cos C \sum 2 \sin A \sin(\alpha' - \alpha)$$
$$= 2 \cos A \cos B \cos C (\Sigma_c - \Sigma'_c)$$
$$= 2k \cos A \cos B \cos C . \cos S, \text{ from (2 . 1)}$$

Hence $V = 4 \sin A \sin B \sin C . \cos S$.

$$\therefore \Delta\, OPP' = \frac{R^3 . \sin A \sin B \sin C}{\sigma . \sin\alpha \sin\beta \sin\gamma} . \cos S$$
$$= R\rho \cos(\alpha + \beta + \gamma).$$

Note:— (i) It can be deduced from (6 . 1) that

$$\Delta\, APP' = 2R\rho (\cos \alpha' \sin \beta' \sin \gamma' - \cos \alpha \cos \beta \sin \gamma)$$

(ii) Hence $\sin 2A . \Delta APP' + \sin 2B . \Delta BPP' + \sin 2C . \Delta CPP' = (\sum \sin 2A) . \Delta OPP$.

(iii) *In fact, if Q be any point whose absolute areal co-ordinates are* (x, y, z) ; *and P, P' any two points (not necessarily isogonal)*

$$x.\ \Delta\ APP' + y.\ \Delta\ BPP' + z.\ \Delta\ CPP' = \Delta\ QPP'.$$

(iv) If G be the centroid of ABC,

$$\Delta\ APP' + \Delta\ BPP' + \Delta\ CPP' = 3\ \Delta\ GPP'.$$

10. *To prove* $(R^2 - OP^2)\ (R^2 - OP'^2) = 4R^2\rho'^2$.

By the result cited in § 8,

$$(R^2 - OP^2)\ \sin A \sin B \sin C = \Sigma y_1 z_1 \sin A,$$

and
$$(R^2 - OP'^2) \sin A \sin B \sin C = \Sigma y_2 z_2 \sin A.$$

$$\therefore\ (R^2 - OP^2)\ (R^2 - OP'^2) \sin^2 A \sin^2 B \sin^2 C$$
$$= (\Sigma y_1 z_1 \sin A)\ (\Sigma y_2 z_2 \sin A).$$
$$= x_1 x_2\ y_1 y_2\ z_1 z_2 \left(\sum \frac{\sin A}{x_1}\right) \left(\sum \frac{\sin A}{x_2}\right)$$
$$= \rho'^2 \left(\sum \rho'^2 \frac{\sin A}{x_1}\right) \left(\sum \rho'^2 \frac{\sin A}{x^2}\right)$$
$$= \rho'^2\ (\Sigma x_2 \sin A)\ (\Sigma x_1 \sin A).$$

Hence
$$(R^2 - OP^2)\ (R^2 - OP'^2) = 4R^2\ \rho'^2.$$

11. *To prove* (*a*) $OP \cdot OP' = 2R \cdot d$, where d is the distance of the N. P. centre from the centre of the pedal circle
—Mr. V. Ramaswami Iyer.

(*b*) *The radical axis of the N. P. circle and the pedal circle cuts the former at the angle POP'*
—Mr. M. Bhimasena Rao.

Let $OP \cdot OP' = 2\lambda R$, where λ is to be hereafter determined.

Now *in the triangle OPP'*,

$$PP'^2 = 4\ (\rho^2 - \rho'^2).$$

From § 10, $(R^2 - OP^2)\ (R^2 - OP'^2) = 4\ R^2\ \rho'^2$

$$\therefore \qquad R^2 - OP^2 - OP'^2 + 4\lambda^2 = 4\rho'^2 \qquad \ldots\ (11{\cdot}1)$$

$$\therefore\ R^2 + 4\lambda^2 - 4\rho^2 = OP^2 + OP'^2 - 4\ (\rho^2 - \rho'^2)$$
$$= OP^2 + OP'^2 - PP'^2$$
$$= 2 \cdot OP \cdot OP' \cos P\hat{O}P'.$$

$$\therefore\ R^2 + 4\lambda^2 - 4\lambda R \cos P\hat{O}P' = 4\rho^2. \qquad \ldots \qquad \ldots\ (11{\cdot}2)$$

Again from § 9, $\Delta OPP' = R\rho \cos(\alpha + \beta + \gamma)$

$$\therefore \quad OP \cdot OP' \sin P\hat{O}P' = 2R\rho \cos(\alpha + \beta + \gamma).$$

$$\therefore \quad \lambda \sin P\hat{O}P' = \rho \cos(\alpha + \beta + \gamma). \quad \ldots \quad (11{\cdot}3)$$

Again if N be the N. P. centre, and W the point of intersection of the N. P. circle and the pedal circle of P, then *in the triangle* NWS,

$NW = R/2$; $WS = \rho$; $NS = d$; $N\hat{W}S = \frac{\pi}{2} - (\alpha + \beta + \gamma)$ by § 8,

Hence $$R^2 + 4d^2 - 4dR \cos SNW = 4\rho^2, \quad \ldots \quad (11{\cdot}4)$$

and $$d \sin S\hat{N}W = \rho \cos(\alpha + \beta + \gamma). \quad \ldots \quad (11{\cdot}5)$$

Hence from (11·2), (11·3), (11·4), (11·5),

$$\lambda = d, \quad OP \cdot OP' = 2Rd,$$

and $$P\hat{O}P' = S\hat{N}W.$$

12. *To prove*: "*In any triangle, the radical axis of the Director circle of any inscribed conic and the nine-point circle cuts the latter at an angle equal to the angle TOT', where OT, OT' are the tangents from the circum-centre O to the inconic.*"

Proof. From Salmon: *Conic Sections, p. 212*, we have

$$2OP \cdot OP' \cos TOT' = OP^2 + OP'^2 - 4\rho^2.$$

Now remembering that $OP \cdot OP' = 2Rd$,

and $$R^2 + 4d^2 = OP^2 + OP'^2 + 4\rho'^2, \quad (\textit{vide} \S\ 11),$$

we have $$R^2 + 4d^2 - 4Rd \cos T\hat{O}T' = 4(\rho^2 + \rho'^2). \quad \ldots \quad (12.1)$$

Again, if the Director circle cuts the N. P. circle at V, in the triangle NVS, $NS = d$, $NV = R/2$, $SV^2 = \rho^2 + \rho'^2$.

$$\therefore \quad R^2 + 4d^2 - 4Rd \cos S\hat{N}V = \rho^2 + \rho'^2. \quad \ldots \quad (12.2)$$

Hence from (12.1) and (12.2) we obtain

$$S\hat{N}V = T\hat{O}T'.$$

Corollary: (i) *If the in-conic passes through the circum-centre O, the Director circle touches the N. P. circle.*
This is a well-known theorem.

Corollary: (ii) *If the Director-circle of any in-conic passes through the circum-centre, then it passes through the ends of a diameter of the N. P. circle*; and *conversely*,

if the Director-circle of any in-conic passes through the ends of a diameter of the N. P. circle, then it passes through the circum-centre as well.

[The result stated in § 12 may be considered as an extension of corollary (1) which is a particular case of the following theorem of Mr. V. Ramaswami Iyer published in the Edinburgh Mathematical Society's *Proceedings*, 1896 :—

If a conic is inscribed in a triangle and one of its confocals passes through the circum-centre, then the mutual orthoptic circle of the two conics touches the nine-points circle of the triangle.

The following extension corresponding to this theorem may be stated :—

The radical axis of the mutual orthoptic circle of a conic inscribed in a triangle and a confocal, and the nine-points circle of the triangle, cuts the latter at an angle equal to that subtended by the confocal at the circum-centre of the triangle.

Geometrical proofs of the above extension and Mr. Gulasekharam's Theorem of § 12 easily follow from the result (stated in § 11 (*b*) of this article) by the application of Question 192 of Mr. V. Ramaswami Aiyar, or Question 253. M. B.]

MATHEMATICS IN INDIA—THEN AND NOW.

By V. Sankaran, Mangalore.

It is a characteristic of the present times in India that we feel as if we have been shut out from all knowledge of the glories of this ancient land in the palmy days of yore. The cry is now "Look back into your past history and you will learn more than what you can by merely studying what you are taught in schools." In every department of knowledge this looking back into the past is necessary. We have begun to realize how advanced our fore-fathers were in the study of the Humanities, Political and Social Sciences, Arts and Philosophy. It is therefore, but natural to expect that in other departments of knowledge also, they should have reached a tolerably high standard of advancement and efficiency.

The task of making available to the world at large the ancient Mss. concerning Mathematics does not seem so far to have been taken up in right earnest.

We are indeed thankful to persons like the late Prof. M. Rangacharya and Mr. Trivedi, who spared no pains to bring out Government editions of *Ganita Sára Sangraha* and *Rékhá Ganita*. At present we want more men like these to make known to the people the existence of ancient mathematical treatises and their contents.

The place of the 'Preface' in modern treatises was taken up in our old works by the 'introductory verses' Thus, in the first of the above-mentioned works, viz., *Ganita Sara Sangraha*, Mahávirachárya says :—

"In all those transactions which relate to worldly, Vedic or (other) similarly religious affairs, calculation is of use. In the science of love, in the science of wealth, in music and in the drama, in the art of cooking, and similarly in medicine and in things like the knowledge of architecture.

In prosody, in poetics and poetry, in logic and grammar and such other things and in relation to all that constitutes the peculiar value of the arts—in all these, the science of computation is held in high esteem.

In relation to the movements of the sun and other heavenly bodies, in connection with eclipses and the conjunction of planets, and in connection with *triprasna* and the course of the moon—indeed in all these mathematics is utilised.

In short, whatever there is in all the three worlds, possessed of moving and non-moving beings, cannot exist as apart from measurement."

It is thus clear that mathematics was studied in those days because, it was found indispensable in ordinary life and in its study the practical aspect was always kept in view. No branch of mathematics was studied which did not have its application in life at the time in which it was written. Further, things were studied only to that degree of accuracy which was consistent with the practical needs of the people. People in those days were not interested in calculating the value of π to 707 places. For, they realized that decimal fractions to more places than 3 or 4 are useless for all practical purposes. That calculations were not carried beyond 2 or 3 places of decimals can be easily understood from a perusal of the contents of some of the mathematical treatises in Tamil and Sanskrit.

There seem to have been about a dozen mathematical works in Tamil and Sanskrit, of which Kari Nayanar says he intends writing a summary. Among mathematical books in Sanskrit, the author mentions *Anjanam*, *Daivam*, *Bhuvanadípam* and *Ganitaratnam*. Among the treatises in Tamil are mentioned *Govindanarpadi*, *Kanakkadhikáram Orambam*, *Kalambakam Tribhuvanatilakam*, *Kanakaratnam* and *Chirukanakku*. Of these, only the '*Kanakkadhikaram*' is now available and very little is known about the others. No mention is made, however, of *Lílaváti* or *Ganita Sára Sangraha*, thus showing that the '*Kanakkadhikaram* belongs to an earlier age.

Bhaskaracharya, the author of the *Lilavati*, says towards the end of his *Bijaganitam*: ब्रह्माह्वयश्रीधरपद्मनाभबीजानि यस्मादतिविस्तृतानि suggesting that there were many authors before him who had written treatises on mathematics—chief among them being Brahma Gupta, Sridhara and Padmanabha. Of these Brahma Gupta alone seems to have gained recognition, the works of the others being almost unknown. Mr. Sudhakara Dwivedi in his '*Ganakatarangini*' gives a big list of mathematicians. But as it is written in Sanskrit, its contents are not widely known.

Confining our attention to *Lilavati* and *Kanakkadikaram*, we may begin by noticing that these are works on 'general mathematics'—the analogue of what we now call 'elementary mathematics.'

It may be useful to compare them with modern treatises as regards the general treatment of the subject.

We know that most of the modern treatises contain a good many examples after each type. The type is explained and worked out and a number of questions are given on the same or mainly the same model, though with some verbal changes and with some chapters on what are

known as 'miscellaneous examples.' But in the *Lilavati* as also in the *Kanakkadikaram*, we find only one or two examples after each type. The reason for this is stated by Bhaskara towards the end of his *Bijaganita*, where he says :

नह्युदाहरणान्तोऽस्ति स्तोकमुक्तं इदं यतः ॥

* * *

प्राज्ञे शास्त्रं स्वयं याति विस्तारं वस्तुशक्तितः ॥

"Examples can be easily multiplied by the intelligent student. Hence only a few examples are given."

Again, the idea of suggesting miscellaneous problems towards the end of a treatise is not solely a modern one. Though the *Lilavati* lacks it, the same author has suggested several problems in the last chapter of the *Siddanta Siromani*, his masterpiece on Astronomy.

A second feature of the modern text-book is that it does not give ready made rules for types, while the *Lilavati* and *Kanakkadikaram* give them. The reason is not far to seek. The people in the early ages used to compress most of their learning into short mnemonic verses or sutras. Those were not the days of elaborate treatises written *in extenso*. The art of printing had not then been invented. The teacher was expected in each case to arrive at the method by independent work and in the end to give the result in the form of a rule; then, read the corresponding sloka or stanza and finally elicit from the student himself its explanation. Again, a last point with regard to the treatises of the Middle Ages is that they are almost always written in verse thus throwing some side-light on the literary advancement of the times.

Now proceeding to the contents, we find that the *Lilavati* of Bhaskara and *Ganita Sara Sangraha* of Mahavira are essentially treatises on elementary mathematics as it is understood to-day. Till very recently, the several departments of mathematical knowledge, arithmetic, algebra and geometry were regarded as watertight compartments. But now, the angle of vision has changed and instead of regarding them as separate and mutually exclusive subjects, we have begun to see in them common elements which give them a corporate and inter-dependent existence. A question is first solved arithmetically, generalisations are made therefrom, the results are then expressed algebraically and, whenever possible, illustrated geometrically. Though to most of us this might appear as an idea of recent date, we have ample grounds to believe that the notion of the inter-dependence of these subjects was in existence and has been applied in India from the earliest times. Even

to-day, if we go to any old gentlemen without any experience of western education, it will be interesting to know how he finds out the square of any number. The process by which he works out the result will suggest to any thoughtful man the application of what is known as the distributive law for multiplication. For example, to find out the square of 27, the working will be as follows: "20 times 20, 400; 20 times 7, 140; 540; 20 times 7, 140; 680; 7 times 7, 49; 729—so that the square of 27 is 729." Again, taking up another instance, the multiplication of mixed fractions, say $3\frac{1}{2}$, $4\frac{1}{2}$, the process is more patent. "3 times 4, 12; 3 times $\frac{1}{2}$, $1\frac{1}{2}$, $13\frac{1}{2}$; 4 times $\frac{1}{2}$, 2, $15\frac{1}{2}$; $\frac{1}{2}$ of $\frac{1}{2}$, $\frac{1}{4}$, $15\frac{3}{4}$." In working questions by this method, we see how unconsciously the distributive law is applied. In the "*Lilavati*," the author does not introduce the word "bija" anywhere in his book, though free use is made of the rules of algebra. For example, in the 9th sloka he says,

समद्विघातः कृतिरुच्यतेथ स्थाप्योऽन्त्यवर्गोगुणान्त्यनिघ्नः ॥

* * * *

खण्डद्वयस्याभिहतिर्द्विनिघ्नी तत्खण्डवर्गैक्य युताकृतिर्वा ॥

"The square of a quantity is obtained by multiplying it by itself. To find this, to the square of the last digit, add twice the product of this and the preceding number; and to all this add the square of the preceding; and proceed like this"—evidently applying the general formula

$$(a + b + c)^2 = a^2 + 2a\,(b + c) + (b + c)^2.$$

Another method is: "divide the number into two parts and to twice the product of these add the squares of the parts."

Again in the 13th sloka, the author says:

समत्रिघातश्चघनः प्रदिष्टः स्थाप्योघनोन्त्यस्य ततोऽन्त्यवर्गः

आदित्रिनिघ्नः तत आदिवर्गः त्रयन्त्याहतो थादिघनश्चसर्वे ॥

* * * *

खण्डाभ्यां वाहगे राशिः त्रिघ्नः खण्डघनैक्ययुक् ॥

In the first part the author applies the expansion $a^3 + 3a^2 b + 3ab^2 + b^3$ and in the second $a^3 + b^3 + 3ab.(a+b)$ for the cube of the binomial $(a+b)$. Thus we find that the rules and formulæ of algebra are freely applied in the work.

Next, taking up the contents of the *Lilavati* in greater detail, the author begins with "abhinnaparikarmáshtakam," comprising the four rules applied to integers and raising them to the powers 2, 3, $\frac{1}{2}$ and $\frac{1}{3}$. Then comes "játichatushtayam," or conversion of mixed fractions into vulgar

fractions reducing them to equivalent fractions with common numerator or denominator and comparing them, followed by "bhinna" and "sunya pari. karmashtakam"—the eight operations aforesaid on fractions and zero. Here it will be interesting to note that the result of dividing a number by zero is left as undetermined and indeterminate being simply called "khahasa." Then follow the chapters on problems leading to simple and quadratic equations, "trairasika" and "pancharasika" (proportion, simple and compound, direct and inverse), "bhandapratibhanda" or barter, "misravyavahara" or alligations or mixtures. Then the author takes up "sredi"—simple arithmetical and geometrical progressions, the sum of the squares and cubes of the first n natural numbers. This is followed by "kshétra vyavahára"—mensuration of the triangle, quadrilateral, circle, sphere, rectangular parallelpiped, frusta on rectangular bases, and square and cubic measure. The last two chapters are devoted to "kuttaka" or continued fractions, and "ankapasa" or permutations and combinations. The above give in brief the contents of the *Lilavati* and we can see how far these compare with the elementary mathematics of to-day. Before we take up the other works for consideration, one thing that could be said in favour of the *Lilavati* is that it meets almost all the requirements of every-day life in India. Each one of the sections above enumerated has got an application in life. The questions set forth are mostly practical and no question is *dry*. Each has got a bearing, historical, social or religious. Thus while suggesting questions for mixtures, the author takes up the mixing up and melting together of different qualities of gold in varying proportions. A question on quadratics runs thus :

पार्थः कर्णवधाय मार्गणगणं क्रुद्धो रणे संदधे
तस्यार्धेन निवार्य तच्छरगणं मूलैश्चतुर्भिर्हयान् ।
शल्यं षड्भिरथेषुभिस्त्रिभिरपि छत्रं ध्वजं कार्मुकं
चिच्छेदास्य शिरः शरेण कतिते यानर्जुनः संदधे ॥

Again a question on proportion takes the form :

प्राप्नोति चेत् षोडशवत्सरा स्त्री द्वात्रिंशतं विंशतिवत्सरा किम् ।

"If a maid-servant 16 years old gets Rs. 32 what will a maid-servant of 20 get?" Here some western historian of mathematics takes this verse to mean : "If the cost of a maid of 16 be Rs. 32, what will be the cost of a maid of 20?" and goes on to infer the social condition of the period and says that slave trade was prevalent in India in those times. The meaning itself is far-fetched and the inference unwarranted.

A question on permutations reads:

पाशांकुशाहिडमस्ककपालशूलैः खट्वाङ्गशि शरचापयुतैर्भवन्ति ।
अन्योन्यहस्तकलितैः कतिमूर्तिभेदाः शभीर्हरेरिव गदारिसरोजशंखै ॥

"How many different idols of Siva and Vishnu can be made by interchange of àyudhas, it being given that Siva has ten and Vishnu four different àyudhas."

These questions are essentially Indian and these appeal to the Indian reader much better than the dry and unpractical examples given in most English text-books.

Thus in point of contents, modern elementary mathematics includes the same portions as are dealt with in the *Lilavati* with something more of formal geometry and algebra.

(To be continued.)

SHORT NOTES.

Transformation of $(x^p - 1)/(x - 1)$.

In 'Nature' of June 9, 1921, Prof. G. B. Matthews announced the polynomials y and z in x in connection with the well-known transformation of the polynomial

$$\frac{4(x^p - 1)}{x - 1} \text{ in the form } y^2 - (-1)^{\frac{1}{2}(p-1)} pz^2,$$

(p being an odd prime number) for the value $p = 37$. Sometime later, Prof. Pocklington, F. R. S., announced the polynomials y and z for the primes 41, 43, 47, 53, 59 and 61, following the method given on pp. 216, 217 in Art. 194 of Matthew's *Theory of Numbers*, Part I.

The following is perhaps a simpler and readier method of arriving successively at the co-efficients of the powers of x in y and z and does not involve any high knowledge of the theory of numbers.

1. We may write $X = \dfrac{x^p - 1}{x - 1}$, ($p$ being an odd prime)

$$y = 2x^{\frac{1}{2}(p-1)} + P_1 x^{\frac{1}{2}(p-3)} + P_2 x^{\frac{1}{2}(p-5)} + \ldots\ldots,$$

and $$z = Q_1 x^{\frac{1}{2}(p-3)} + Q_2 x^{\frac{1}{2}(p-5)} + \ldots\ldots$$

Now, following the notation of Prof. Matthews in his *Theory of Numbers*, let us put

(i). $$P_n = -\frac{1}{n}(S_n + p_1 S_{n-1} + p_2 S_{n-2} + \ldots\ldots + p_{n-1} S_1)$$

where $p_r = a_r + b_r \eta_0$, η_0 and η_1 being the roots of the equation

(ii) $$\eta^2 + \eta + \frac{1 - (-1)^{\frac{1}{2}(p-1)} p}{4} = 0;$$

$S_h = \eta_0$ or η_1 according as h is a residue of p or is a non-residue of p that is, $h\ R\ p$ or $h\ N\ p$.

Also (iii) $$X_1 = x^{\frac{p-1}{2}} + p_1 x^{\frac{p-3}{2}} + p_2 x^{\frac{p-5}{2}} + \ldots\ldots + p_{\frac{p-3}{2}} x + p_{\frac{p-}{2}}$$

$$\equiv U + \eta_0 V,$$

where U and V are polynomials in x with integral co-efficients.

(iv) $y = 2U - V\,;\ z = -V$

2. From (iii) and (iv), we infer

(v) $y = 2\,[X_1]_{\eta_0} = -\frac{1}{2}$

Equating the co-efficients of powers of 'x' in (iv), we deduce

(vi) $\begin{cases} P_n = 2\,[p_n]_{\eta_0} = -\frac{1}{2} \\ Q_n = -b_n) \end{cases}$

Evidently $[S_h]_{\eta_0} = -\frac{1}{2} = -\frac{1}{2}$,

whether $h\ R\ p$, or $h\ N\ p$, since $\eta_0 + \eta_1 = -1$

When $(n-r)\ Rp$, $p_r\ S_{n-r} = (a_r + b_r\,\eta_0)\ \eta_0$

$$= (a_r - b_r)\,\eta_0 - \frac{b_r}{4}\left\{1 - (-1)^{\frac{1}{2}(p-1)}\ p\right\}$$

When $(n-r)\ Np$, $p_r\ S_{n-r} = (a_r + b_r\,\eta_0)(-1-\eta_0)$

$$= -a_r\eta_0 - a_r + \frac{b_r}{4}\left\{1 - (-1)^{\frac{1}{2}(p-1)}\ p\right\}$$

$$\therefore\ [p_r\ S_{n-r}]_{\eta_0} = -\tfrac{1}{2} = -\tfrac{1}{2}\,a_r + \frac{b_r}{4}\left\{1 \pm (-1)^{\frac{1}{2}(p-1)}\ p\right\},$$

the + sign being taken when $(n-r)\ Rp$ and the — sign when $(n-r)\ Np$

(vii) $$= -\tfrac{1}{2}\,[p_r]_{\eta_0} = -\tfrac{1}{2} - \frac{(-1)^{\frac{1}{2}(p-1)}\ p}{4}\ e_{n-r}\ (+b_r)$$

$$= -\frac{1}{4}\,P_r - \frac{(-1)^{\frac{1}{2}(p-1)}\ p}{4}\,Q_r\,e_{n-r}$$ from (vi)

where $e_{n-r} = \{(n-r) \mid p\}$, with the usual notation.

Hence, we get the reduction formula

$$P_n = -\frac{2}{n}\left(-\frac{1}{2} - \frac{1}{4}\sum_{1}^{n-1} P_r - \frac{(-1)^{\frac{1}{2}(p-1)}\ p}{4}\sum_{1}^{n-1} Q\ \ e_{n-r}\right)$$

$$= \frac{1}{2n}\left\{\sum_{0}^{n-1} P_r + (-1)^{\frac{1}{2}(p-1)}\ p\ Q_r\ e_{n-r}\right\}$$

(viii) $= \frac{1}{2n} \sum_{0}^{n-1} \left\{ P_r + (-1)^{\frac{1}{2}(p-1)} \, pQ_r e_{n-r} \right\}$, assuming $P_0 = 2$.

3. Again, since $y + z = 2(U - V) = 2\,[X_1]_{\eta_0} = -1$

equating co-efficients of like powers of x on both sides, we get

(ix) $$P_n + Q_n = 2\,[p_n]_{\eta_0} = -1$$

When $(n-r)\ Rp$,

$$[p_r S_{n-r}]_{\eta_0 = -1} = -(a_r - b_r) - b_r \frac{1 - (-1)^{\frac{1}{2}(p-1)}}{4} p$$

$$= -[p_r]_{\eta_0 = -1} + Q_r \frac{1 - (-1)^{\frac{1}{2}(p-1)}}{4} p$$

$$= -\frac{1}{2}(P_r + Q_r) + Q_r e_{r1-r} \frac{1 - (-1)^{\frac{1}{2}(p-1)}}{4} p$$

When $(n-r)\ Np$,

$$[p_r S_{n-r}] = \frac{b_r}{4}(1 - (-1)^{\frac{1}{2}(p-1)} p)$$

$$= + \frac{e_{n-r} Q_r}{4}(1 - (-1)^{\frac{1}{2}(p-1)} p)$$

$$\therefore P_n + Q_n = -\frac{2}{n}\left[S_n + \frac{1}{4}\left\{ 1 - (-1)^{\frac{1}{2}(p-1)} \, p \right\} \sum_{1}^{n-1} Q_r e_{n-r} \right.$$

$$\left. - \frac{1}{4} \sum_{1}^{n-1} (P_r + Q_r)(1 + e_{n-r}) \right]$$

since when $\frac{1 + e_{n-r}}{2} = 1$ or 0, according as $(n-r)\ Rp$ or $(n-r)\ Np$.

But $[S_h]_{\eta_0 = -1} = -1$ or 0 according as $h\ R\ p$ or $h\ N\ p$.

$\therefore -1 - 2\,[S_h]_{\eta_0 = -1} = +1$ or -1, accg. as $h\ R\ p$ or $h\ N\ p$

$= (h/p) = e_h^{\lambda}$, with the usual notation.

$$(x) \quad \therefore P_n + Q_n = \frac{1}{2n} \sum_{0}^{n-1} \Big\{ (P_r + Q_r) + P_r\, e_{n-r} + (-1)^{\frac{1}{2}(p-1)}\, p\, Q_r\, e_{n-r} \Big\}$$

since $S_n = -\frac{1+e_n}{2}$, $P_0 = 2$, and $Q_0 = 0$.

4. Subtracting (viii) from (x), we get

$$(xi) \qquad Q_n = \frac{1}{2n} \sum_{0}^{n-1} \{ Q_r + P_r\, e_{n-r} \}.$$

Thus, we get two convenient reduction formulæ for P_n and Q_n, viz.

$$P_n = \frac{1}{2n} \sum_{0}^{n-1} \Big\{ P_r + (-1)^{\frac{1}{2}(p-1)}\, p.\, Q_r\, e_{n-r} \Big\}, \qquad \ldots \quad I$$

and $$Q_n = \frac{1}{2n} \sum_{0}^{n-1} \Big\{ Q_r + P_r\, e_{n-r} \Big\}. \qquad \ldots \quad \ldots \quad II$$

5. From the formulæ I and II, we can easily deduce the following two formulæ which are more suitable for numerical work :

$$2n\, P_n - (2n-1)\, P_{n-1} = (-1)^{\frac{1}{2}(p-1)}\, p \{ Q_{n-1} - 2 \Sigma Q_{n-r} + 2 \Sigma Q_{n-r-1} \}$$

and $$2n\, Q_n - (2n-1)\, Q_{n-1} = P_{n-1} + 2 \{ e_n - e^{n-1} - \Sigma P_{n-r} + \Sigma P_{n-r-1} \}$$

the summation on the left-hand side extending over all the values of $r < n$ and non-residues of p.

Since the mth co-efficients of y and z from the beginning and the end are always equal numerically, it will be sufficient if we calculate the first $\frac{p+3}{4}$ co-efficients in y and the first $\frac{p-1}{4}$ co-efficients in z, when $p \equiv 1 \pmod 4$; and the first $\frac{p+1}{4}$ co-efficients in y and the first $\frac{p-3}{4}$ co-efficients in z, when $p \equiv 3 \pmod 4$.

6. The following table gives the values of P_n and Q_n for the primes 67, 71, 73, and 79.

p.	P	Q
67	2, 1, — 16, 9, 33, — 44, — 18, 79, — 39 — 48, 75, — 35, — 14, 69, — 89, 10 106	1, 0, — 3, 3, 4, — 8, 1, 9 — 8, — 1, 7, — 8, 5, 5, — 14, 8
71	2, 1, — 17, — 26, — 5, 31, 58, 64, 60 33, — 29, — 89, — 106, — 91, — 70, — 44 — 16, — 3	1, 1, — 2, — 5, — 5, — 2, 2 6, 11, 13, 9, 2, — 3, — 6, — 8, — 8, — 7
73	2, 1, 19, 28, 61, 106, 158, 251, 322, 442, 544, 652, 783, 868, 983, 1050, 1113, 1164, 1156	1, 1, 4, 7, 12, 20, 27, 40 50, 64, 78, 89, 104, 113, 124, 131, 134, 138
79	2, 1, — 19, — 29, 24, 69, 4, — 64, 19 75, — 67, — 125, 81, 166, — 60, — 137 87, 110, — 152, — 148	1, 1, — 3, — 6, 1, 8, 0, — 7 5, 11, — 9, — 17, 8, 18, — 7 — 13, 14, 14, — 18.

A. A. KRISHNASWAMI AIYANGAR.

ASTRONOMICAL NOTES.

Interference Method in Astronomy.

1. The recent announcement of the successful application of this method at Mount Wilson, to the measurement of the angular diameters of some of the brightest stars has created such a widespread interest that a brief account of the method may not be out of place in the astronomical page of this journal. Following so closely, as it does, the theoretical researches of Eddington[1], Wilsing[2], and other investigators, it forms one of the many remarkable instances in the history of Astronomy in which theory and observation have advanced side by side, the one helping and checking the other.

2. The method itself is not in any way a recent discovery; in fact the possibilities of the method were indicated by Fizeau so early as 1868 and in 1874 Stephan made at the Marseilles Observatory some attempts to measure the angular diameters of some of the brightest stars; these, however, were not quite successful on account of the small aperture of the telescope used.

3. The principles involved in the method may be summarised thus:—

In an astronomical telescope the image of a star (a point source of light) formed in the focal plane, is not also a point, but consists generally of a small circular bright disc in the centre surrounded by a series of diffraction rings—the angular diameter of the first ring calculated from theory is $1.22 \frac{\lambda}{D}$, where λ is the mean wave length of light and D the aperture of the telescope. If a star, which is a close double, is observed, the images will appear just separated if the angular distance between them is such that one image falls in the first diffraction ring of the other. Lord Rayleigh has thus deduced an expression $5''/D$ for the resolving power of a telescope, where D is the aperture of the object glass in inches, *i.e.*, a telescope D″ in aperture will just be able to separate the two components of a double star whose singular distance apart is $5''/D$

4. If a narrow slit were placed in front of the object glass, the diffraction rings will be transformed into a series of fringes parallel to the edges of the slit. Michelson and later on M. Hamy have investigated the theory of the phenomena which occur when two narrow slits parallel to each other are placed in front of the object glass. It may be interesting to recount some of the results obtained by Michelson (*Phil. Mag.* 1890).

[1] Presidential Address: Section A, British Association, 1920.

[2] Publications of the Potsdam Observatory, No. 76.

(1) If the object observed is a bright circular disc, the diffraction fringes will vanish for the distances of the slits given by $l = (1{\cdot}22,\ 2{\cdot}24\ 3.26\ \ldots)\ \lambda/\alpha$, where λ is the mean wave length of light and α the angular diameter of the object.

(2) In the case of a double star with components of the same brightness close to each other, if the slits are placed in such a way that their length is perpendicular to the line joining the centres of the component stars, the fringes will vanish for the distances

$$l = \left(\frac{1}{2}, \frac{3}{2}, \ldots\ldots\ldots\ldots\right) \frac{\lambda}{\alpha},$$

where α is the angular distance between the component stars.

5. These two results have been used with remakable success in measuring the angular distances between some very close double stars and the angular diameters of some of the brightest giant stars. The 100-inch telescope at Mount Wilson was specially directed towards this research and some very good measures have been obtained. Capella was the star first chosen for trial. Campbell and Newall had discovered its binary character by spectroscopic methods, but hitherto the largest telescope could not separate the components. By the interference method it was found possible to obtain satisfactory measures of the distance and position angle which are in sensible accord with the results obtained by spectroscopic methods.

6. Michelson then extended the method by using an interferomete) for the measurement of stellar diameters. The giant red stars with a low density and considerable luminosity were found most suitable for the purpose. The stars whose diameters have been so far measured with the 100-inch telescope at Mount Wilson are α Orionis (Betelguese)[1] and Scorpii (Antares). The angular diameter of the former was found to be 0″.047, and as its parallax has been determined as 0″.018, its linear diameter should be about 240,000,000 miles, with an uncertainty of only about 10 per cent. The results for Antares indicate that this star is of even larger linear dimensions. The angular diameter measured by the interference method is 0″.040, but as its parallax is exceedingly small, about 0″.009, its linear diameter proves to be about 400×10^6 miles. The method seems to be of immense possibilities and it may confidently be expected that further devolopments will be made, so that angular distances down to 0″.01 may be accurately measured.

T. P. Bhaskara Sastri.

1 *Astrophysical Journal*, May 1921.

SOLUTIONS.

Question 937.

(H. R. KAPADIA) :—If n have any positive integral value except unity, and r be any positive integer which is not a perfect power, show that

$$\Sigma(n-1)/(r^n-1) = \pi^2/6\,;$$

and if $d(n)$ denote the number of divisors of n, that $\Sigma\{d(n)-1\}/r^n = 1$; also that $\Sigma(n-1)/r = \Sigma 1/(r-1)^2$.

Solution by N. B. Mitra.

Consider the series $s_n = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \ldots\ldots$ to ∞ (1), where n is any positive integer > 1.

The series is absolutely convergent and we are, therefore, at liberty to rearrange the series in any manner.

Let r be any integer which is not a perfect power ($r > 1$).

Now the series (1) can be arranged in groups like these :—

$$\left(\frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{8^n} + \ldots\ldots\right) + \left(\frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{27^n} + \ldots\ldots\right) + \ldots\ldots$$
$$+ \left(\frac{1}{r^n} + \frac{1}{r^{2n}} + \frac{1}{r^{3n}} + \ldots\ldots\right) + \ldots\ldots$$

where $r = 2, 3, 5, 6, 7, 10, 11\ldots$

Now the group beginning with $\frac{1}{r^n} = 1/(r^n - 1)$. Hence giving to r all possible values, we get $s_n = \Sigma 1/(r^n - 1)$, where Σ refers to summation with respect to r. (2)

Hence $$\sum \frac{n-1}{r^n - 1} = (n-1)\,s_n.$$

Substituting 2, 3, 4 in succession, for n, in this we get,

$$\sum \frac{2-1}{r^2-1} = (2-1)s_2 = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots \text{ to } \infty$$

$$\sum \frac{3-1}{r^3-1} = (3-1)s_3 = \frac{2}{2^3} + \frac{2}{3^3} + \frac{2}{4^3} + \ldots\ldots \text{ to } \infty$$

$$\sum \frac{4-1}{r^4-1} = (4-1)s_4 = \frac{3}{2^4} + \frac{3}{3^4} + \frac{3}{4^4} + \ldots\ldots \text{ to } \infty$$

and so on.

Each of these series is absolutely convergent, for s_n is so if $n > 1$; and it is easy to see that the double series on the right satisfies Cauchy's test for absolute convergence.

Adding up these, the addition on the right being performed column by column, and observing that the sum of the p^{th} column

$$= \frac{1}{(p+1)^2}\left\{1 + \frac{2}{p+1} + \frac{3}{(p+1)^2} + \ldots\ldots \infty\right\}$$

$$= \frac{1}{(p+1)^2}\left\{1 - \frac{1}{p+1}\right\}^{-2} = \frac{1}{p^2},$$

we get,
$$\sum_{n=2}^{n=\infty}\left\{\sum^{n-1}\frac{n-1}{r^n-1}\right\} = \sum_{p=1}^{p=\infty}\frac{1}{p^2} = \frac{\pi^2}{6}.$$

NOTE.—The summation on the left-side is to be performed doubly, with respect to r as well as to n; hence it would have been better to use two Σ's instead of one as printed in the question.

Let the notation $\sum_p$ denote summation with respect to p.

(ii) It is known that $x + x^2 d(2) + x^3 d(3) + \ldots\ldots$ is equal to the Lambert's series $\frac{x}{1-x} + \frac{x^2}{1-x^2} + \frac{x^3}{1-x^3} + \ldots\ldots (|x| < 1)$.

Hence if n be as defined in the question, we have

$$x + \sum_n d(n)x^n = \frac{x}{1-x} + \sum_n \frac{x^n}{1-x^n}$$

or
$$\sum_n d(n)x^n = \frac{x^2}{1-x} + \sum_n \frac{x^n}{1-x^n}.$$

In this, put $x = 1/r^n$, where r is as defined in the question, so that $r > 1$. We get

$$\sum_n d(n)/r^n = \frac{1}{r(r-1)} + \sum_n \frac{1}{r^n - 1}.$$

Also $\sum_n 1/r^n = 1/r^2 + 1/r^3 + \ldots\ldots = 1/r(r-1)$.

Hence $\sum_n \{d(n) - 1\}/r^n = \sum_n 1/(r^n - 1)$,

and $\therefore$ $$\sum_r \sum_n \{d(n) - 1\}/r^n = \sum_r \sum_n \frac{1}{r^n - 1}.$$

Now the double series on the right can be easily shown to be absolutely convergent.

Hence $\sum_r \sum_n 1/(r^n - 1) = \sum_n \sum_r 1/(r^n - 1) = \sum_n s_n$ by (2).

Now $$\sum_n s_n = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots$$
$$+ \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots\ldots$$
$$+ \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots\ldots$$

The double series on the right is evidently absolutely convergent and so we shall arrive at the same result in whatever way we proceed to sum it.

Adding by columns and observing that the p^{th} column

$$= \frac{1}{(p+1)^2} + \frac{1}{(p+1)^3} + \ldots\ldots = \frac{1}{p(p+1)},$$

we get

$$\sum_n s_n = \sum 1/p(p+1) = 1/1.2 + 1/2.3 + 1/3.4 + \ldots\ldots \text{ to } \infty = 1.$$

Hence $$\sum_r \sum_n \{ d(n) - 1 \}/r^n = 1.$$

(iii) Lastly

$$1/(r-1)^2 = \frac{1}{r^2}\left(1 - \frac{1}{r}\right)^{-2} = \frac{1}{r^2} + \frac{2}{r^3} + \frac{3}{r^4} + \ldots\ldots = \sum_n (n-1)/r^n.$$

$$\therefore \quad \sum_r \frac{1}{(r-1)^2} = \sum_r \sum_n \frac{n-1}{r^n}.$$

NOTE.—There appears to be a printing mistake in the 3rd part of the question, where on the left side of the equality r should be replaced by r^n.

Question 1011.

(SELECTED) :—In an ellipse the tangent at P cuts the directrices in Z, Z', and the remaining tangents from Z,Z' to the ellipse meet at T. Show that PT is normal to the ellipse and bisected by the minor axis.

Solution by N. Sundaram Aiyar, S. M. Malurkar and several others.

Let the equation to the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let TZ and TZ' touch the ellipse at Q and R respectively. Let the eccentric angles of P, Q and R be θ, θ_1, and θ_2 respectively.

Then, the co-ordinates of T are $\left\{\dfrac{a\cos\dfrac{\theta_1+\theta_2}{2}}{\cos\dfrac{\theta_1-\theta_2}{2}},\ \dfrac{b\sin\dfrac{\theta_1+\theta_2}{2}}{\cos\dfrac{\theta_1-\theta_2}{2}}\right\}$ (i)

Also $$\frac{a}{e}=\frac{a\cos\left(\dfrac{\theta+\theta_1}{2}\right)}{\cos\dfrac{\theta-\theta_1}{2}}, \qquad \ldots \text{(ii)}$$

and $$-\frac{a}{e}=\frac{a\cos\left(\dfrac{\theta+\theta_2}{2}\right)}{\cos\dfrac{\theta-\theta_2}{2}}, \ldots \qquad \ldots \text{(iii)}$$

where e is the eccentricity of the ellipse.

Putting t, t_1 and t_2, for $\tan\frac{\theta}{2}$, $\tan\frac{\theta_1}{2}$ and $\tan\frac{\theta_2}{2}$ respectively, we have by adding (ii) and (iii)

$$\frac{1-tt_1}{1+tt_1}+\frac{1-tt_2}{1+tt_2}=0, \text{ or } t_1t_2=\frac{1}{t^2}. \qquad \ldots \qquad \ldots \text{(iv)}$$

Again from (ii) and (iii), $-\dfrac{1}{e^2}=\dfrac{(1-tt_1)\ (1-tt_2)}{(1+tt_1)\ (1+tt_2)}=\dfrac{2-t\ (t_1+t_2)}{2+t\ (t_1+t_2)}$.

$$\therefore \qquad -\frac{1+e^2}{1-e^2}=\frac{t(t_1+t_2)}{2}.$$

The co-ordinates of T now become $\left\{\dfrac{a\ (1-t_1t_2)}{1+t_1\ t_2},\ \dfrac{b\ (t_1+t_2)}{1+t_1\ t_2}\right\}$,

or $$\left\{-\frac{a\ (1-t^2)}{1+t^2},\ \frac{-2bt\ (1+e^2)}{(1-e^2)\ (1+t^2)}\right\}$$

i.e. $$\left\{-a\cos\theta,\ -b\sin\theta\ \frac{(2a^2-b^2)}{b^2}\right\} \ldots \qquad \ldots \text{(v)}$$

Since the x — co-ordinates of P and T are respectively $a\cos\theta$ and $-a\cos\theta$, PT is bisected by the y-axis which is the minor axis.

It is now obvious that the normal at P, the equation to which is $\dfrac{ax}{\cos\theta}-\dfrac{by}{\sin\theta}=a^2-b^2$, contains T, the co-ordinates of which are given in (v).

So, PT is normal to the ellipse and bisected by the minor axis.

Question 1091.

(ENQUIRER):—If a, b, c denote the radii of three circles which touch one another externally and r_1 and r_2 be the radii of the two circles which can be drawn to touch these three, prove that

$$\frac{1}{r_1} - \frac{1}{r_2} = 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Additional Solution by G. A. Srinivasan.

This has been solved in various ways in the pages of this Journal (*e.g.* Vol. XII, pp. 237-8; Vol. XIII, pp. 22-4); an additional solution, however, may not be out of place.

Let A, B, C be the centres of the given circles, a, b, c their radii; let (A) and (B) touch at P. Invert with respect to P. The inverse figure is as shown. Let p and q be the distances from P of the inverses of (A) and (B).

Then $$2a.p = 2b\,q = R^2, \quad \ldots \quad \ldots \quad \text{(i)}$$

where R is the radius of inversion.

The two circles, of radii r_1 and r_2, which touch the three given circles, and the circle (C) invert into equal circles, of radius ρ say. Suppose the distances from P of the centres of these three inverse circles are in order d_1, d_2, and d.

Then, we have at once,

$$r_1 = \frac{R^2\rho}{d_1{}^2 - \rho^2}, \; r_2 = \frac{R^2\rho}{\rho^2 - d_2{}^2}, \; c = \frac{R^2\rho}{d^2 - \rho^2} \quad \ldots \text{(ii)}$$

and $$2\rho = p + q. \quad \ldots \quad \ldots \quad \ldots \text{(iii)}$$

We have, from (ii),

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{d_1{}^2 + d_2{}^2 - 2\rho^2}{R^2\rho} = \frac{2d^2 + 2(2\rho)^2 - 2\rho^2}{R^2\rho}$$

(from the triangle PO_1O_2, in which O is the middle point of O_1O_2)

$$= \frac{2(d^2 - \rho^2)}{R^2\rho} + \frac{8\rho}{R^2}$$

$$= \frac{2}{c} + \frac{4(p+q)}{R^2}, \quad \text{from (ii) and (iii)}$$

$$= \frac{2}{c} + \frac{2}{a} + \frac{2}{b}, \quad \text{from (i).}$$

Question 1109.

(Saldhana and Madhava).—Prove that

$$e^{\lambda_1}\left[f(x)\right] = \left(1 + \frac{1}{x}\right) f(x+1)$$

where $\lambda_1 = \left(\frac{d}{dx} + \frac{1}{x}\right)$; (Bombay, B.A.)

and extend the result as follows :—

$$e^{\lambda_2}\left[\tfrac{1}{2} f(x)\right] = \left(\frac{1}{2} + \frac{1}{x} + \frac{1}{2x^2}\right) f(x+1),$$

$$e^{\lambda_3}\left[\tfrac{1}{3} f(x)\right] = \left(\frac{1}{3} + \frac{1}{x} + \frac{1}{3x^3}\right) f(x+1),$$

where $\lambda_2 = \frac{d}{dx} + \frac{2}{x}$ and $\lambda_3 = \frac{d}{dx} + \frac{3}{x}$.

Remarks by Balakram.

The results quoted in the question are (as pointed out by several gentlemen) special cases of a general theorem. This general theorem is capable of being generalized still further as follows :—

If F (k) is a function that can be expanded in ascending and descending integral powers of k, then

$$\mathrm{F}(\Delta)\, y = x^{-r}\, \mathrm{F}(\mathrm{D}) \,.\, (x^r y)$$

where Δ is the operator $\mathrm{D} + r/x = d/dx + r/x$.

Proof :—

$$\Delta y = x^{-r}\left[x^r \mathrm{D}y + y \,.\, r x^{r-1}\right] = x^{-r}\, \mathrm{D}\,(x^r y),$$

$$\Delta^2 y = x^{-r}\, \mathrm{D}\,(x^r \,.\, \Delta y) = x^{-r}\, \mathrm{D}^2\,(x^r y),$$

$$\Delta^3 y = x^{-r}\, \mathrm{D}^3\,(x^r y),$$

and so on, for all integral (positive) values of n.

If $\Delta^{-n} \,.\, y = z$,

$$y = \Delta^n z = x^{-r}\, \mathrm{D}^n\,(x^r z).$$

$\therefore$ $$x^{-r}\, \mathrm{D}^{-n}\,(x^r y) = z = \Delta^{-n}\, y.$$

Also $$\Delta^0\, y = y = x^{-r}\,(x^r y) = x^{-r}\, \mathrm{D}^0\,(x^r y).$$

Hence for all integral values of n

$$\Delta^n y = x^{-r}\, \mathrm{D}^n\,(x^r y),$$

and consequently

$$\mathrm{F}(\Delta) y = x^{-r}\, \mathrm{F}(\mathrm{D}) \,.\, (x^r y)$$

Taking F to be the exponential function, we get the results given in the question.

Question 1123.

(G. S. M.) :—Find a formula for the number of regions into which space of three dimensions is divided by n planes.

Solution by N. B. Mitra, and also by the proposer.

We shall consider the most general case ; viz the case in which no wo of the planes are parallel, no three of them pass through the same straight line and no four of them have a common point. Let F (N) denote the required number. Also let $f(n)$ denote the number into which a plane is divided by n straight lines in it, no two of which are parallel and no three of which meet in a point.

Now consider the nth plane. It is cut by the other planes in $n-1$ straight lines and these $n-1$ lines divide the nth plane into $f(n-1)$ parts. And these parts are such that if any one of them were removed, two regions into which space is divided would become one.

Hence $$F(n) = F(n-1) + f(n-1). \quad \ldots \quad (1)$$

Now to find $f(n)$. We observe that the nth straight line is cut by each of the remaining $n-1$ lines and therefore it is divided into n parts Now there are two parts of the plane on two sides of each of these parts of the nth line which would become one if the nth line were removed.

Hence $$f(n) = f(n-1) + n.$$

Putting n equal to $n-1$, $n-2$,...,...2 in succession in this and adding all the equations, we get

$$f(n) = n + (n-1) \ldots\ldots + 2 + f(1) \text{ and } f(1) = 2.$$

Thus $$f(n) = 1 + \tfrac{1}{2}n(n+1).$$

Hence from (1), $F(n) = F(n-1) + \frac{1}{2}n(n-1) + 1.$

Putting $n-1$, $n-2$, ... 3, 2 in succession for n in this and adding all the results, we get

$$\begin{aligned} F(n) &= \tfrac{1}{2}\Sigma n(n-1) + n - 1 + F(1) \\ &= (n-1)n(n+1)/6 + n + 1, \qquad \text{since } F(1) = 2 \\ &= (n^3 + 5n + 6)/6. \end{aligned}$$

QUESTIONS FOR SOLUTION.

1195. (K. J. Sanjana, M.A.):—From the point P (p, q, r) perpendiculars PM, PN are drawn to the planes given by the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0:$$

prove that the length of MN is given by

$$\frac{\{A^{-1}(qG-rH)^2+B^{-1}(rH-pF)^2+C^{-1}(pF-qG)^2\}^{\frac{1}{2}}}{\{A+B+C-\frac{1}{4}(a+b+c)^2\}^{\frac{1}{2}}}$$

where A, B, C, F, G, H have their usual values.

1196. (K. J. Sanjana, M.A.):—Solve the equation

$$\frac{d^3y}{dx^3}-3nx^{n-1}\frac{d^2y}{dx^2}+\{3n^2x^{2n-2}-3n(n-1)x^{n-2}\}\frac{dy}{dx}$$

$$-\{n^3x^{3n-3}-3n^2(n-1)x^{2n-3}+n(n-1)(n-2)x^{n-3}\}y=0.$$

1197. (N. B. Mitra):—If $2n+1$ is a prime p, prove that the numerator of $2^{4n}+p\left(1+\frac{1}{2}\ldots\ldots+\frac{1}{n}\right)-1\equiv 0 \pmod{p^2}$.

1198. (F. H. V. Gulasekharam):—The major axis of an inscribed conic of a triangle ABC is given by the trilinear equation

$$p\alpha+q\beta+r\gamma=0.$$

(i) Prove that the equation of the conic is

$$\Sigma\{p(p^2-q^2-r^2+2qr\cos A)\}^{\frac{1}{2}}=0.$$

(ii) If the axis be fixed in direction, the centre traces out a rectangular hyperbola through the in-centre and the e-centre, *w. r. t.* which the original triangle is self-polar.

1199. (F. H. V. Gulasekharam):—Prove that the radius of the pedal circle of a point P with respect to a triangle ABC is given by

$$\frac{R\sin A\sin B\sin C}{\cos PAB\,.\cos PBC\,.\cos PCA+\cos PAC\,.\cos PCB\,.\cos PBA},$$

where R is the circum-radius of ABC.

If the pedal triangle of P is in perspective with ABC, prove that the above expression reduces to

$$\tfrac{1}{2}R\,.\,\frac{\sin A\sin B\sin C}{\cos PAB\,.\cos PBC\,.\cos PCA}.$$

1200. (Martyn M. Thomas):—Show that the parabola

$$(a^2y^2-b^2x^2)\sin 2\theta-2abxy\cos 2\theta+b^2x^2+a^2y^2-4a^2b^2$$
$$+2\sqrt{2ab}\{(bx-ay)\sin\theta+(bx+ay)\cos\theta\}=0,$$

where θ is variable, osculates its envelope.

1201. (MARTYN M. THOMAS) :—A periodic comet when at an angular distance θ from the perihelion of its orbit, of eccentricity e, suddenly encounters a resistance which brings it to a stand-still. Show that it will fall into the sun in time $\frac{t}{4\sqrt{2}}\left(\frac{1-e^2}{1+e\cos\theta}\right)^{\frac{3}{2}}$ where t is the periodic time of the comet.

1202. (SELECTED) :—Being given that

$$x = \frac{(1+uv)(u-v)}{(1-uv)^2},\quad y = \frac{2\{uv(1-u^2)(1-v^2)\}^{\frac{1}{2}}}{(1-uv)^2},$$

find what $dx^2 + dy^2$ becomes, in terms of u, v, du, dv; and explain accurately the geometrical meaning of the result.

1203. (C. N. SREENIVASA IYENGAR) :—If $Q(x)$ be the quotient, and $R(x)$ the remainder when $f(x)$ is divided by its $(2n+1)^{th}$ derivative, then the roots of $Q(x).R(x) = 0$ separate those of $f(x) = 0$, supposed to be all real.

1204. (A. A. KRISHNASWMI AIYANGAR) :—Conics are drawn having quadritactic contact with the Folium of Descartes at one point and bitactic contact elsewhere. Show that the chord of contact envelopes a rectangular hyperbola.

1205. (A. A. KRISHNASWAMI AIYANGAR) :—If n be an even number, prove that all the non-repeated factors of $(2^n - 1)$ are prime to n and that the repeated factors occur once oftener than in n.

1206. (V. RAMASWAMI IYER) :—Three collinear points F_1, F_2, F_3, being the foci of a confocal system of Cartesian ovals, prove (1) that there are two ovals T_1, T_2 of the system that have vertices C_1, C_2 respectively, which are points of undulation ; (2) that the vertices of equilateral triangles described on F_1F_2, F_2F_3, F_3F_1 both ways, are six of the points of intersection of T_1 and T_2 ; (3) that the centres of these equilateral triangles lie on a circle X whose centre is the mean centre of F_1, F_2, F_3 and which cuts the axis in C_1, C_2 ; (4) that the circle Y concentric with X and of double its radius cuts the axis in O_1, O_2 the triple foci of T_1, T_2 ; (5) that F_1, F_2, F_3 are the projections on the axis of the vertices of two equilateral triangles inscribable in Y ; and (6) that the two remaining intersections of T_1, T_2 are the points with respect to which F_1, F_2, F_3 invert into the vertices of equilateral triangles.

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XII – 1, 3, 4, 6

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